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Volume 4: Hilbert Spaces

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C*-Algebras

Volume 4: Hilbert Spaces

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Preface

Functional analysis plays an important role in the program of studies at the Swiss Federal Institute of Technology. At present, courses entitled Functional Analysis I and II are taken during the fifth and sixth semester, respectively. I have taught these courses several times and after a while typewritten lecture notes resulted that were distributed to the students. During the academic year 1987/88, I was fortunate enough to have an eager enthusiastic group of students that I had already encountered previously in other lecture courses. These students wanted to learn more in the area and asked me to design a continuation of the courses. Accordingly, I proceeded during the academic year, following, with a series of special lectures, Functional Analysis III and IV, for which I again distributed typewritten lecture notes. At the end I found that there had accumulated a mass of textual material, and I asked myself if I should not publish it in the form of a book. Unfortunately, I realized that the two special lecture series (they had been given only once) had been badly organized and contained material that should have been included in the first two portions. And so I came to the conclusion that I should write everything anew – and if at all – then preferably in English. Little did I realize what I was letting myself in for! The number of pages grew almost imperceptibly and at the end it had more than doubled. Also, the English language turned out to be a stumbling block for me; I would like to take this opportunity to thank Prof. Imre Bokor and Prof. Edgar Reich for their help in this regard. Above all I must thank Mrs. Barbara Aquilino, who wrote, first a WordMARC™, and then a LaTeX™ version with great competence, angelic patience, and utter devotion, in spite of illness. My thanks also go to the Swiss Federal Institute of Technology that generously provided the infrastructure for this extensive enterprise and to my colleagues who showed their understanding for it.

Corneliu Constantinescu

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Table of Contents of Volume 4

Introduction xix

5 Hilbert Spaces 3
5.1 Pre–Hilbert Spaces 3
5.1.1 General Results 3
5.1.2 Examples 14
5.1.3 Hilbert sums 19
5.2 Orthogonal Projections of Hilbert Space 24
5.2.1 Projections onto Convex Sets 24
5.2.2 Orthogonality 29
5.2.3 Orthogonal Projections 33
5.2.4 Mean Ergodic Theorems 54
5.2.5 The Fréchet–Riesz Theorem 63
5.3 Adjoint Operators 72
5.3.1 General Results 72
5.3.2 Supplementary Results 86
5.3.3 Selfadjoint Operators 108
5.3.4 Normal Operators 123
5.4 Representations 130
5.4.1 Cyclic Representation 130
5.4.2 General Representations 146
5.4.3 Example of Representations 156
5.5 Orthonormal Bases 166
5.5.1 General Results 166
5.5.2 Hilbert Dimension 191
5.5.3 Standard Examples 206
5.5.4 The Fourier–Plancherel Operator 218
5.5.5 Operators and Orthonormal Bases 223
5.5.6 Self–normal Compact Operators 243
5.5.7 Examples of Real C^*–Algebras 258

5.6 Hilbert right C^*–Modules 286
5.6.1 Some General Results 286
5.6.2 Self–duality 310
5.6.3 Von Neumann right W^*–modules 341
5.6.4 Examples 373
5.6.5 $\mathcal{K}_E$ 430
5.6.6 Matrices over C^*–algebras 477
5.6.7 Type I W^*–algebras 515

Name Index 535

Subject Index 539

Symbol Index 567

Contents of All Volumes

Table of Contents of Volume 1

Introduction xix

Some Notation and Terminology 1

1 Banach Spaces 7
1.1 Normed Spaces 7
1.1.1 General Results 7
1.1.2 Some Standard Examples 12
1.1.3 Minkowski's Theorem 31
1.1.4 Locally Compact Normed Spaces 35
1.1.5 Products of Normed Spaces 37
1.1.6 Summable Families 40
Exercises 58
1.2 Operators 61
1.2.1 General Results 61
1.2.2 Standard Examples 74
1.2.3 Infinite Matrices 92
1.2.4 Quotient Spaces 113
1.2.5 Complemented Subspaces 123
1.2.6 The Topology of Pointwise Convergence 134
1.2.7 Convex Sets 138
1.2.8 The Alaoglu-Bourbaki Theorem 148
1.2.9 Bilinear Maps 150
Exercises 153
1.3 The Hahn-Banach Theorem 159
1.3.1 The Banach Theorem 159
1.3.2 Examples in Measure Theory 171
1.3.3 The Hahn-Banach Theorem 180
1.3.4 The Transpose of an Operator 191

1.3.5 Polar Sets 199
1.3.6 The Bidual 211
1.3.7 The Krein–Šmulian Theorem 228
1.3.8 Reflexive Spaces 240
1.3.9 Completion of Normed Spaces 245
1.3.10 Analytic Functions 246
Exercises 254
1.4 Applications of Baire's Theorem 256
1.4.1 The Banach–Steinhaus Theorem 256
1.4.2 Open Mapping Principle 264
Exercises 280
1.5 Banach Categories 281
1.5.1 Definitions 281
1.5.2 Functors 288
1.6 Nuclear Maps 308
1.6.1 General Results 308
1.6.2 Examples 322
1.7 Ordered Banach spaces 334
1.7.1 Ordered normed spaces 334
1.7.2 Order Continuity 340

Name Index 357

Subject Index 359

Symbol Index 371

Table of Contents of Volume 2

Introduction xix

2 Banach Algebras 3
2.1 Algebras 3
2.1.1 General Results 3
2.1.2 Invertible Elements 13
2.1.3 The Spectrum 17
2.1.4 Standard Examples 32
2.1.5 Complexification of Algebras 51
Exercises 65
2.2 Normed Algebras 69
2.2.1 General Results 69
2.2.2 The Standard Examples 82
2.2.3 The Exponential Function and the Neumann Series . . . 114
2.2.4 Invertible Elements of Unital Banach Algebras 125
2.2.5 The Theorems of Riesz and Gelfand 153
2.2.6 Poles of Resolvents 161
2.2.7 Modules 174
Exercises 197
2.3 Involutive Banach Algebras 201
2.3.1 Involutive Algebras 201
2.3.2 Involutive Banach Algebras 241
2.3.3 Sesquilinear Forms 275
2.3.4 Positive Linear Forms 287
2.3.5 The State Space 305
2.3.6 Involutive Modules 322
Exercises 328
2.4 Gelfand Algebras 331
2.4.1 The Gelfand Transform 331
2.4.2 Involutive Gelfand Algebras 343

2.4.3 Examples 358
2.4.4 Locally Compact Additive Groups 365
2.4.5 Examples 378
2.4.6 The Fourier Transform 390
Exercises 396

3 Compact Operators 399
3.1 The General Theory 399
3.1.1 General Results 399
3.1.2 Examples 419
3.1.3 Fredholm Operators 437
3.1.4 Point Spectrum 468
3.1.5 Spectrum of a Compact Operator 477
3.1.6 Integral Operators 489
Exercises 517
3.2 Linear Differential Equations 518
3.2.1 Boundary Value Problems for Differential Equations . . . 518
3.2.2 Supplementary Results 530
3.2.3 Linear Partial Differential Equations 549
Exercises 563

Name Index 565

Subject Index 568

Symbol Index 588

Table of Contents of Volume 3

Introduction xix

4 C^*-Algebras 3

4.1 The General Theory 3

4.1.1 General Results 4

4.1.2 The Symmetry of C^*-Algebra 30

4.1.3 Functional calculus in C^*-Algebras 56

4.1.4 The Theorem of Fuglede-Putnam 75

4.2 The Order Relation 92

4.2.1 Definition and General Properties 92

4.2.3 Examples 116

4.2.4 Powers of Positive Elements 123

4.2.5 The Modulus 143

4.2.6 Ideals and Quotients of C^*-Algebras 150

4.2.7 The Ordered Set of Orthogonal Projections 162

4.2.8 Approximate Unit 178

4.3 Supplementary Results on C^* Algebras 208

4.3.1 The Exterior Multiplication 208

4.3.2 Order Complete C^*-Algebras 215

4.3.3 The Carrier 243

4.3.4 Hereditary C^*-Subalgebras 263

4.3.5 Simple C^*-algebras 276

4.3.6 Supplementary Results Concerning Complexification . . 286

4.4 W^*-Algebras 297

4.4.1 General Properties 297

4.4.2 F as an E submodule of E' 309

4.4.3 Polar Representation 335

4.4.4 W^*-Homomorphisms 361

Name Index 385

Subject Index 388

Symbol Index 411

Table of Contents of Volume 4

Introduction xix

5 Hilbert Spaces 3
5.1 Pre-Hilbert Spaces 3
5.1.1 General Results 3
5.1.2 Examples 14
5.1.3 Hilbert sums 19
5.2 Orthogonal Projections of Hilbert Space 24
5.2.1 Projections onto Convex Sets 24
5.2.2 Orthogonality 29
5.2.3 Orthogonal Projections 33
5.2.4 Mean Ergodic Theorems 54
5.2.5 The Fréchet-Riesz Theorem 63
5.3 Adjoint Operators 72
5.3.1 General Results 72
5.3.2 Supplementary Results 86
5.3.3 Selfadjoint Operators 108
5.3.4 Normal Operators 123
5.4 Representations 130
5.4.1 Cyclic Representation 130
5.4.2 General Representations 146
5.4.3 Example of Representations 156
5.5 Orthonormal Bases 166
5.5.1 General Results 166
5.5.2 Hilbert Dimension 191
5.5.3 Standard Examples 206
5.5.4 The Fourier-Plancherel Operator 218
5.5.5 Operators and Orthonormal Bases 223
5.5.6 Self-normal Compact Operators 243
5.5.7 Examples of Real C^*-Algebras 258

5.6 Hilbert right C^*-Modules 286
5.6.1 Some General Results 286
5.6.2 Self-duality 310
5.6.3 Von Neumann right W^*-modules 341
5.6.4 Examples 373
5.6.5 $\mathcal{K}_E$ 430
5.6.6 Matrices over C^*–algebras 477
5.6.7 Type I W^*–algebras 515

Name Index 535

Subject Index 539

Symbol Index 567

Table of Contents of Volume 5

Introduction xix

6 Selected Chapters of C^*-Algebras 3
6.1 $\mathcal{L}^p$-Spaces 3
6.1.1 Characteristic Families of Eigenvalues 3
6.1.2 Characteristic Sequences 10
6.1.3 Properties of the $\mathcal{L}^p$-spaces 21
6.1.4 Hilbert-Schmidt Operators 46
6.1.5 The Trace 56
6.1.6 Duals of $\mathcal{L}^p$-spaces 72
6.1.7 Exterior Multiplication and $\mathcal{L}^p$-Spaces 79
6.1.8 The Canonical Projection of the Tridual of $\mathcal{K}$ 102
6.1.9 Integral Operators on Hilbert Spaces 116
6.2 Selfadjoint Linear Differential Equations 124
6.2.1 Selfadjoint Boundary Value Problems 125
6.2.2 The Regular Sturm–Liouville Theory 139
6.2.3 Selfadjoint Linear Differential Equations on $\mathbf{T}$ 150
6.2.4 Associated Parabolic and Hyperbolic Evolution Equations 153
6.2.5 Selfadjoint Linear Partial Differential Equations 184
6.2.6 Associated Parabolic and Hyperbolic Evolution Equations 192
6.3 Von Neumann Algebras 202
6.3.1 The Strong Topology 203
6.3.2 Bidual of a C^*-algebra 218
6.3.3 Extension of the Functional Calculus 263
6.3.4 Von Neumann-Algebras 283
6.3.5 The Commutants 293
6.3.6 Irreducible Representations 299
6.3.7 Commutative von Neumann Algebras 320
6.3.8 Representations of W^*-Algebras 325
6.3.9 Finite-dimensional C^*-algebras 334

6.3.10 A generalization 355

7 C^* algebras Generated by Groups 369
7.1 Projective Representations of Groups 369
7.1.1 Schur functions 369
7.1.2 Projective Representations 404
7.1.3 Supplementary Results 431
7.1.4 Examples 466
7.2 Clifford Algebras 492
7.2.1 General Clifford Algebras 492
7.2.2 $\mathcal{C}\ell_{p,q}$ 518
7.2.3 $\mathcal{C}\ell(\mathbb{N})$ 538

Name Index 559

Subject Index 563

Symbol Index 592

Introduction

This book has evolved from the lecture course on Functional Analysis I had given several times at the ETH. The text has a strict logical order, in the style of "Definiton – Theorem – Proof – Example – Exercises". The proofs are rather thorough and there are many examples.

The first part of the book (the first three chapters, resp. the first two volumes) is devoted to the theory of Banach spaces in the most general sense of the term. The purpose of the first chapter (resp. first volume) is to introduce those results on Banach spaces which are used later or which are closely connected with the book. It therefore only contains a small part of the theory, and several results are stated (and proved) in a diluted form. The second chapter (which together with Chapter 3 makes the second volume) deals with Banach algebras (and involutive Banach algebras), which constitute the main topic of the first part of the book. The third chapter deals with compact operators on Banach spaces and linear (ordinary and partial) differential equations – applications of the theory of Banach algebras.

The second part of the book (the last four chapters, resp. the last three volumes) is devoted to the theory of Hilbert spaces, once again in the general sense of the term. It begins with a chapter (Chapter 4, resp. Volume 3) on the theory of C^* algebras and W^* algebras which are essentially the focus of the book. Chapter 5 (resp. Volume 4) treats Hilbert spaces for which we had no need earlier. It contains the representation theorems, i.e. the theorems on isometries between abstract C^*-algebras and the concrete C^*-algebras of operators on Hilbert spaces. Chapter 6 (which together with Chapter 7 makes Volume 5) presents the theory of $\mathcal{L}^p$ spaces of operators, its application to the self adjoint linear (ordinary and partial) differential equations, and the von Neumann algebras. Finally, Chapter 7 presents examples of C^* algebras defined with the aid of groups, in particular the Clifford algebras. Many important domains of C^* algebras are ignored in the present book. It should be emphasized that the whole theory is constructed in parallel for the real and for the complex numbers, i.e. the C^* algebras are real or complex.

In addition to the above (vertical) structure of the book, there is also a second (horizontal) division. It consists of a main strand, eight branches, and additional material. The results belonging to the main strand are marked with (0). Logically speaking, a reader could restrict himself/herself to these and ignore the rest. Results on the eight subsidiary branches are marked with (1), (2), (3), (4), (5), (6), (7), and (8). The key is

1. Infinite Matrices
2. Banach Categories
3. Nuclear Maps
4. Locally Compact Groups
5. Differential Equations
6. Laurent Series
7. Clifford Algebras
8. Hilbert C^*–Modules

These are (logically) independent of each other, but all depend on the main strand. Finally, the results which belong to the additional material have no marking and – from a logical perspective – may be ignored. So the reader can shorten for himself/herself this very long book using the above marks. Also, since the proofs are given with almost all references, it is possible to get into the book at any level and not to read it linearly.

We assume that the reader is familiar with classical analysis and has rudimentary knowledge of set theory, linear algebra, point set topology, and integration theory. The book addresses itself mainly to mathematicians, or to physicists interested in C^* algebras.

I would like to apologize for any omissions in citations occasioned by the fact that my acquaintance with the history of functional analysis is, unfortunately, very restricted. For this history we recommend the following texts.

1. BIRKHOFF, G. and KREYSZIG, E., The Establishment of Functional Analysis, Historia Mathematica 11 (1984), 258–321.

2. BOURBAKI, N., Elements of the History of Mathematics, (21. Topological Vector Spaces), Springer-Verlag (1994).

3. DIEUDONNÉ, J., History of Functional Analysis, North–Holland (1981).

4. DIEUDONNÉ, J., A Panorama of Pure Mathematics (Chapter C III: Spectral Theory of Operators), Academic Press (1982).

5. HEUSER, H., Funktionalanalysis, 2. Auflage (Kapitel XIX: Ein Blick auf die werdende Functionalanalysis), Teubner (1986), 3. Auflage (1992).

6. KADISON, R.V., Operator Algebras, the First Forty Years, in: Proceedings of Symposia in Pure Mathematics 38 I (1982), 1–18.

7. MONNA, A.F., Functional Analysis in Historical Perspective, John Wiley & Sons (1973).

8. STEEN, L.A., Highlights in the History of Spectral Theory, Amer. Math. Monthly 80 (1973), 359 382.

There is no shortage of excellent books on C^* algebras. Nevertheless, we hope that this book will be also of some utility to the mathematics community.

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VOLUME 4:
HILBERT SPACES

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5. Hilbert Spaces

Most books on functional analysis treat the theory of Hilbert spaces before the theory of C^*–algebras if they treat the latter at all. This is didactically justifiable since the former is simpler and is the main source of examples for the latter. But in a strictly logical sense, the theory of C^*–algebras has priority. The development of this theory does not require even the definition of a Hilbert space. Conversely the theory of Hilbert spaces benefits from the theory of C^*–algebras because the set of operators on a Hilbert space is a C^*–algebra in a natural way. There is a didactic principle which says that in a book on mathematics the degree of difficulty should be roughly speaking an increasing function of the page number. While we do not wish to underestimate the importance of this principle, we have chosen in this instance to let logical priority be our guide and to rely on the reader not to completely ignore Hilbert spaces. This inversion of the customary order does have substantial benefits, for when dealing with the algebra of operators on a Hilbert space we may avail ourselves of all the results from the theory of C^*–algebras.

5.1 Pre–Hilbert Spaces

A number of general results, constructions, and examples from the theory of Hilbert spaces are pesented in this first section.

5.1.1 General Results

Definition 5.1.1.1 (0) (von Neumann, 1928) *Let E be a vector space. A* ***scalar product on*** *E is a positive sesquilinear form f on E (Proposition 2.3.3.3 e)), such that for every $x \in E$,*

$$f(x,x) = 0 \Longrightarrow x = 0\,.$$

*A **pre-Hilbert space** is a vector space endowed with a scalar product. A pre-Hilbert space is called **real (complex)** if the ground field in $\mathbb{R}$ ($\mathbb{C}$). We shall usually denote a scalar product in a pre-Hilbert space E by*

$$E \times E \longrightarrow \mathbb{K}, \quad (x,y) \longmapsto \langle x|y\rangle \quad \textbf{\textit{(scalar product of } } x \textbf{\textit{ and }} y \textbf{\textit{)}}$$

and for $x \in E$, we define

$$\|x\| := \sqrt{\langle x|x\rangle} \in \mathbb{R}_+ .$$

J. von Neumann defined only spearable Hilbert spaces. Non-separable Hilbert spaces were considered by H. Löwing (1934) and F. Rellich (1935).

Proposition 5.1.1.2 (0) *If E is a pre-Hilbert space, then the map*

$$E \longrightarrow \mathbb{R}_+, \quad x \longmapsto \|x\|$$

is a norm and

$$|\langle x|y\rangle| \leq \|x\| \, \|y\| \quad \textbf{\textit{(Schwarz's Inequality)}}$$

*for all $x, y \in E$. The above norm is called **the canonical norm on the pre-Hilbert space** E. Unless otherwise stated, we always regard a pre-Hilbert space as endowed with its canonical norm. A **Hilbert space** is a complete pre-Hilbert space. A **real (complex) Hilbert space** is a complete real (complex) pre-Hilbert space.*

The assertion follows immediately from Proposition 2.3.3.9. ■

Corollary 5.1.1.3 (0) *If E is a pre-Hilbert space, then the map*

$$E \times E \longrightarrow \mathbb{K}, \quad (x,y) \longmapsto \langle x|y\rangle$$

is continuous.

By Schwarz's Inequality,

$$|\langle x|y\rangle| \leq \|x\| \, \|y\|$$

for all $(x,y) \in E \times E$ and the assertion now follows from Proposition 1.2.9.2 c $\Rightarrow$ a. ■

Corollary 5.1.1.4 (0) *Let E, F be pre-Hilbert spaces and $u : E \to F$ a bijective linear map. u is an isometry of the normed spaces E and F iff*

$$\langle ux|uy\rangle = \langle x|y\rangle$$

for all $x, y \in E$.

The necessity follows from Proposition 2.3.3.7 a $\Rightarrow$ b and Proposition 2.3.3.8 a $\Rightarrow$ c & d. The sufficiency follows from

$$\|ux\|^2 = \langle ux|ux\rangle = \langle x|x\rangle = \|x\|^2$$

for every $x \in E$. ■

Corollary 5.1.1.5 *Let $E \neq \{0\}$ be a pre-Hilbert space. Take $u \in \mathcal{L}(E)$, $\alpha \in \mathbb{K}$, and $x \in E^{\#}$. If*

$$|\alpha| = \|u\|\,, \quad \langle ux|x\rangle = \alpha\,,$$

then $x \in \operatorname{Ker}(\alpha 1 - u)$ and $\alpha \in \sigma_p(u)$.

Now

$$\|(\alpha 1 - u)x\|^2 = \|\alpha x - ux\|^2 = |\alpha|^2\|x\|^2 - 2\operatorname{re}\alpha\langle x|ux\rangle + \|ux\|^2 \leq$$

$$\leq |\alpha|^2 - 2\alpha\overline{\alpha} + \|u\|^2 = 0$$

(Corollary 2.3.3.4). Hence

$$(\alpha 1 - u)x = 0\,,$$

$$x \in \operatorname{Ker}(\alpha 1 - u)\,.$$

If $u \neq 0$, then $\alpha \neq 0$ and $x \neq 0$. Thus $\alpha \in \sigma_p(u)$. If $u = 0$, then

$$\alpha = 0 \in \sigma_p(u)\,.$$ ■

Theorem 5.1.1.6 (0) (Jordan, von Neumann) *Let E be a normed space. The following are equivalent:*

a) *There is a scalar product on E generating the norm of E.*

b) $x, y \in E \Rightarrow \|x + y\|^2 + \|x - y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$.

c) $x, y \in E \Rightarrow \|x+y\|^2 + \|x-y\|^2 \geq 2(\|x\|^2 + \|y\|^2)$.

If these conditions are fulfillled, then the scalar product in a) is unique.

The uniqueness of the scalar product follows from Proposition 2.3.3.7 a $\Rightarrow$ b and Proposition 2.3.3.8 a $\Rightarrow$ c & d.

a $\Rightarrow$ b & c follows from the parallelogram law (Proposition 2.3.3.2 b)).

b $\Rightarrow$ c (resp. c $\Rightarrow$ b). Put

$$\begin{cases} a := x + y \\ b := x - y \,. \end{cases}$$

Then

$$\begin{cases} x = \frac{1}{2}(a+b) \\ y = \frac{1}{2}(a-b) \end{cases},$$

so

$$\|x+y\|^2 + \|x-y\|^2 = \|a\|^2 + \|b\|^2 \geq (\text{resp. } \leq)$$

$$\geq (\text{resp. } \leq)\frac{1}{2}(\|a+b\|^2 + \|a-b\|^2) = 2(\|x\|^2 + \|y\|^2)\,.$$

b & c $\Rightarrow$ a.

Case 1 $\quad \mathbb{K} = \mathbb{R}$

Given $x, y \in E$, put

$$\langle x|y\rangle := \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)\,.$$

Then, for every $x, y \in E$,

$$\langle x|y\rangle = \langle y|x\rangle$$

and

$$\langle x|x\rangle = \|x\|^2\,.$$

Thus only linearity in the first variable needs further proof.

Step 1 $\quad x, y, z \in E \Rightarrow \langle x+y|z\rangle = \langle x|z\rangle + \langle y|z\rangle$

Now

$$\langle x|z\rangle + \langle y|z\rangle = \frac{1}{4}(\|x+z\|^2 - \|x-z\|^2 + \|y+z\|^2 - \|y-z\|^2) =$$

$$= \frac{1}{8}\left(\|x+z+y\|^2 + \|x+z-y\|^2 - 2\|y\|^2 - \|x-z+y\|^2 - \|x-z-y\|^2 + \right.$$

$$+2\|y\|^2 + \|x+z+y\|^2 + \|x-z-y\|^2 - 2\|x\|^2 - \|x+y-z\|^2 -$$

$$\left. - \|x-y+z\|^2 + 2\|x\|^2\right) = \frac{1}{4}\left(\|x+y+z\|^2 - \|x+y-z\|^2\right) = \langle x+y|z\rangle\,.$$

Step 2 $\quad n \in \mathbb{N} \cup \{0\}\,,\ x, y \in E \Rightarrow \langle nx|y\rangle = n\langle x|y\rangle\,.$

The assertion follows from Step 1, by complete induction.

Step 3 $\quad x, y \in E \Rightarrow \langle -x|y\rangle = -\langle x|y\rangle$

By Step 1,

$$0 = \langle 0|y\rangle = \langle x-x|y\rangle = \langle x|y\rangle + \langle -x|y\rangle\,.$$

Hence

$$\langle -x|y\rangle = -\langle x|y\rangle\,.$$

Step 4 $\quad \alpha \in \mathbb{Q}\,,\ x, y \in E \Rightarrow \langle \alpha x|y\rangle = \alpha\langle x|y\rangle$

There are $m \in \mathbb{Z}\,,\ n \in \mathbb{N}$ with $\alpha = \frac{m}{n}$. By Steps 2 and 3,

$$n\langle \alpha x|y\rangle = \langle n\alpha x|y\rangle = \langle mx|y\rangle = m\langle x|y\rangle\,.$$

Hence

$$\langle \alpha x|y\rangle = \frac{m}{n}\langle x|y\rangle = \alpha\langle x|y\rangle\,.$$

Step 5 $\quad x, y \in E \Rightarrow |\langle x|y\rangle| \le \|x\|\,\|y\|$

$$\langle x|y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \le \frac{1}{4}\left((\|x\| + \|y\|)^2 - (\|x\| - \|y\|)^2\right) = \|x\|\,\|y\|\,,$$

$$-\langle x|y\rangle = \langle -x|y\rangle \le \|-x\|\,\|y\| = \|x\|\,\|y\|$$

by the triangle inequality and Step 3. Thus

$$|\langle x|y\rangle| \leq \|x\| \, \|y\|$$

Step 6 $\alpha \in \mathbb{R}$, $x, y \in E \Rightarrow \langle \alpha x|y\rangle = \alpha\langle x|y\rangle$

Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Q}$ converging to α. Then

$$|\langle \alpha x|y\rangle - \langle \alpha_n x|y\rangle| = |\langle \alpha x|y\rangle + \langle -\alpha_n x|y\rangle = |\langle (\alpha - \alpha_n)x|y\rangle| \leq$$

$$\leq \|(\alpha - \alpha_n)x\| \, \|y\| = |\alpha - \alpha_n| \, \|x\| \, \|y\|$$

by Steps 3, 1, and 5. Thus

$$\langle \alpha x|y\rangle = \lim_{n\to\infty} \langle \alpha_n x|y\rangle = \lim_{n\to\infty} \alpha_n \langle x|y\rangle = \alpha\langle x|y\rangle$$

by Step 4.

Case 2 $\mathbb{K} = \mathbb{C}$

Given $x, y \in E$, define

$$\mathrm{re}\langle x|y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right),$$

$$\mathrm{im}\langle x|y\rangle := \mathrm{re}\langle x|iy\rangle .$$

Step 1 $x \in E \Rightarrow \langle x|x\rangle = \|x\|^2$

Now

$$\mathrm{re}\langle x|x\rangle = \|x\|^2,$$

$$\mathrm{im}\langle x|x\rangle = \mathrm{re}\langle x|ix\rangle = \frac{1}{4}\left(\|x+ix\|^2 - \|x-ix\|^2\right) = \frac{1}{4}\left(2\|x\|^2 - 2\|x\|^2\right) = 0,$$

$$\langle x|x\rangle = \mathrm{re}\langle x|x\rangle + i\,\mathrm{im}\langle x|x\rangle = \|x\|^2 .$$

Step 2 $x, y \in E \Rightarrow \langle x|y\rangle = \langle y|x\rangle$

$$\mathrm{re}\langle x|y\rangle = \mathrm{re}\langle y|x\rangle ,$$

$$\mathrm{im}\langle x|y\rangle = \mathrm{re}\langle x|iy\rangle = \frac{1}{4}\left(\|x+iy\|^2 - \|x-iy\|^2\right) =$$

$$= \frac{1}{4}\left(\| - ix + y\|^2 - \|ix + y\|^2\right) = -\mathrm{re}\langle y|ix\rangle = -\mathrm{im}\langle y|x\rangle\,.$$

Hence

$$\langle x|y\rangle = \langle y|x\rangle\,.$$

Step 3 $\quad \alpha, \beta \in \mathbb{R}\,,\ x, y, z \in E \Rightarrow \langle \alpha x + \beta y|z\rangle = \alpha\langle x|z\rangle + \beta\langle y|z\rangle$

By Case 1,

$$\langle \alpha x + \beta y|z\rangle = \mathrm{re}\langle \alpha x + \beta y|z\rangle + i\,\mathrm{re}\langle \alpha x + \beta y|iz\rangle =$$

$$= \alpha\,\mathrm{re}\langle x|y\rangle + \beta\mathrm{re}\langle y|z\rangle + i(\alpha\mathrm{re}\langle x|iz\rangle + \beta\mathrm{re}\langle y|iz\rangle =$$

$$= \alpha(\mathrm{re}\langle x|z\rangle + i\,\mathrm{re}\langle x|iz\rangle) + \beta(\mathrm{re}\langle y|z\rangle + i\,\mathrm{re}\langle y|iz\rangle = \alpha\langle x|z\rangle + \beta\langle y|z\rangle\,.$$

Step 4 $\quad x, y \in E \Rightarrow \langle ix|y\rangle = i\langle x|y\rangle$

By Steps 2 and 3,

$$\mathrm{re}\langle ix|y\rangle = \mathrm{re}\langle y|ix\rangle = \mathrm{im}\langle y|x\rangle = -\mathrm{im}\langle x|y\rangle\,,$$

$$\mathrm{im}\langle ix|y\rangle = -\mathrm{im}\langle y|ix\rangle = -\mathrm{re}\langle y| - x\rangle = -\mathrm{re}\langle -x|y\rangle = \mathrm{re}\langle x|y\rangle\,.$$

Hence

$$\langle ix|y\rangle = \mathrm{re}\langle ix|y\rangle + i\,\mathrm{im}\langle ix|y\rangle = -\mathrm{im}\langle x|y\rangle + i\,\mathrm{re}\langle x|y\rangle =$$

$$= i(\mathrm{re}\langle x|y\rangle + i\,\mathrm{im}\langle x|y\rangle) = i\langle x|y\rangle\,.$$

Step 5 $\quad \alpha, \beta \in \mathbb{C}\,,\ x, y, z \in E \Rightarrow \langle x|\alpha y + \beta z\rangle = \overline{\alpha}\langle x|y\rangle + \overline{\beta}\langle x|z\rangle\,.$

By Steps 2, 3, and 4,

$$\langle x|\alpha y + \beta z\rangle = \langle \alpha y + \beta z|x\rangle = \alpha\langle y|x\rangle + \beta\langle z|x\rangle =$$

$$= \overline{\alpha}\,\langle y|x\rangle + \overline{\beta}\,\langle z|x\rangle = \overline{\alpha}\langle x|y\rangle + \overline{\beta}\langle x|y\rangle\,. \qquad \blacksquare$$

Remark. This theorem makes it possible to define pre-Hilbert (Hilbert) spaces as normed (Banach) spaces satisfying the parallelogram law.

Corollary 5.1.1.7 (0) *Let E be a pre-Hilbert space and $\widetilde{E}$ the completion of the associated normed space. There is a unique scalar product on $\widetilde{E}$ generating the norm of $\widetilde{E}$. $\widetilde{E}$ with this scalar product is called* ***the completion of*** *E. The scalar product of E is the restriction to E of the scalar product of $\widetilde{E}$.*

Take $x, y \in \widetilde{E}$. There are sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in E converging to x and y, respectively. By the parallelogram law (Proposition 2.3.3.2 b)),

$$\|x+y\|^2 + \|x-y\|^2 = \lim_{n\to\infty} \|x_n+y_n\|^2 + \lim_{n\to\infty} \|x_n-y_n\|^2 =$$

$$= \lim_{n\to\infty} \left(\|x_n+y_n\|^2 + \|x_n-y_n\|^2\right) =$$

$$= \lim_{n\to\infty} 2\left(\|x_n\|^2 + \|y_n\|^2\right) = 2\left(\|x\|^2 + \|y\|^2\right) .$$

By Theorem 5.1.1.6, there is a unique scalar product on $\widetilde{E}$ generating its norm. The restriction to E of this scalar product generates the norm of E. Hence it coincides with the original scalar product of E (Theorem 5.1.1.6). ■

Corollary 5.1.1.8 *Let E, F be pre-Hilbert spaces and take $u \in \mathcal{L}(E,F)$. Then the following are equivalent:*

a) $\|ux\| = \|x\|$ *for every* $x \in E$.

b) $\langle ux|uy\rangle = \langle x|y\rangle$ *for all* $x, y \in E$.

a ⇒ b. The map

$$E \longrightarrow \operatorname{Im} E, \quad x \longmapsto ux$$

is an isometry and the assertion follows from Theorem 5.1.1.6.

b ⇒ a is trivial. ■

Proposition 5.1.1.9 (0) *Let E be a real (complex) pre-Hilbert space, f a sesquilinear form on E, and take $\alpha \in \mathbb{R}_+$. If*

$$|\widetilde{f}(x)| \leq \alpha\|x\|^2$$

for every $x \in E$, then

$$|f(x,y) + f(y,x)| \leq 2\alpha\|x\|\,\|y\|,$$

$$(|f(x,y)| + |f(y,x)| \leq 2\alpha\|x\|\,\|y\|)$$

for all $x, y \in E$.

We may assume without loss of generality that $x \neq 0$ and $y \neq 0$. Consider

$$\beta := \sqrt{\frac{\|x\|}{\|y\|}}.$$

By Proposition 2.3.3.2 b),c),

$$|f(x,y) + f(y,x)| = \frac{1}{2}|\tilde{f}(x+y) - \tilde{f}(x-y)| \leq$$

$$\leq \frac{1}{2}\left(|\tilde{f}(x+y)| + |\tilde{f}(x-y)|\right) \leq \frac{\alpha}{2}\left(\|x+y\|^2 + \|x-y\|^2\right) = \alpha\left(\|x\|^2 + \|y\|^2\right).$$

Thus

$$|f(x,y) + f(y,x)| = f\left(\frac{1}{\beta}x, \beta y\right) + f\left(\beta y, \frac{1}{\beta}x\right) \leq$$

$$\leq \alpha\left(\left\|\frac{1}{\beta}x\right\|^2 + \|\beta y\|^2\right) = \alpha\left(\frac{\|y\|}{\|x\|}\|x\|^2 + \frac{\|x\|}{\|y\|}\|y\|^2\right) = 2\alpha\|x\|\,\|y\|.$$

Suppose $\mathbb{K} = \mathbb{C}$. Take $\gamma, \delta \in \mathbb{R}$ with

$$f(x,y) = e^{2i\gamma}|f(x,y)|,$$

$$f(y,x) = e^{2i\delta}|f(y,x)|.$$

Then

$$|f(x,y)| + |f(y,x)| = |e^{-2i\gamma}f(x,y) + e^{-2i\delta}f(y,x)| =$$

$$= |e^{i(\delta-\gamma)}f(x,y) + e^{i(\gamma-\delta)}f(y,x)| =$$

$$= |f(e^{i\delta}x, e^{i\gamma}y) + f(e^{i\gamma}y, e^{i\delta}x)| \leq 2\alpha\|e^{i\delta}x\|\,\|e^{i\gamma}y\| = 2\alpha\|x\|\,\|y\|. \quad \blacksquare$$

Proposition 5.1.1.10 *Let E be a pre-Hilbert space and take $x \in E$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E. The following are equivalent:*

a) $\lim_{n\to\infty} x_n = x$.

b) $\lim_{n\to\infty} \langle x_n | x\rangle = \lim_{n\to\infty} \|x_n\|^2 = \|x\|^2$.

a $\Rightarrow$ b is trivial.

b $\Rightarrow$ a. By Corollary 2.3.3.4,

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\mathrm{re}\langle x_n|x\rangle + \|x\|^2$$

for every $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} \|x_n - x\| = 0\,, \quad \lim_{n\to\infty} x_n = x\,. \qquad \blacksquare$$

Proposition 5.1.1.11 (0) *Let E be a pre-Hilbert space and take $x, y \in E$. Put:*

a) $:\Leftrightarrow \|x + y\| = \|x\| + \|y\|$.

b) $:\Leftrightarrow \mathrm{re}\langle x|y\rangle = \|x\|\,\|y\|$.

c) $:\Leftrightarrow \|y\|x = \|x\|y$.

d) $:\Leftrightarrow$ *x and y are linearly dependent.*

e) $:\Leftrightarrow |\langle x|y\rangle| = \|x\|\,\|y\|$.

Then

$a \Leftrightarrow b \Leftrightarrow c \Rightarrow d \Leftrightarrow e$.

a $\Leftrightarrow$ b. By Corollary 2.3.3.4,

$$\|x + y\|^2 = \|x\|^2 + 2\mathrm{re}\langle x|y\rangle + \|y\|^2\,,$$

from which the assertion follows.

b $\Rightarrow$ c. By Corollary 2.3.3.4,

$$\|\,\|y\|x - \|x\|y\,\|^2 = \|y\|^2\|x\|^2 - 2\|x\|\,\|y\|\mathrm{re}\langle x|y\rangle + \|x\|^2\|y\|^2 = 0\,.$$

Hence

$$\|y\|x - \|x\|y = 0\,.$$

c $\Rightarrow$ a & d and d $\Rightarrow$ e are trivial.

e $\Rightarrow$ d. Take $\theta \in \mathbb{R}$ with

$$\langle x|y\rangle = |\langle x|y\rangle|e^{i\theta}\,.$$

Then

$$\langle x|e^{i\theta}y\rangle = e^{-i\theta}\langle x|y\rangle = |\langle x|y\rangle| = \|x\|\,\|y\|\,.$$

By b $\Rightarrow$ c, x and $e^{i\theta}y$ are linearly dependent. Hence x and y are lineraly dependent. ■

Remark. a) It is easy to see that d) does not imply b) (take $y := -x$).

b) In ℓ^1,

$$\|e_1 + e_2\|_1 = \|e_1\|_1 + \|e_2\|_1$$

and so a) does not imply c) in the case of arbitrary Banach spaces.

5.1.2 Examples

Example 5.1.2.1 (0) *Let $(T, \mathfrak{T}, \mu)$ be a measure space. The map*

$$L^2(\mu) \times L^2(\mu) \longrightarrow \mathbb{K}\,, \quad (x,y) \longmapsto \int x\overline{y}d\mu$$

is a scalar product. This scalar product renders $L^2(\mu)$ a Hilbert space with norm

$$L^2(\mu) \longrightarrow \mathbb{R}_+\,, \quad x \longmapsto \left(\int |x|^2 d\mu\right)^{\frac{1}{2}}\,. \qquad \blacksquare$$

By taking μ to be the counting measure in the above example, we get:

Example 5.1.2.2 (0) *Let T be a set. The map*

$$\ell^2(T) \times \ell^2(T) \longrightarrow \mathbb{K}\,, \quad (x,y) \longmapsto \sum_{t\in T} x(t)\overline{y(t)}$$

is a scalar product, which renders $\ell^2(T)$ a Hilbert space with norm

$$\ell^2(T) \longrightarrow \mathbb{R}_+\,, \quad x \longmapsto \|x\|_2\,. \qquad \blacksquare$$

Example 5.1.2.3 *The map*

$$\ell^2 \times \ell^2 \longrightarrow \mathbb{K}\,, \quad (x,y) \longmapsto \sum_{n\in\mathbb{N}} x_n\overline{y_n}$$

is a scalar product. ℓ^2 endowed with this scalar product is a Hilbert space (it is called ***the Hilbert space of square summable sequences****). Its norm is*

$$\ell^2 \longrightarrow \mathbb{R}_+\,, \quad x \longmapsto \|x\|_2\,. \qquad \blacksquare$$

Example 5.1.2.4 *Take $n \in \mathbb{N}$. The map*

$$\mathbb{K}^n \times \mathbb{K}^n \longrightarrow \mathbb{K}\,, \quad (x,y) \longmapsto \sum_{k=1}^{n} x_k\overline{y_k}$$

is a scalar product (it is called ***the canonical scalar product of*** *$\mathbb{K}^n$). It generates the Euclidean norm.* $\blacksquare$

Example 5.1.2.5 *Let T be a locally compact group, $\mathcal{M}_b(T)$ the involutive unital Banach algebra defined in Proposition 2.3.2.28 a), λ a left invariant Haar measure on T, and take $\mu \in \mathcal{M}_b(T)$. For $x, y \in L^2(\lambda)$, define*

$$\langle x|y\rangle := \int x\overline{y}d\lambda$$

$$u : L^2(\lambda) \longrightarrow L^2(\lambda), \quad x \longmapsto \mu^* * \mu * x$$

(Proposition 2.2.2.14 a)), and

$$f : L^2(\lambda) \times L^2(\lambda) \longrightarrow \mathbb{K}, \quad (x, y) \longmapsto \langle ux|y\rangle .$$

Then f is a scalar product iff $\sigma_p(u) \subset]0, \infty[$. This condition is fulfilled whenever μ is invertible.

By Proposition 2.3.3.10, f is a positive sesquilinear form and by Example 3.1.4.14, $\sigma_p(u) \subset \mathbb{R}_+$.

If $0 \in \sigma_p(u)$, then

$$ux = 0$$

for some $x \in L^2(\lambda)\backslash\{0\}$. Thus

$$f(x, x) = \langle ux|x\rangle = 0$$

and f is not a scalar product.

Assume that $0 \notin \sigma_p(u)$. Take $x \in L^2(\lambda)$ with

$$f(x, x) = 0 .$$

Then

$$0 = \langle ux|x\rangle = \langle \mu^* * \mu * x|x\rangle = \langle \mu * x|\mu * x\rangle$$

(Proposition 2.3.2.28 c), Example 2.3.2.27 b)). Thus

$$\mu * x = 0 ,$$

$$ux = \mu^* * \mu * x = 0 ,$$

$$x = 0 .$$

Hence f is a scalar product.

If μ is invertible, then μ^* is invertible (Proposition 2.3.1.14), $\mu^* * \mu$ is invertible, and u is invertible (Proposition 2.2.2.14 e)). Hence $0 \notin \sigma_p(u)$. ■

Example 5.1.2.6 *Take $p \in [1, \infty]\backslash\{2\}$ and let T be a set. If the norm of $\ell^p(T)$ is generated by a scalar product then* $\operatorname{Card} T \leq 1$.

Assume $\operatorname{Card} T > 1$. Let s, t be two distinct elements of T. Then

$$\|e_s + e_t\|_p^2 + \|e_s - e_t\|_p^2 = 2 \cdot 2^{\frac{2}{p}} \neq 4 = 2\left(\|e_t\|_p^2 + \|e_s\|_p^2\right).$$

By the parallelogram law (Proposition 2.3.3.2 b)), the norm of $\ell^p(T)$ is not generated by a scalar product. ■

Example 5.1.2.7 *Let $n \in \mathbb{N}$ and take $a \in \operatorname{Re} \mathbb{K}_{n,n}$. The map*

$$f : \mathbb{K}^n \times \mathbb{K}^n \longrightarrow \mathbb{K}, \quad (x, y) \longmapsto \sum_{i,j=1}^{n} a_{ij} x_i \overline{y}_j$$

is a scalar product iff $\sigma(a) \subset]0, \infty[$ and every scalar product on $\mathbb{K}^n$ is of this form.

By Example 2.3.3.6, f is a positive sesquilinear form iff $\sigma(a) \subset \mathbb{R}_+$.

Assume that $0 \in \sigma(a)$. Then there is an $x \in \mathbb{K}^n \backslash \{0\}$ with

$$\sum_{j=1}^{n} a_{ij} x_j = 0$$

and so

$$\widetilde{f}(\overline{x}) = \sum_{i,j=1}^{n} a_{ij} \overline{x}_i x_j = \sum_{i,j=1}^{n} \overline{a}_{ji} \overline{x}_i x_j = 0.$$

Hence f is not a scalar product.

Now assume that $0 \notin \sigma(a)$. Take $x \in \mathbb{K}^n$ with

$$\|x\| = 1, \quad \widetilde{f}(x) = \inf\left\{\widetilde{f}(y) \,|\, y \in \mathbb{K}^n, \|x\| = 1\right\},$$

where $\|\cdot\|$ is the Euclidean norm. Then, by Lagrange's Theorem, there is an $\alpha \in \mathbb{R}$ with

$$\sum_{j=1}^{n} a_{ij} \overline{x}_j = \frac{\partial \widetilde{f}}{\partial x_i}(x) = \alpha \overline{x}_i$$

for every $i \in \mathbb{N}_n$. Thus $\alpha \in \sigma(a)$ and

$$\widetilde{f}(x) = \sum_{i,j=1}^{n} a_{ij} x_i \overline{x}_j = \alpha \sum_{i=1}^{n} |x_i|^2 = \alpha > 0.$$

It follows that

$$\widetilde{f}(y) = \|y\|^2 \widetilde{f}\left(\frac{1}{\|y\|} y\right) \geq \alpha \|y\|^2 > 0$$

for every $y \in \mathbb{K}^n \backslash \{0\}$. Hence f is a scalar product.

Let g be a scalar product on $\mathbb{K}^n$. Given $i, j \in \mathbb{N}_n$, put

$$b_{ij} := g(e_i, e_j).$$

Then

$$g(x,y) = g\left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j\right) = \sum_{i,j=1}^{n} g(e_i, e_j) x_i \overline{y}_j = \sum_{i,j=1}^{n} b_{ij} x_i \overline{y}_j$$

for all $x, y \in \mathbb{K}^n$, i.e. g is of the above form. ■

Example 5.1.2.8 *Every infinite-dimensional vector space E admits two scalar products, which generate non-equivalent norms.*

Let $(x_\iota)_{\iota \in I}$ be an algebraic basis of E and f a strictly positive real function on I such that

$$\inf_{\iota \in I} f(\iota) = 0, \quad \sup_{\iota \in I} f(\iota) = \infty.$$

Then

$$E \times E \longrightarrow \mathbb{K}, \quad \left(\sum_{\iota \in I} \alpha_\iota x_\iota, \sum_{\iota \in I} \beta_\iota x_\iota\right) \longmapsto \sum_{\iota \in I} \alpha_\iota \overline{\beta}_\iota,$$

$$E \times E \longrightarrow \mathbb{K}, \quad \left(\sum_{\iota \in I} \alpha_\iota x_\iota, \sum_{\iota \in I} \beta_\iota x_\iota\right) \longmapsto \sum_{\iota \in I} f(\iota) \alpha_\iota \overline{\beta}_\iota$$

are two scalar products on E which generate non-equivalent norms. ■

Proposition 5.1.2.9 (0) *Let E be a vector space and f a positive sesquilinear form on E. Define*

$$F := \{x \in E \mid f(x,x) = 0\}.$$

a) $(x,y) \in E \times F \Rightarrow f(x,y) = 0$.

b) *F is a vector subspace of E. We write $u : E \to E/F$ for the quotient map.*

c) *There is a unique map*

$$g : E/F \times E/F \longrightarrow \mathbb{K}$$

such that

$$g(ux, uy) = f(x,y)$$

for every $x, y \in E$.

d) g *is a scalar product, called* ***the scalar product associated to*** f .

a) is a consequence of Schwarz's Inequality (Proposition 2.3.3.9).
b) follows from a) and Corollary 2.3.3.4.
c) Take $X, Y \in E/F$ and $x_1, x_2 \in X$, $y_1, y_2 \in Y$. Then

$$x_1 - x_2, y_1 - y_2 \in F,$$

so that

$$f(x_1, y_1) - f(x_2, y_2) = f(x_1, y_1 - y_2) + f(x_1 - x_2, y_2) = 0,$$

$$f(x_1, y_1) = f(x_2, y_2),$$

by a). This proves the existence of g. The uniqueness is trivial.
d) is easy to check. ■

Example 5.1.2.10 *The map*

$$\mathbb{C} \longrightarrow \mathbb{H}, \quad \alpha + \beta i \longmapsto (\alpha, \beta, 0, 0)$$

is an injective homomorphism of unital real algebras. Identifying $\mathbb{C}$ *with its image with respect to the above map,* $\mathbb{H}$ *becomes a two-dimensional complex vector space and the map*

$$\mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{C},$$

$$((\alpha, \beta, \gamma, \delta), (\alpha', \beta', \gamma', \delta')) \longmapsto (\alpha + \beta i)(\alpha' - \beta' i) + (\gamma + \delta i)(\gamma' - \delta' i)$$

is a scalar product generating the euclidean norm on $\mathbb{H}$.

The proof is a straightforward verification. ■

5.1.3 Hilbert sums

Proposition 5.1.3.1 (0) *Let $(E_\iota)_{\iota\in I}$ be a family of pre-Hilbert spaces and put*

$$\bigoplus_{\iota\in I} E_\iota := \left\{ x \in \prod_{\iota\in I} E_\iota \;\middle|\; \sum_{\iota\in I} \|x_\iota\|^2 < \infty \right\} .$$

a) $\bigoplus_{\iota\in I} E_\iota$ *is a vector subspace of* $\prod_{\iota\in I} E_\iota$.

b) *The map*

$$\left(\bigoplus_{\iota\in I} E_\iota\right) \times \left(\bigoplus_{\iota\in I} E_\iota\right) \longrightarrow \mathbb{K}, \quad (x,y) \longmapsto \sum_{\iota\in I} \langle x_\iota | y_\iota \rangle$$

is a scalar product with associated norm

$$\bigoplus_{\iota\in I} E_\iota \longrightarrow \mathbb{R}_+, \quad x \longmapsto \left(\sum_{\iota\in I} \|x_\iota\|^2 \right)^{\frac{1}{2}} .$$

$\bigoplus_{\iota\in I} E_\iota$ *with this scalar product is called* ***the Hilbert sum of the family*** $(E_\iota)_{\iota\in I}$.

c) *If each E_ι $(\iota \in I)$ is complete then $\bigoplus_{\iota\in I} E_\iota$ is also complete.*

d) *If each E_ι $(\iota \in I)$ is separable and if I is countable, then $\bigoplus_{\iota\in I} E_\iota$ is also separable.*

a) Take $x, y \in \bigoplus_{\iota\in I} E_\iota$ and $\alpha, \beta \in \mathbb{K}$. Then

$$\left(\sum_{\iota\in I} \|(\alpha x + \beta y)_\iota\|^2 \right)^{\frac{1}{2}} = \left(\sum_{\iota\in I} \|\alpha x_\iota + \beta y_\iota\|^2 \right)^{\frac{1}{2}} \leq$$

$$\leq \left(\sum_{\iota\in I} (|\alpha| \, \|x_\iota\| + |\beta| \, \|y_\iota\|)^2 \right)^{\frac{1}{2}} \leq$$

$$\leq |\alpha| \left(\sum_{\iota\in I} \|x_\iota\|^2 \right)^{\frac{1}{2}} + |\beta| \left(\sum_{\iota\in I} \|y_\iota\|^2 \right)^{\frac{1}{2}} < \infty .$$

Hence $\alpha x + \beta y \in \bigoplus_{\iota\in I} E_\iota$, i.e. $\bigoplus_{\iota\in I} E_\iota$ is a vector subspace of $\prod_{\iota\in I} E_\iota$.

b) Take $x, y, z \in \bigoplus_{\iota \in I} E_\iota$ and $\alpha, \beta \in \mathbb{K}$. By Schwarz's Inequality,

$$\sum_{\iota \in I} |\langle x_\iota | y_\iota \rangle| \leq \sum_{\iota \in I} \|x_\iota\| \, \|y_\iota\| \leq \left(\sum_{\iota \in I} \|x_\iota\|^2 \right)^{\frac{1}{2}} \left(\sum_{\iota \in I} \|y_\iota\|^2 \right)^{\frac{1}{2}} < \infty .$$

We have

$$\sum_{\iota \in I} \langle (\alpha x + \beta y)_\iota | z_\iota \rangle = \sum_{\iota \in I} \langle \alpha x_\iota + \beta y_\iota | z_\iota \rangle =$$

$$= \sum_{\iota \in I} (\alpha \langle x_\iota | z_\iota \rangle + \beta \langle y_\iota | z_\iota \rangle) = \alpha \sum_{\iota \in I} \langle x_\iota | z_\iota \rangle + \beta \sum_{\iota \in I} \langle y_\iota | z_\iota \rangle ,$$

$$\sum_{\iota \in I} \langle y_\iota | x_\iota \rangle = \sum_{\iota \in I} \overline{\langle x_\iota | y_\iota \rangle} = \overline{\sum_{\iota \in I} \langle x_\iota | y_\iota \rangle} ,$$

$$\sum_{\iota \in I} \langle x_\iota | x_\iota \rangle = \sum_{\iota \in I} \|x_\iota\|^2 \in \mathbb{R}_+ ,$$

$$\sum_{\iota \in I} \langle x_\iota | x_\iota \rangle = 0 \Longrightarrow x = 0$$

and the assertion follows.

c) Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\bigoplus_{\iota \in I} E_\iota$. For every $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ such that

$$\|x_m - x_n\| < \varepsilon$$

for all $m, n \in \mathbb{N}$ with $m \geq n_\varepsilon$, $n \geq n_\varepsilon$. It follows that for every $\iota \in I$, $(x_n(\iota))_{n \in \mathbb{N}}$ is a Cauchy sequence, and hence a convergent sequence. Define

$$x : I \longrightarrow \bigcup_{\lambda \in I} E_\lambda , \quad \iota \longmapsto \lim_{n \to \infty} x_n(\iota) .$$

Then

$$x_n - x \in \bigoplus_{\iota \in I} E_\iota , \quad \|x_n - x\| \leq \varepsilon$$

for every $\varepsilon > 0$ and $n \in \mathbb{N}$, $n \geq n_\varepsilon$. Hence

$$x \in \bigoplus_{\iota \in I} E_\iota$$

and

$$\lim_{n\to\infty} x_n = x\,.$$

d) Given $\iota \in I$, let A_ι be a countable dense set of E_ι with $0 \in A_\iota$. Define

$$A := \left\{ x \in \prod_{\iota\in I} A_\iota \mid \{\iota \in I | x_\iota \neq 0\} \text{ is finite} \right\}\,.$$

It is easy to see that A is a countable dense set of $\bigoplus_{\iota\in I} E_\iota$. ■

Remark. If $E_\iota = \mathbb{K}$ for every $\iota \in I$, then

$$\bigoplus_{\iota\in I} E_\iota = \ell^2(I)\,.$$

Corollary 5.1.3.2 *If $(E_\iota)_{\iota\in I}$ is a finite family of pre-Hilbert spaces, then the Euclidean norm on $\prod_{\iota\in I} E_\iota$ is generated by the scalar product*

$$\left(\prod_{\iota\in I} E_\iota\right) \times \left(\prod_{\iota\in I} E_\iota\right) \longrightarrow \mathbb{K}\,, \quad (x,y) \longmapsto \sum_{\iota\in I} \langle x_\iota | y_\iota \rangle\,.$$

If each E_ι $(\iota \in I)$ is a Hilbert space, then $\prod_{\iota\in I} E_\iota$ is also a Hilbert space. ■

Proposition 5.1.3.3 (0) *Let $(E_\iota)_{\iota\in I}$, $(F_\iota)_{\iota\in I}$ be two families of pre-Hilbert spaces and take $(u_\iota)_{\iota\in I} \in \prod_{\iota\in I} \mathcal{L}(E_\iota, F_\iota)$ with*

$$\sup_{\iota\in I} \|u_\iota\| < \infty\,.$$

a) $x \in \bigoplus_{\iota\in I} E_\iota \Rightarrow (u_\iota x_\iota)_{\iota\in I} \in \bigoplus_{\iota\in I} F_\iota$. *We define*

$$\bigoplus_{\iota\in I} u_\iota : \bigoplus_{\iota\in I} E_\iota \longrightarrow \bigoplus_{\iota\in I} F_\iota\,, \quad x \longmapsto (u_\iota x_\iota)_{\iota\in I}\,.$$

b) $\bigoplus_{\iota\in I} u_\iota \in \mathcal{L}\left(\bigoplus_{\iota\in I} E_\iota\,,\ \bigoplus_{\iota\in I} F_\iota\right)$ *and* $\| \bigoplus_{\iota\in I} u_\iota\| = \sup_{\iota\in I} \|u_\iota\|$.

Given $x \in \bigoplus_{\iota\in I} E_\iota$,

$$\sum_{\iota\in I} \|u_\iota x_\iota\|^2 \leq \sum_{\iota\in I} \|u_\iota\|^2 \|x_\iota\|^2 \leq$$

$$\leq \left(\sup_{\iota\in I}\|u_\iota\|^2\right)\sum_{\iota\in I}\|x_\iota\|^2 = \left(\sup_{\iota\in I}\|u_\iota\|^2\right)\|x\|^2 < \infty\,,$$

which proves a). Since $\bigoplus_{\iota\in I} u_\iota$ is linear we deduce further that

$$\bigoplus_{\iota\in I} u_\iota \in \mathcal{L}\left(\bigoplus_{\iota\in I} E_\iota\,,\ \bigoplus_{\iota\in I} F_\iota\right)\,,$$

$$\|\bigoplus_{\iota\in I} u_\iota\| \leq \sup_{\iota\in I}\|u_\iota\|\,.$$

Take $\lambda \in I$ and let $x_\lambda \in E_\lambda^{\#}$. Define

$$x : I \longrightarrow \bigcup_{\iota\in I} E_\iota\,,\quad \iota \longmapsto \begin{cases} x_\lambda & if\ \iota = \lambda \\ 0 & if\ \iota \neq \lambda\,. \end{cases}$$

Then $x \in \bigoplus_{\iota\in I} E_\iota$ and

$$\|x\| = \|x_\lambda\| \leq 1\,,$$

so that

$$\|\bigoplus_{\iota\in I} u_\iota\| \geq \left\|\left(\bigoplus_{\iota\in I} u_\iota\right)x\right\| = \|u_\lambda x_\lambda\|\,.$$

Since x_λ is arbitrary,

$$\|\bigoplus_{\iota\in I} u_\iota\| \geq \|u_\lambda\|\,.$$

Since λ is also arbitrary,

$$\left\|\bigoplus_{\iota\in I} u_\iota\right\| \geq \sup_{\iota\in I}\|u_\iota\|$$

and

$$\left\|\bigoplus_{\iota\in I} u_\iota\right\| = \sup_{\iota\in I}\|u_\iota\|\,. \qquad \blacksquare$$

Proposition 5.1.3.4 (0) *Let* $(E_\iota)_{\iota\in I}$, $(F_\iota)_{\iota\in I}$, *and* $(G_\iota)_{\iota\in I}$ *be families of pre-Hilbert spaces. Take* $\alpha, \beta \in \mathbb{K}$,

$$(u_\iota)_{\iota\in I}\,,\ (v_\iota)_{\iota\in I} \in \prod_{\iota\in I}\mathcal{L}(E_\iota, F_\iota)\,,$$

and

$$(w_\iota)_{\iota\in I} \in \prod_{\iota\in I} \mathcal{L}(F_\iota, G_\iota)$$

with

$$\sup_{\iota\in I} \|u_\iota\| < \infty\,, \quad \sup_{\iota\in I} \|v_\iota\| < \infty\,, \quad \sup_{\iota\in I} \|w_\iota\| < \infty\,.$$

Then

$$\bigoplus_{\iota\in I}(\alpha u_\iota + \beta v_\iota) = \alpha \bigoplus_{\iota\in I} u_\iota + \beta \bigoplus_{\iota\in I} v_\iota\,,$$

$$\bigoplus_{\iota\in I}(w_\iota \circ u_\iota) = \left(\bigoplus_{\iota\in I} w_\iota\right) \circ \left(\bigoplus_{\iota\in I} u_\iota\right)\,.$$

These relations are easy to check. ■

5.2 Orthogonal Projections of Hilbert Space

If F is a closed vector subspace of a Hilbert space, then we can define an orthogonal projection onto F, just as in the case of Euclidean spaces. This projection is used to prove the Fréchet–Riesz Theorem that every continuous linear form on a Hilbert space can be written as $\langle \cdot | x \rangle$, with $||\langle \cdot | x \rangle|| = ||x||$. Thus the dual of a Hilbert space may be identified with the Hilbert space itself, whereby the isometry in question is a conjugate linear one.

5.2.1 Projections onto Convex Sets

Proposition 5.2.1.1 (0) *Let E be a pre-Hilbert space, $x \in E$, and A a convex set of E.*

a) $y, z \in A \Rightarrow \|y - z\|^2 \leq 2\|x - y\|^2 + 2\|x - z\|^2 - 4d_A(x)^2$.

b) *Every sequence $(x_n)_{n \in \mathbb{N}}$ in A with*

$$\lim_{n \to \infty} \|x - x_n\| = d_A(x)$$

is a Cauchy sequence. If it converges, then

$$\left\| x - \lim_{n \to \infty} x_n \right\| = d_A(x).$$

a) $\frac{1}{2}(y + z)$ belongs to A. By the parallelogram law (Proposition 2.3.3.2 b)),

$$\|y - z\|^2 = \|(y - x) + (x - z)\|^2 =$$

$$= 2\|y - x\|^2 + 2\|x - z\|^2 - \|(y - x) - (x - z)\|^2 =$$

$$= 2\|x - y\|^2 + 2\|x - z\|^2 - 4\|x - \frac{1}{2}(y + z)\|^2 \leq$$

$$\leq 2\|x - y\|^2 + 2\|x - z\|^2 - 4d_A(x)^2.$$

b) follows from a). ■

Theorem 5.2.1.2 (0) (O. Nikodym, 1931, F. Riesz, 1934) *Let E be a pre-Hilbert space. Take $x \in E$ and let A be a nonempty convex set of E. If*

A is complete with respect to the induced metric, then there is a unique $y \in A$ such that

$$\|x - y\| = d_A(x) .$$

y is characterized by the property

$$z \in A \Longrightarrow \mathrm{re}\langle x - y | y - z\rangle \geq 0 .$$

We define

$$\pi_A(x) := y .$$

Step 1 Uniqueness

Take $y, z \in A$ with

$$\|x - y\| = \|x - z\| = d_A(x) .$$

By Proposition 5.2.1.1 a),

$$\|y - z\|^2 \leq 2\|x - y\| + 2\|x - z\| - 4d_A(x) = 0 .$$

Hence

$$y = z .$$

Step 2 Existence

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A with

$$\lim_{n\to\infty} \|x_n - x\| = d_A(x) .$$

By Proposition 5.2.1.1 b), $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A. Since A is complete, $(x_n)_{n\in\mathbb{N}}$ converges to some $y \in A$. By Proposition 5.2.1.1 b),

$$\|x - y\| = d_A(x) .$$

Step 3 $y, z \in A$, $\|x - y\| = d_A(x) \Rightarrow \mathrm{re}\langle x - y | y - z\rangle \geq 0$

Take $\alpha \in]0, 1[$. Then $\alpha z + (1 - \alpha)y \in A$ and so, by Corollary 2.3.3.4,

$$\|x - y\|^2 = d_A(x)^2 \leq \|x - (\alpha z + (1 - \alpha)y)\|^2 =$$

$$= \|(x-y)+\alpha(y-z)\|^2 = \|x-y\|^2 + 2\alpha \mathrm{re}\langle x-y|y-z\rangle + \alpha^2\|y-z\|^2 ,$$

$$\mathrm{re}\langle x-y|y-z\rangle + \frac{\alpha}{2}\|y-z\|^2 \geq 0 .$$

Since α is arbitrary,

$$\mathrm{re}\langle x-y|y-z\rangle \geq 0 .$$

Step 4 $y \in A$, $(z \in A \Rightarrow \mathrm{re}\langle x-y|y-z\rangle \geq 0) \Rightarrow \|x-y\| = d_A(x)$

By Corollary 2.3.3.4,

$$\|x-z\|^2 = \|(x-y)+(y-z)\|^2 =$$

$$\|x-y\|^2 + 2\mathrm{re}\langle x-y|y-z\rangle + \|y-z\|^2 \geq \|x-y\|^2$$

for every $z \in A$. Hence

$$\|x-y\| = d_A(x) . \qquad \blacksquare$$

Remark. The above theorem does not hold for general Banach spaces. Let A be the closed convex hull of $\left\{\frac{n+1}{n}e_n \,|\, n \in \mathbb{N}\right\}$ in ℓ^∞. Then

$$d_A(0) = 1$$

but

$$\|x\| > 1$$

for every $x \in A$.

Proposition 5.2.1.3 (0) *Let E be a pre-Hilbert space and A a nonempty convex set of E which is complete with respect to the induced metric.*

a) $x, y \in E \Rightarrow \|\pi_A(x) - \pi_A(y)\| \leq \|x-y\|$.

b) π_A *is uniformly continuous.*

c) $\pi_A \circ \pi_A = \pi_A$.

a) By Theorem 5.2.1.2,

$$\mathrm{re}\langle x - \pi_A(x)\,|\,\pi_A(x) - \pi_A(y)\rangle \geq 0 ,$$

$$\mathrm{re}\langle y - \pi_A(y) \,|\, \pi_A(y) - \pi_A(x)\rangle \geq 0\,,$$

It follows that

$$\mathrm{re}\langle \pi_A(y) - y \,|\, \pi_A(x) - \pi_A(y)\rangle \geq 0\,,$$

$$0 \leq \mathrm{re}\langle x - \pi_A(x) \,|\, \pi_A(x) - \pi_A(y)\rangle + \mathrm{re}\langle \pi_A(y) - y \,|\, \pi_A(x) - \pi_A(y)\rangle =$$

$$= \mathrm{re}\langle (x - y) - (\pi_A(x) - \pi_A(y)) \,|\, \pi_A(x) - \pi_A(y)\rangle =$$

$$= \mathrm{re}\langle x - y \,|\, \pi_A(x) - \pi_A(y)\rangle - \mathrm{re}\langle \pi_A(x) - \pi_A(y) \,|\, \pi_A(x) - \pi_A(y)\rangle \leq$$

$$\leq \|x - y\|\,\|\pi_A(x) - \pi_A(y)\| - \|\pi_A(x) - \pi_A(y)\|^2$$

and so

$$\|\pi_A(x) - \pi_A(y)\| \leq \|x - y\|\,.$$

b) follows from a).

c) $\pi_A(x) \in A$,

so that

$$\pi_A \circ \pi_A(x) = \pi_A(\pi_A(x)) = \pi_A(x)$$

for every $x \in E$. Hence

$$\pi_A \circ \pi_A = \pi_A\,. \qquad \blacksquare$$

Proposition 5.2.1.4 *Let E be a pre-Hilbert space, $\mathfrak{A}$ a nonempty downward (upward) directed set of nonempty convex sets of E which are complete in the induced metric, $\mathfrak{F}$ the lower (upper) section filter of $\mathfrak{A}$, and put*

$$B := \bigcap_{A \in \mathfrak{A}} A \qquad \left(B := \overline{\bigcup_{A \in \mathfrak{A}} A} \right).$$

If B is nonempty and complete in the induced metric, then

$$\lim_{A,\mathfrak{F}} \pi_A(x) = \pi_B(x)$$

for every $x \in E$.

We first remark that B is convex. Consider

$$\varphi : \mathfrak{A} \longrightarrow E\,, \quad A \longmapsto \pi_A(x)\,.$$

We have

$$\|\varphi(A) - \varphi(A')\|^2 = \|\pi_A(x) - \pi_{A'}(x)\|^2 \leq$$

$$\leq 2\|x - \pi_A(x)\|^2 + 2\|x - \pi_{A'}(x)\|^2 - 4d_A(x)^2 = 2\left(d_{A'}(x)^2 - d_A(x)^2\right)$$

for all $A, A' \in \mathfrak{A}$ with $A' \subset A$ (Proposition 5.2.1.1 a)). Since $(d_A(x))_{A\in\mathfrak{A}}$ is an increasing bounded (decreasing) family in $\mathbb{R}_+$, it follows that $\varphi(\mathfrak{F})$ is a Cauchy filter. Hence $\lim\limits_{A,\mathfrak{F}} \pi_A(x)$ exists and belongs to B. Now

$$d_B(x) \leq \|x - \lim_{A,\mathfrak{F}} \pi_A(x)\| \leq \lim_{A,\mathfrak{F}} \|x - \pi_A(x)\| = \lim_{A,\mathfrak{F}} d_A(x) \leq d_B(x)\,,$$

$$\left(\left\|x - \lim_{A,\mathfrak{F}} \pi_A(x)\right\| = \lim_{A,\mathfrak{F}} \|x - \pi_A(x)\| = \lim_{A,\mathfrak{F}} d_A(x) = d_B(x)\right)\,,$$

so that

$$\lim_{A,\mathfrak{F}} \pi_A(x) = \pi_B(x)\,,$$

by Theorem 5.2.1.2. ■

5.2.2 Orthogonality

Definition 5.2.2.1 (0) *Let E be a pre-Hilbert space. Given $x, y \in E$ and subsets A, B of E, define*

$$x \perp y :\Leftrightarrow \langle x|y\rangle = 0 \quad (x \text{ and } y \text{ are } \textbf{orthogonal}),$$

$$A \perp B :\Leftrightarrow ((x,y) \in A \times B \Rightarrow x \perp y) \quad (A \text{ and } B \text{ are } \textbf{orthogonal}),$$

$$A^{\perp} := \{x \in E \,|\, y \in A \Rightarrow x \perp y\} \quad (\textbf{the orthogonal set of } A).$$

Proposition 5.2.2.2 (0) *Let E be a pre-Hilbert space. Take $x \in E$ and A, B subsets of E.*

a) *$A^{\perp}$ is a closed vector subspace of E.*

b) $A \subset A^{\perp\perp}$.

c) $A \subset B \Rightarrow B^{\perp} \subset A^{\perp}$.

d) $A^{\perp} = A^{\perp\perp\perp}$.

e) $x \perp x \Leftrightarrow x = 0$.

a) Given $x \in A$, define

$$\widetilde{x} := \langle \cdot | x \rangle .$$

By Corollary 5.1.1.3, $\widetilde{x} \in E'$ for every $x \in A$. Since

$$A^{\perp} = \bigcap_{x \in A} \operatorname{Ker} \widetilde{x},$$

it follows that $A^{\perp}$ is a closed vector subspace of E.

b) and c) follow from the definition.

d) By b) and c),

$$A \subset A^{\perp\perp},$$

$$A^{\perp} \subset A^{\perp\perp\perp} \subset A^{\perp},$$

i.e.

$$A^{\perp} = A^{\perp\perp\perp}.$$

e)

$$(x \perp x) \Longleftrightarrow (\langle x|x\rangle = 0) \Longleftrightarrow (x = 0).$$

■

Proposition 5.2.2.3 (0) (Pythagoras' Theorem) *Let E be a pre-Hilbert space and $(x_\iota)_{\iota\in I}$ a finite family of pairwise orthogonal elements of E. Then*

$$\left\|\sum_{\iota\in I} x_\iota\right\|^2 = \sum_{\iota\in I} \|x_\iota\|^2 .$$

First consider $I = \{1,2\}$. By Corollary 2.3.3.4,

$$\|x_1 + x_2\|^2 = \|x_1\|^2 + 2\mathrm{re}\langle x_1|x_2\rangle + \|x_2\|^2 = \|x_1\|^2 + \|x_2\|^2 .$$

Now let I be arbitrary. We prove the relation by complete induction with respect to $\mathrm{Card}\, I$. Take $\lambda \in I$ and put $J := I\backslash(\lambda)$. By Proposition 5.2.2.2 a),

$$\left(\sum_{\iota\in J} x_\iota\right) \perp x_\lambda .$$

Hence, by the above proof and inductive hypothesis,

$$\left\|\sum_{\iota\in I} x_\iota\right\|^2 = \left\|\left(\sum_{\iota\in J} x_\iota\right) + x_\lambda\right\|^2 = \left\|\sum_{\iota\in J} x_\iota\right\|^2 + \|x_\lambda^2\| =$$

$$\sum_{\iota\in J} \|x_\iota\|^2 + \|x_\lambda\|^2 = \sum_{\iota\in I} \|x_\iota\|^2 . \qquad \blacksquare$$

Corollary 5.2.2.4 (0) *Let E be a Hilbert space and $(x_\iota)_{\iota\in I}$ a family of pairwise orthogonal elements of E. Then $(x_\iota)_{\iota\in I}$ is summable iff $(\|x_\iota\|^2)_{\iota\in I}$ is summable and in this case*

$$\left\|\sum_{\iota\in I} x_\iota\right\|^2 = \sum_{\iota\in I} \|x_\iota\|^2 .$$

By Pythagoras' Theorem,

$$\sum_{\iota\in J} \|x_\iota\|^2 = \left\|\sum_{\iota\in J} x_\iota\right\|^2$$

for every $J \in \mathfrak{P}_f(I)$, and the assertion now follows from Proposition 1.1.6.6. $\blacksquare$

Corollary 5.2.2.5 (0) *Let E be a Hilbert space and A a subset of E, such that given $x, y \in A$,*

$$\langle x|y\rangle = \delta_{x,y} .$$

Take $f \in \mathbb{K}^A$. Then $(f(x)x)_{x\in A}$ is summable iff $f \in \ell^2(A)$ and in this case

$$\left\|\sum_{x\in A} f(x)x\right\|^2 = \|f\|_2^2 . \qquad \blacksquare$$

Corollary 5.2.2.6 (5) *Let E be a Hilbert space and A a subset of E such that*

$$\langle x|y\rangle = \delta_{x,y}$$

for all $x, y \in A$. Take $f \in \ell^2(A)$. Let further T be a topological space and take $g \in \mathbb{K}^{T\times A}$ such that $g(\cdot, x)$ is continuous and

$$\sup_{t\in T} |g(t,x)| \leq |f(x)|$$

for every $x \in A$. Then the map

$$T \longrightarrow E\,, \quad t \longmapsto \sum_{x\in A} g(t,x)x$$

(Corollary 5.2.2.5) is continuous.

Take $t_0 \in T$ and $\varepsilon > 0$. There is a $B \in \mathfrak{P}_f(A)$ such that

$$\sum_{x\in A\setminus B} |f(x)|^2 < \frac{\varepsilon}{5}\,.$$

Further there is a neighbourhood U of t_0 such that for each $t \in U$,

$$\sum_{x\in B} |g(t,x) - g(t_0,x)|^2 < \frac{\varepsilon}{5}\,.$$

It follows from Corollary 5.2.2.5, that

$$\left\| \sum_{x\in A} g(t,x)x - \sum_{x\in A} g(t_0,x)x \right\|^2 = \sum_{x\in A} |g(t,x) - g(t_0,x)|^2 \leq$$

$$\leq \sum_{x\in B} |g(t,x) - g(t_0,x)|^2 + 4 \sum_{x\in A\setminus B} |f(x)|^2 < \varepsilon$$

for every $t \in U$. ■

Corollary 5.2.2.7 *Let E be a pre-Hilbert space and $(x_\iota)_{\iota\in I}$ a family of pairwise orthogonal elements of E. Take $x \in E$ such that*

$$\sum_{\iota\in I} \langle x_\iota|y\rangle = \langle x|y\rangle$$

for every $y \in E$. Then $(x_\iota)_{\iota\in I}$ is summable and

$$\sum_{\iota\in I} x_\iota = x\,.$$

We may assume that E is complete (Corollary 5.1.1.7). Given $\iota \in I$,

$$\langle x|x_\iota\rangle = \|x_\iota\|^2,$$

and so

$$\|x\|^2 = \langle x|x\rangle = \sum_{\iota\in I}\langle x_\iota|x\rangle = \sum_{\iota\in I}\|x_\iota\|^2.$$

By Corollary 5.2.2.4, $(x_\iota)_{\iota\in I}$ is summable. Since

$$\langle x|y\rangle = \sum_{\iota\in I}\langle x_\iota\,|\,y\rangle = \left\langle \sum_{\iota\in I} x_\iota\,|\,y \right\rangle$$

for every $y \in E$,

$$\left\| x - \sum_{\iota\in I} x_\iota \right\|^2 = \left\langle x - \sum_{\iota\in I} x_\iota \,\middle|\, x - \sum_{\iota\in I} x_\iota \right\rangle =$$

$$= \left\langle x \,\middle|\, x - \sum_{\iota\in I} x_\iota \right\rangle - \left\langle \sum_{\iota\in I} x_\iota \,\middle|\, x - \sum_{\iota\in I} x_\iota \right\rangle = 0,$$

$$\sum_{\iota\in I} x_\iota = x. \qquad \blacksquare$$

5.2.3 Orthogonal Projections

Proposition 5.2.3.1 (0) *Let E be a pre-Hilbert space, F a complete subspace of E and take $(x, y) \in E \times F$. Then the following are equivalent:*

a) $y = \pi_F(x)$.

b) $x - y \in F^\perp$.

If these conditions hold, then

$$\|y\|^2 = \langle x|y\rangle \leq \|x\|^2 = \|x - y\|^2 + \|y\|^2 .$$

a $\Rightarrow$ b. Take $z \in F$ and $\alpha \in \mathbb{K}$. Since $y - \alpha z \in F$, it follows from Theorem 5.2.1.2 that

$$0 \leq \mathrm{re}\langle x - y|y - (y - \alpha z)\rangle = \mathrm{re}\langle x - y|\alpha z\rangle = \mathrm{re}\,\overline{\alpha}\langle x - y|z\rangle .$$

If we set

$$\alpha := -\langle x - y|z\rangle$$

then we get successively

$$0 \leq -|\langle x - y|z\rangle|^2 ,$$

$$\langle x - y|z\rangle = 0 ,$$

$$(x - y) \perp z ,$$

$$x - y \in F^\perp .$$

b $\Rightarrow$ a. Take $z \in F$. Then $y - z \in F$ and so

$$(x - y) \perp (y - z) ,$$

$$\mathrm{re}\langle x - y|y - z\rangle = 0 \geq 0 ,$$

and

$$y = \pi_F(x) ,$$

by Theorem 5.2.1.2.

Now we prove the last assertion. By b) and Pythagoras' Theorem (Proposition 5.2.2.3),

$$\|x\|^2 = \|x - y\|^2 + \|y\|^2 \geq \|y\|^2 = \langle y - x|y\rangle + \langle x|y\rangle = \langle x|y\rangle . \qquad \blacksquare$$

Corollary 5.2.3.2 (0) *If E is a pre-Hilbert space and F a complete subspace of E, then π_F is a projection of E onto F with $\|\pi_F\| \leq 1$. It is called* ***the orthogonal projection of E onto F*** *.*

Take $x, y \in E$ and $\alpha, \beta \in \mathbb{K}$. By Proposition 5.2.3.1 a $\Rightarrow$ b,

$$x - \pi_F(x)\,,\ y - \pi_F(y) \in F^{\perp}\,.$$

It follows that

$$(\alpha x + \beta y) - (\alpha\pi_F(x) + \beta\pi_F(y)) = \alpha(x - \pi_F(x)) + \beta(y - \pi_F(y)) \in F^{\perp}$$

(Proposition 5.2.2.2 a)). Thus

$$\alpha\pi_F(x) + \beta\pi_F(y) = \pi_F(\alpha x + \beta y)\,,$$

by Proposition 5.2.3.1 b $\Rightarrow$ a. Hence π_F is linear. By Proposition 5.2.1.3 a), c), π_F is a projection of E onto F with $\|\pi_F\| \leq 1$. ■

Remark. We show below (Theorem 5.3.1.13) that, if E is a Hilbert space, then $\mathcal{L}(E)$ is in a natural way a C^*-algebra. In Proposition 5.3.2.8, we show further that the above notion of "orthogonal projection" coincides with the one introduced in Definition 4.1.2.18.

Proposition 5.2.3.3 (0) *Let E be a normed space, F a pre-Hilbert space, and $u : E \to F$ a compact operator of the underlying real normed spaces. Let $\mathfrak{G}$ be an upward directed set of complete subspaces of F the union of which is dense in F and let $\mathfrak{F}$ be the upper section filter of $\mathfrak{G}$. Then*

$$\lim_{G,\mathfrak{F}} \pi_G \circ u = u\,.$$

Put

$$K := \overline{u(E^{\#})}$$

and given $G \in \mathfrak{G}$, define

$$f_G : K \longrightarrow \mathbb{R}\,, \quad y \longmapsto \|y - \pi_G y\|\,.$$

Then f_G is a downward directed family of continuous functions with infimum 0. Since K is compact, $(f_G)_{G\in\mathfrak{G}}$ converges to 0 uniformly (Dini's Theorem). Hence

$$\|\pi_G \circ u - u\| = \sup_{x\in E^{\#}} \|\pi_G(ux) - ux\| = \sup_{y\in K} f_G(y)$$

for every $G \in \mathfrak{G}$ and the assertion now follows. ■

Corollary 5.2.3.4 (0) *If E is a normed space and F a Hilbert space, then*

$$\overline{\mathcal{L}_f(E,F)} = \mathcal{K}(E,F)\,.$$

By Proposition 3.1.1.4,

$$\overline{\mathcal{L}_f(E,F)} \subset \mathcal{K}(E,F)\,.$$

Take $u \in \mathcal{K}(E,F)$ and let $\mathfrak{G}$ be the set of finite-dimensional subspaces of F. Then $\mathfrak{G}$ is upward directed and

$$\bigcup_{G\in\mathfrak{G}} G = F\,.$$

By Proposition 5.2.3.3

$$\lim_{G,\mathfrak{F}} \pi_G \circ u = u\,,$$

where $\mathfrak{F}$ denotes the upper section filter of $\mathfrak{G}$. Since for every $G \in \mathfrak{G}$, $\pi_G \circ u$ belongs to $\mathcal{L}_f(E,F)$,

$$u \in \overline{\mathcal{L}_f(E,F)}\,,$$

$$\mathcal{K}(E,F) = \overline{\mathcal{L}_f(E,F)}\,. \qquad \blacksquare$$

Proposition 5.2.3.5 (0) *Let E be a pre-Hilbert space and F a complete subspace of E. Let u be a projection of E onto F. Then the following are equivalent:*

a) $u = \pi_F$.

b) $\langle ux|x\rangle \in \mathbb{R}_+$ *for every* $x \in E$.

c) $\operatorname{Ker} u \subset F^\perp$.

d) $\operatorname{Ker} u = F^\perp$.

e) $\|u\| \leq 1$.

f) $\|u\| \in \{0,1\}$.

a $\Rightarrow$ b follows from Proposition 5.2.3.1.

b $\Rightarrow$ c. Take $x \in \operatorname{Ker} u$ and $y \in F$. Then

$$0 \leq \langle u(x+\alpha y)|x+\alpha y\rangle = \langle \alpha y|x+\alpha y\rangle = \alpha\langle y|x\rangle + \alpha^2\|y\|^2$$

for every $\alpha \in \mathbb{R}$, and so

$$\langle y|x\rangle = 0\,,$$

$$x \in F^\perp\,,$$

$$\operatorname{Ker} u \subset F^\perp\,.$$

c $\Rightarrow$ d. Take $x \in F^\perp$. Then

$$\langle ux|x\rangle = 0\,,$$

$$ux - x \in \operatorname{Ker} u \subset F^\perp\,.$$

We deduce, successively, that

$$\|ux\|^2 = \langle ux|ux\rangle - \langle ux|x\rangle = \langle ux|ux - x\rangle = 0\,,$$

$$ux = 0\,,$$

$$x \in \operatorname{Ker} u\,,$$

$$F^\perp \subset \operatorname{Ker} u\,.$$

d $\Rightarrow$ e. Take $x \in E$. Then

$$x - ux \in \operatorname{Ker} u = F^\perp\,.$$

It follows that

$$(x-ux) \perp ux\,,$$

$$\|x\|^2 = \|(x-ux)+ux\|^2 = \|x-ux\|^2 + \|ux\|^2 \geq \|ux\|^2$$

(Pythagoras' Theorem). Hence

$$\|u\| \leq 1 .$$

e $\Rightarrow$ f. Assume that $u \neq 0$. Then $F \neq \{0\}$. Take $x \in F \backslash \{0\}$. It follows from

$$\|x\| = \|ux\| \leq \|u\| \, \|x\| ,$$

that

$$\|u\| \geq 1 ,$$

$$\|u\| = 1 .$$

f $\Rightarrow$ a. Take $x \in E$, $y \in F$, and $\alpha \in \mathbb{K}$. Then

$$u(x - ux + \alpha y) = ux - u^2 x + \alpha u y = \alpha y ,$$

so that

$$|\alpha|^2 \|y\|^2 = \|\alpha y\|^2 = \|u(x - ux + \alpha y)\|^2 \leq \|u\|^2 \|x - ux + \alpha y\|^2 \leq$$

$$\leq \|x - ux\|^2 + 2\mathrm{re}\, \overline{\alpha} \langle x - ux | y \rangle + |\alpha|^2 \|y\|^2$$

(Corollary 2.3.3.4). If we define

$$\alpha := \beta \langle x - ux | y \rangle ,$$

where $\beta \in \mathbb{R}$, then

$$2\beta |\langle x - ux | y \rangle|^2 + \|x - ux\|^2 \geq 0 .$$

Since β is arbitrary, we deduce that

$$\langle x - ux | y \rangle = 0 ,$$

and so

$$x - ux \in F^{\perp} ,$$

$$ux = \pi_F x$$

(Proposition 5.2.3.1 b $\Rightarrow$ a). Hence $u = \pi_F$. ■

Corollary 5.2.3.6 (0) *Let E be a pre-Hilbert space and F, G, H complete subspaces of E. If $\pi_F \pi_G$ is a projection of E onto H, then*

$$\pi_F \pi_G = \pi_H .$$

By Corollary 5.2.3.2,

$$\|\pi_F \pi_G\| \leq \|\pi_F\| \, \|\pi_G\| \leq 1$$

and the assertion follows now from Proposition 5.2.3.5 e $\Rightarrow$ a. ■

Proposition 5.2.3.7 (0) *Let E be a pre-Hilbert space. If F is a complete subspace of E, then*

$$E = F \oplus F^{\perp}, \quad F^{\perp\perp} = F ,$$

and the map

$$F \bigoplus F^{\perp} \longrightarrow E, \quad (x, y) \longmapsto x + y$$

is an isometry. If E is also complete then

$$1_E = \pi_F + \pi_{F^{\perp}} .$$

By Corollary 5.2.3.2, π_F is a projection of E onto F and by Proposition 5.2.3.5 a $\Rightarrow$ d,

$$\operatorname{Ker} \pi_F = F^{\perp} .$$

By Murray's Theorem (Theorem 1.2.5.8 b $\Rightarrow$ a),

$$E = F \oplus F^{\perp} .$$

By Pythagoras' Theorem, the map

$$F \bigoplus F^{\perp} \longrightarrow E, \quad (x, y) \longmapsto x + y$$

is an isometry.

We have $F \subset F^{\perp\perp}$ (Proposition 5.2.2.2 b)). Take $x \in F^{\perp\perp}$. There is a pair $(y, z) \in F \times F^{\perp}$ with

$$x = y + z .$$

From

$$z = x - y \in F^{\perp\perp}$$

(Proposition 5.2.2.2 a)) we deduce that $z = 0$. (Proposition 5.2.2.2 e)), and

$$x = y \in F .$$

Hence $F^{\perp\perp} \subset F$, $F^{\perp\perp} = F$.

By Murray's Theorem (Theorem 1.2.5.8) and the above considerations, $1_E - \pi_F$ is a projection of E onto $F^{\perp}$ with

$$\mathrm{Ker}\,(1_E - \pi_F) = F = F^{\perp\perp} .$$

Thus, by Proposition 5.2.3.5 d $\Rightarrow$ a,

$$1_E - \pi_F = \pi_{F^{\perp}} .$$

Hence

$$1_E = \pi_F + \pi_{F^{\perp}} . \qquad \blacksquare$$

Corollary 5.2.3.8 (0) *Every closed vector subspace of a Hilbert space is a complemented subspace.* ■

Corollary 5.2.3.9 (0) *Let E be a pre-Hilbert space and A a subset of E. If E is complete or if A is finite, then $A^{\perp\perp}$ is the closed vector subspace of E generated by A.*

Let F be the closed vector subspace of E generated by A. Since $A^{\perp\perp}$ is a closed vector subspace of E containing A (Proposition 5.2.2.2 a),b)), $F \subset A^{\perp\perp}$.

If E is complete, then F is complete as well. If A is finite, then F is finite-dimensional and therefore complete. Take $x \in A^{\perp\perp}$. By Proposition 5.2.3.7, there is a pair $(y, z) \in F \times F^{\perp}$ with

$$x = y + z .$$

By the above considerations, $y \in A^{\perp\perp}$. By Proposition 5.2.2.2 a),b),e), we deduce further

$$z \in A^{\perp} ,$$

$$z = x - y \in A^{\perp\perp} ,$$

$$z = 0\,.$$

Hence

$$x = y \in F\,,$$

$$A^{\perp\perp} \subset F\,,$$

$$A^{\perp\perp} = F\,. \qquad \blacksquare$$

Remark. The hypothesis "E complete or A finite" cannot be relinquished, as the following example shows. Put

$$x := \left(\frac{1}{n}\right)_{n\in\mathbb{N}}\,, \quad A := \{e_n \,|\, n \in \mathbb{N}\backslash\{1\}\}\,,$$

and let E denote the vector subspace of ℓ^2 generated by $A \cup \{x\}$. Then

$$A^{\perp} = \{0\}\,, \quad A^{\perp\perp} = E\,,$$

and x does not belong to the closed vector subspace of E generated by A.

Corollary 5.2.3.10 (0) *Let E be a Hilbert space and F a vector subspace of E. Then the following are equivalent:*

a) $\overline{F} = E$.

b) $F^{\perp} = \{0\}$.

From

$$F^{\perp\perp} = \overline{F}$$

(Corollary 5.2.3.9) we deduce that

$$E = F^{\perp} \oplus \overline{F}$$

(Proposition 5.2.3.7) and the assertion now follows. $\blacksquare$

Definition 5.2.3.11 (0) *Let E be a Hilbert space and take $u \in \mathcal{L}(E)$. A closed vector subspace F of E is said* ***to reduce*** *u if F and $F^{\perp}$ are u-invariant.*

Corollary 5.2.3.12 (0) *Let E be a Hilbert space, F a closed vector subspace of E, and take $u \in \mathcal{L}(E)$.*

a) *F is u-invariant iff*

$$\pi_F u \pi_F = u \pi_F .$$

b) *F reduces u iff*

$$\pi_F u = u \pi_F .$$

a) is easy to see.

b) If F reduces u, then by a) and Proposition 5.2.3.7,

$$\pi_F u = \pi_F u(\pi_F + \pi_{F^\perp}) = \pi_F u \pi_F + \pi_F u \pi_{F^\perp} = u \pi_F + \pi_F \pi_{F^\perp} u \pi_{F^\perp} = u \pi_F .$$

Suppose that

$$\pi_F u = u \pi_F .$$

Take $x \in F$ and $y \in F^\perp$. Then

$$ux = u \pi_F x = \pi_F u x \in F ,$$

$$\pi_F u y = u \pi_F y = 0$$

(Proposition 5.2.3.5 a $\Rightarrow$ d),

$$uy \in \operatorname{Ker} \pi_F = F^\perp$$

(Proposition 5.2.3.5 a $\Rightarrow$ d), so that

$$u(F) \subset F , \quad u(F^\perp) \subset F^\perp .$$

Thus F reduces u. ■

Proposition 5.2.3.13 (0) *Let F, G be closed vector subspaces of the Hilbert space E. Then the following are equivalent:*

a) $F \subset G$.

b) $\pi_G \pi_F = \pi_F$.

c) $\pi_F \pi_G = \pi_F$.

d) $x \in E \Rightarrow \|\pi_F x\| \leq \|\pi_G x\|$.

e) $x \in E \Rightarrow \langle \pi_F x | x \rangle \leq \langle \pi_G x | x \rangle$.

f) $\pi_G - \pi_F = \pi_{G \cap F^\perp}$.

g) $\pi_G - \pi_F$ *is a projection in* E.

a $\Leftrightarrow$ b is trivial.

a & b $\Rightarrow$ c. By b),

$$(\pi_F \pi_G)^2 = \pi_F \pi_G \pi_F \pi_G = \pi_F \pi_G ,$$

i.e. $\pi_F \pi_G$ is a projection in E. By a), it is a projection onto F and so the assertion follows from Corollary 5.2.3.6.

c $\Rightarrow$ d. By Corollary 5.2.3.5 a $\Rightarrow$ e,

$$\|\pi_F x\| = \|\pi_F \pi_G x\| \leq \|\pi_G x\| .$$

d $\Leftrightarrow$ e. By Proposition 5.2.3.1,

$$\|\pi_F x\|^2 = \langle \pi_F x | x \rangle , \quad \|\pi_G x\|^2 = \langle \pi_G x | x \rangle .$$

d $\Rightarrow$ a. Take $x \in F$. By Proposition 5.2.3.1,

$$\|x\|^2 = \|\pi_F x\|^2 \leq \|\pi_G x\|^2 = \|x\|^2 - \|x - \pi_G x\|^2 \leq \|x\|^2 .$$

Thus

$$x = \pi_G x \in G$$

and

$$F \subset G .$$

a & b & c & e $\Rightarrow$ f. We have

$$(\pi_G - \pi_F)^2 = \pi_G^2 - \pi_G \pi_F - \pi_F \pi_G + \pi_F^2 = \pi_G - \pi_F ,$$

i.e. $\pi_G - \pi_F$ is a projection in E. Take $x \in E$. Then

$$x - \pi_G x \in G^\perp \subset F^\perp , \quad x - \pi_F x \in F^\perp$$

(Proposition 5.2.3.1 a $\Rightarrow$ b). Hence

$$\pi_G x - \pi_F x = (x - \pi_F x) - (x - \pi_G x) \in F^\perp$$

(Proposition 5.2.2.2 a)). Thus

$$\mathrm{Im}\,(\pi_G - \pi_F) \subset G \cap F^\perp .$$

Since

$$(\pi_G - \pi_F)x = \pi_G x - \pi_F x = x ,$$

whenever $x \in G \cap F^\perp$ (Corollary 5.2.3.5 a $\Rightarrow$ d), it follows that $\pi_G - \pi_F$ is a projection of E onto $G \cap F^\perp$. By e),

$$\langle (\pi_G - \pi_F)x | x \rangle = \langle \pi_G x | x \rangle - \langle \pi_F x | x \rangle \geq 0$$

for every $x \in E$. By Corollary 5.2.3.5 b $\Rightarrow$ a,

$$\pi_G - \pi_F = \pi_{G \cap F^\perp} .$$

f $\Rightarrow$ g is trivial.

g $\Rightarrow$ a. Since

$$\pi_G - \pi_F = (\pi_G - \pi_F)^2 = \pi_G^2 - \pi_G\pi_F - \pi_F\pi_G + \pi_F^2 = \pi_G - \pi_G\pi_F - \pi_F\pi_G + \pi_F ,$$

it follows that

$$\pi_F = \frac{1}{2}(\pi_F\pi_G + \pi_G\pi_F) .$$

Take $x \in F$. Then

$$\|x\| = \|\pi_F x\| = \frac{1}{2}\|\pi_F\pi_G x + \pi_G\pi_F x\| \leq \frac{1}{2}\left(\|\pi_F\pi_G x\| + \|\pi_G x\|\right) \leq$$

$$\leq \frac{1}{2}\left(\|\pi_G x\| + \|\pi_G x\|\right) = \|\pi_G x\|$$

(Corollary 5.2.3.5 a $\Rightarrow$ e). Thus

$$\|x - \pi_G x\|^2 = \|x\|^2 - \|\pi_G x\|^2 \leq 0$$

(Proposition 5.2.3.1)

$$x = \pi_G x \in G ,$$

$$F \subset G .$$

■

Proposition 5.2.3.14 (0) *Let F, G be closed vector subspaces of a Hilbert space E and $H := F \cap G$. Then the following are equivalent:*

a) $\pi_F\pi_G = \pi_G\pi_F$.

b) $\pi_F \pi_G \pi_F = \pi_G \pi_F$.

c) $\pi_G \pi_F = \pi_H$.

d) $(F \cap H^\perp) \perp (G \cap H^\perp)$.

e) $(F \cap H^\perp) \perp G$.

f) $F = H \oplus (F \cap G^\perp)$.

If these conditions hold, then $F + G$ *is closed and*

$$\pi_{F+G} = \pi_F + \pi_G - \pi_H .$$

a $\Rightarrow$ b is trivial.

b $\Rightarrow$ c. Since

$$(\pi_G \pi_F)^2 = \pi_G \pi_F \pi_G \pi_F = \pi_G \pi_G \pi_F = \pi_G \pi_F ,$$

$\pi_G \pi_F$ is a projection in E. Since

$$x \in E \Longrightarrow \pi_G \pi_F x = \pi_F \pi_G \pi_F x \in F \cap G = H ,$$

$$x \in H \Longrightarrow \pi_G \pi_F x = x ,$$

$\pi_G \pi_F$ is a projection onto H. By Corollary 5.2.3.6,

$$\pi_G \pi_F = \pi_H .$$

c $\Rightarrow$ d. Take $x \in F \cap H^\perp$. Then

$$\pi_G x = \pi_G \pi_F x = \pi_H x = 0$$

(Proposition 5.2.3.5 a $\Rightarrow$ d), so that

$$x \in \operatorname{Ker} \pi_G \subset G^\perp \subset (G \cap H^\perp)^\perp$$

(Proposition 5.2.3.5 a $\Rightarrow$ c). Hence

$$(F \cap H^\perp) \perp (G \cap H^\perp) .$$

d $\Rightarrow$ e. Take $x \in G$. There is a $(y, z) \in H \times H^\perp$ with

$$x = y + z$$

(Proposition 5.2.3.7). By d),

$$z = x - y \in G \cap H^{\perp} \subset (F \cap H^{\perp})^{\perp}.$$

From

$$F \cap H^{\perp} \subset H^{\perp}$$

it follows that

$$y \in H \subset H^{\perp\perp} \subset (F \cap H^{\perp})^{\perp},$$

$$x = y + z \in (F \cap H^{\perp})^{\perp}$$

(Proposition 5.2.2.2 a),b),c)) and so

$$(F \cap H^{\perp}) \perp G.$$

e $\Rightarrow$ f. By e),

$$F \cap H^{\perp} \subset F \cap G^{\perp}.$$

Since

$$G^{\perp} \subset H^{\perp},$$

it follows that

$$F \cap H^{\perp} = F \cap G^{\perp}.$$

Take $x \in F$. There is a unique $(y, z) \in H \times H^{\perp}$ with

$$x = y + z$$

(Proposition 5.2.3.7). Now

$$z = x - y \in F \cap H^{\perp} = F \cap G^{\perp},$$

so that

$$F = H \oplus (F \cap G^{\perp})$$

(Corollary 1.4.2.6).

f $\Rightarrow$ d. Take $x \in F \cap H^{\perp}$. By f), there is a pair $(y, z) \in H \times (F \cap G^{\perp})$ with

$$x = y + z.$$

From

$$y = x - z \in H^{\perp},$$

it follows that $y = 0$ and

$$x = z \in G^{\perp} \subset (G \cap H^{\perp})^{\perp},$$

i.e.

$$(F \cap H^{\perp}) \perp (G \cap H^{\perp}).$$

f $\Rightarrow$ a. Take $x \in E$. By f), there is a pair $(y, z) \in H \times (F \cap G^{\perp})$ with

$$\pi_F x = y + z.$$

Then

$$\pi_G \pi_F x = y = \pi_H \pi_F x = \pi_H x$$

(Proposition 5.2.3.5 a $\Rightarrow$ d, Proposition 5.2.3.13 a $\Rightarrow$ c), i.e.

$$\pi_G \pi_F = \pi_H.$$

By d $\Leftrightarrow$ f and by the symmetry of d), we deduce from the above equality that

$$\pi_F \pi_G = \pi_H,$$

i.e.

$$\pi_F \pi_G = \pi_G \pi_F.$$

We now prove the final assertion. Put

$$u := \pi_F + \pi_G - \pi_H.$$

By a),c), and Proposition 5.2.3.13 a $\Rightarrow$ b & c,

$$u\pi_F = \pi_F, \quad u\pi_G = \pi_G, \quad u\pi_H = \pi_H.$$

Thus

$$u^2 = u,$$

i.e. u is a projection in E. Take $x \in F + G$. There is a $(y, z) \in F \times G$ with

$$x = y + z\,.$$

By c),

$$ux = y + \pi_G y - \pi_H y + \pi_F z + z - \pi_H z =$$

$$= y + \pi_G\pi_F y - \pi_G\pi_F y + \pi_F\pi_G z + z - \pi_F\pi_G z = y + z = x\,,$$

so that

$$F + G \subset \operatorname{Im} u\,.$$

since the reverse inclusion is obvious, u is a projection of E onto $F+G$. Hence $F + G$ is closed. Given $x \in E$,

$$\langle ux|x\rangle = \langle \pi_F x|x\rangle + \langle \pi_G x|x\rangle - \langle \pi_H x|x\rangle \geq \langle \pi_F x|x\rangle \in \mathbb{R}_+$$

(Proposition 5.2.3.13 a $\Rightarrow$ e), so

$$u = \pi_{F+G}$$

(Proposition 5.2.3.5 b $\Rightarrow$ a). ■

Corollary 5.2.3.15 (0) *Let F, G be closed vector subspaces of the Hilbert space E. Then the following are equivalent:*

a) $F \perp G$.

b) $\pi_F\pi_G = 0$.

c) $F + G$ *is closed and*

$$\pi_{F+G} = \pi_F + \pi_G\,.$$

a $\Leftrightarrow$ b follows from Proposition 5.2.3.5 a $\Rightarrow$ d.
b $\Rightarrow$ c. Clearly,

$$\pi_G\pi_F\pi_G = 0 = \pi_F\pi_G\,.$$

By Proposition 5.2.3.14, $F + G$ is closed and

$$\pi_{F\cap G} = \pi_F\pi_G = 0\,,$$

$$\pi_{F+G} = \pi_F + \pi_G - \pi_F\pi_G = \pi_F + \pi_G\,.$$

c $\Rightarrow$ a. If $x \in F$, then

$$\pi_G x = \pi_{F+G} x - \pi_F x = x - x = 0 .$$

Thus

$$F \subset \operatorname{Ker} \pi_G \subset G^{\perp}$$

(Proposition 5.2.3.5 a $\Rightarrow$ c) and

$$F \perp G . \qquad \blacksquare$$

Corollary 5.2.3.16 *Let $\mathcal{F}$ be a finite set of orthogonal projections in a Hilbert space and*

$$\pi = \sum_{u \in \mathcal{F}} u .$$

π is an orthogonal projection iff

$$uv = 0$$

whenever $u, v \in \mathcal{F}$ are distinct. In this case,

$$\operatorname{Im} u = \sum_{u \in \mathcal{F}} \operatorname{Im} u .$$

The sufficiency of the condition and the last assertion follow from Corollary 5.2.3.15 b $\Rightarrow$ c by complete induction.

We now assume that π is an orthogonal projection. Let u, v be two distinct elements of $\mathcal{F}$ and $x \in \operatorname{Im} u$. Then

$$\|x\|^2 \geq \langle \pi x | x \rangle = \sum_{w \in \mathcal{F}} \langle wx | x \rangle = \sum_{w \in \mathcal{F}} \|wx\|^2 \geq \|ux\|^2 + \|vx\|^2 = \|x\|^2 + \|vx\|^2$$

(Proposition 5.2.3.1), so

$$vx = 0 .$$

It follows

$$vu = 0 . \qquad \blacksquare$$

Proposition 5.2.3.17 *Let F be a closed vector subspace of the Hilbert space E and take $u \in \mathcal{L}(E)$. Then the following are equivalent:*

a) $F^{\perp} \subset \operatorname{Ker} u$ *and* $\operatorname{Im} u \subset F$.

b) $\pi_F u \pi_F = u$.

a $\Rightarrow$ b. It follows immediately from

$$\operatorname{Im} u \subset F,$$

that

$$\pi_F u = u.$$

Take $x \in E$. Then

$$x - \pi_F x \in F^{\perp}$$

(Proposition 5.2.3.1 a $\Rightarrow$ b), so that

$$0 = u(x - \pi_F x) = ux - u\pi_F x,$$

$$u\pi_F x = ux,$$

$$u\pi_F = u.$$

b $\Rightarrow$ a is obvious. ■

Proposition 5.2.3.18 *Let E be a Hilbert space, F a vector subspace of E, and G a Banach space. Take $u \in \mathcal{L}(F, G)$. Then there is a $v \in \mathcal{L}(E, G)$ such that*

$$v|F = u, \quad \|v\| = \|u\|.$$

Since u is uniformly continuous, we may assume that F is closed. By Corollary 5.2.3.2, the map

$$v : E \longrightarrow G, \quad x \longmapsto u\pi_F x$$

has the required properties. ■

Proposition 5.2.3.19 (0) *Let $(x_\iota)_{\iota \in I}$ be a family in the Hilbert space E. Let*

$$\alpha := \inf_{\iota \in I} \|x_\iota\|, \quad \beta := \sup_{\iota \in I} \|x_\iota\| < \infty,$$

and take $\xi \in \ell^2(I)$ such that

$$|\langle x_\iota | x_\lambda \rangle| \leq |\xi(\iota)\xi(\lambda)|$$

for distinct $\iota, \lambda \in I$. *Then, given* $\eta \in \ell^2(I)$, *the family* $(\eta(\iota)x_\iota)_{\iota \in I}$ *is summable in* E *and*

$$\|\eta\|^2(\alpha^2 - \|\xi\|^2) \leq \|\sum_{\iota \in I} \eta(\iota)x_\iota\|^2 \leq \|\eta\|^2(\beta^2 + \|\xi\|^2).$$

Let J be a finite subset of I. Then

$$\sum_{\substack{\iota,\lambda \in J \\ \iota \neq \lambda}} |\langle \eta(\iota)x_\iota \,|\, \eta(\lambda)x_\lambda \rangle| \leq \sum_{\substack{\iota,\lambda \in J \\ \iota \neq \lambda}} |\eta(\iota)|\,|\eta(\lambda)|\,|\xi(\iota)\xi(\lambda)| \leq$$

$$\leq \left(\sum_{\iota \in J} |\eta(\iota)|\,|\xi(\iota)| \right)^2 \leq \left(\sum_{\iota \in J} |\eta(\iota)|^2 \right) \left(\sum_{\iota \in J} |\xi(\iota)|^2 \right).$$

It follows that

$$\left\| \sum_{\iota \in J} \eta(\iota)x_\iota \right\|^2 \leq \sum_{\iota \in J} |\eta(\iota)|^2 \|x_\iota\|^2 + \sum_{\substack{\iota,\lambda \in I \\ \iota \neq \lambda}} |\langle \eta(\iota)x_\iota | \eta(\lambda)x_\lambda \rangle| \leq$$

$$\leq \beta^2 \sum_{\iota \in J} |\eta(\iota)|^2 + \|\xi\|^2 \left(\sum_{\iota \in J} |\eta(\iota)|^2 \right) = (\beta^2 + \|\xi\|^2) \sum_{\iota \in J} |\eta(\iota)|^2,$$

$$\left\| \sum_{\iota \in J} \eta(\iota)x_\iota \right\|^2 \geq \sum_{\iota \in J} |\eta(\iota)|^2 \|x_\iota\|^2 - \sum_{\substack{\iota,\lambda \in I \\ \iota \neq \lambda}} |\langle \eta(\iota)x_\iota | \eta(\lambda)x_\lambda \rangle| \geq$$

$$\geq \alpha^2 \sum_{\iota \in J} |\eta(\iota)|^2 - \|\xi\|^2 \sum_{\iota \in J} |\eta(\iota)|^2 = (\alpha^2 - \|\xi\|^2) \sum_{\iota \in J} |\eta(\iota)|^2.$$

Hence $(\eta(\iota)x_\iota)_{\iota \in I}$ is summable (Proposition 1.1.6.6) and

$$\|\eta\|^2(\alpha^2 - \|\xi\|^2) \leq \left\| \sum_{\iota \in I} \eta(\iota)x_\iota \right\|^2 \leq \|\eta\|^2(\beta^2 + \|\xi\|^2). \qquad \blacksquare$$

Corollary 5.2.3.20 (0) *Let* $(x_\iota)_{\iota \in I}$ *and* $(y_\iota)_{\iota \in I}$ *be families in the Hilbert spaces* E *and* F, *respectively. Take* $\xi, \eta \in \ell^2(I)$ *such that*

$$\|\xi\| < \inf_{\iota \in I} \|x_\iota\| =: \alpha, \quad \beta := \sup_{\iota \in I} \|y_\iota\| < \infty$$

and

$$|\langle x_\iota | x_\lambda \rangle| \leq |\xi(\iota)\xi(\lambda)|\,, \quad |\langle y_\iota | y_\lambda \rangle| \leq |\eta(\iota)\eta(\lambda)|$$

for distinct $\iota, \lambda \in I$. *Then there is a unique* $u \in \mathcal{L}(E,F)$ *such that* $u = 0$ *on* $\{x_\iota | \iota \in I\}^\perp$ *and that*

$$ux_\iota = y_\iota$$

for every $\iota \in I$. *Moreover,*

$$\|u\|^2 \leq \frac{\beta^2 + \|\eta\|^2}{\alpha^2 - \|\xi\|^2}\,.$$

If $\{x_\iota | \iota \in I\}^\perp = 0$ *and*

$$\gamma := \sup_{\iota \in I} \|x_\iota\| < \infty\,, \quad \delta := \inf_{\iota \in I} \|y_\iota\| > \|\eta\|\,,$$

then

$$\|ux\|^2 \geq \frac{\delta^2 - \|\eta\|^2}{\gamma^2 + \|\xi\|^2}\|x\|^2$$

for every $x \in E$.

The uniqueness follows from the fact that $\{x | \iota \in I\}^{\perp\perp}$ is the closed vector subspace of E generated by $\{x_\iota \,|\, \iota \in I\}$ (Corollary 5.2.3.9).

Now we prove the existence and the last assertion. By Proposition 5.2.3.19, the map

$$v : \ell^2(I) \longrightarrow E\,, \quad \zeta \longmapsto \sum_{\iota \in I} \zeta(\iota) x_\iota\,,$$

$$w : \ell^2(I) \longrightarrow F\,, \quad \zeta \longmapsto \sum_{\iota \in I} \zeta(\iota) y_\iota$$

are well-defined and

$$\zeta \in \ell^2(I) \Longrightarrow \begin{cases} \|\zeta\|^2(\alpha^2 - \|\xi\|^2) \leq \|v\zeta\|^2 \leq \|\zeta\|^2(\gamma^2 + \|\xi\|^2) \\ \|\zeta\|^2(\delta^2 - \|\eta\|^2) \leq \|w\zeta\|^2 \leq \|\zeta\|^2(\beta^2 + \|\eta\|^2)\,. \end{cases}$$

Hence $\operatorname{Im} v$ is the closed vector subspace of E generated by $\{x_\iota \,|\, \iota \in I\}$ (Proposition 1.2.1.18 c)). Define

$$u : E \longrightarrow F\,, \quad x \longmapsto w v^{-1} \pi_{\operatorname{Im} v} x\,.$$

Then, by Proposition 1.2.1.18 b),

$$\|ux\|^2 = \|wv^{-1}\pi_{\operatorname{Im} v}x\|^2 \leq (\beta^2 + \|\eta\|^2)\|v^{-1}\pi_{\operatorname{Im} v}x\|^2 \leq$$

$$\leq \frac{\beta^2 + \|\eta\|^2}{\alpha^2 - \|\xi\|^2}\|\pi_{\operatorname{Im} v}x\|^2 \leq \frac{\beta^2 + \|\eta\|^2}{\alpha^2 - \|\xi\|^2}\|x\|^2$$

for every $x \in E$, so that

$$\|u\|^2 \leq \frac{\beta^2 + \|\eta\|^2}{\alpha^2 - \|\xi\|^2}.$$

If $\{x_\iota \,|\, \iota \in I\}^\perp = \{0\}$, then $\operatorname{Im} v = E$ and by Proposition 1.2.1.18 b), given $x \in E$,

$$\|ux\|^2 = \|wv^{-1}x\|^2 \geq (\delta^2 - \|\eta\|^2)\|v^{-1}x\|^2 \geq \frac{\delta^2 - \|\eta\|^2}{\gamma^2 + \|\xi\|^2}\|x\|^2. \qquad \blacksquare$$

Proposition 5.2.3.21 (0) *Let E, F be infinite-dimensional Hilbert spaces. Take $u \in \mathcal{L}(E,F)$, and let $\dot{u}$ be the equivalence class of u in $\mathcal{L}(E,F)/\mathcal{K}(E,F)$. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that*

$$\langle x_m \,|\, x_n\rangle = \delta_{mn}$$

for all $m, n \in \mathbb{N}$ and

$$\|\dot{u}\| = \lim_{n\to\infty} \|ux_n\|.$$

We construct the sequence $(x_n)_{n\in\mathbb{N}}$ inductively such that for every $n \in \mathbb{N}$

$$\|ux_n\| \geq \|\dot{u}\| - \frac{1}{n},$$

$$m \in \mathbb{N}_n \Longrightarrow \langle x_m | x_n\rangle = \delta_{mn}.$$

Take $n \in \mathbb{N}$ and assume the sequence has been constructed up to $n-1$. Let G be the vector subspace of E generated by $\{x_m \,|\, m \in \mathbb{N}_{n-1}\}$. Then $u \circ \pi_G \in \mathcal{K}(E,F)$, so that

$$u - u\circ\pi_G \in \dot{u}, \quad \|\dot{u}\| \leq \|u - u\circ\pi_G\|.$$

Hence there is an $x \in E^\#$ such that

$$\|\dot{u}\| - \frac{1}{n} < \|(u - u\circ\pi_G)x\|.$$

By Proposition 5.2.3.1, $\pi_{G^\perp} x \in E^\#$ and

$$x = \pi_G x + \pi_{G^\perp} x .$$

Hence

$$\|u|G^\perp\| \geq \|u\pi_{G^\perp} x\| = \|ux - u\pi_G x\| = \|(u - u \circ \pi_G)x\| > \|\dot{u}\| - \frac{1}{n} .$$

Then there is an $x_n \in G^\perp$ with

$$\|x_n\| = 1 , \quad \|ux_n\| \geq \|\dot{u}\| - \frac{1}{n} . \qquad \blacksquare$$

Proposition 5.2.3.22 (0) *Let E be a Hilbert space and $\mathfrak{F}$ the set of finite-dimensional vector subspaces of E. Then*

$$\|u\| = \sup_{F \in \mathfrak{F}} \|\pi_F u \pi_F\|$$

for every $u \in \mathcal{L}(E)$.

By Proposition 5.2.3.5 a $\Rightarrow$ e,

$$\sup_{F \in \mathfrak{F}} \|\pi_F u \pi_F\| \leq \|u\| .$$

Take $\alpha \in]0, \|u\|[$. There is an $x \in E^\#$ such that

$$\|ux\| > \alpha .$$

Let F be the vector subspace of E generated by $\{x, ux\}$. Then $F \in \mathfrak{F}$ and

$$\|\pi_F u \pi_F\| \geq \|\pi_F u \pi_F x\| = \|\pi_F ux\| = \|ux\| > \alpha .$$

Since α is arbitrary, it follows

$$\sup_{F \in \mathfrak{F}} \|\pi_F u \pi_F\| \geq \|u\| . \qquad \blacksquare$$

5.2.4 Mean Ergodic Theorems

The main ideas of these results originate in a paper of J. von Neumann.

Proposition 5.2.4.1 *Let E be an unital Banach algebra. Take $x \in E$ with $(x^n)_{n\in\mathbb{N}}$ bounded. Let A be the convex hull of $\{x^{n-1} \mid n \in \mathbb{N}\}$ in E and let $(a_{np})_{n,p\in\mathbb{N}}$ be a family in $\{x\}^c$, such that $(a_{np})_{p\in\mathbb{N}}$ is absolutely summable for every $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty}\left(\|a_{n1}\| + \sum_{p\in\mathbb{N}} \|a_{np} - a_{n,p+1}\|\right) = 0\,.$$

Then for every $n \in \mathbb{N}$, $(a_{np}x^{p-1})_{p\in\mathbb{N}}$ is absolutely summable and for every $y \in A$,

$$\lim_{n\to\infty}(y-1)\sum_{p\in\mathbb{N}} a_{np}x^{p-1} = 0\,.$$

Put

$$\alpha := \sup_{n\in\mathbb{N}} \|x^{n-1}\|\,.$$

Then

$$\sum_{p\in\mathbb{N}} \|a_{np}x^{p-1}\| \le \sum_{p\in\mathbb{N}} \|a_{np}\|\,\|x^{p-1}\| \le \alpha \sum_{p\in\mathbb{N}} \|a_{np}\| < \infty\,.$$

Thus $(a_{np}x^{p-1})_{p\in\mathbb{N}}$ is absolutely summable for every $n \in \mathbb{N}$.

Given $n \in \mathbb{N}$, put

$$x_n := \sum_{p\in\mathbb{N}} a_{np}x^{p-1}\,.$$

Then

$$\|(x-1)x_n\| = \left\|x\sum_{p\in\mathbb{N}} a_{np}x^{p-1} - \sum_{p\in\mathbb{N}} a_{np}x^{p-1}\right\| =$$

$$= \left\|\sum_{p\in\mathbb{N}} a_{np}x^{p} - \sum_{p\in\mathbb{N}\cup\{0\}} a_{n,p+1}x^{p}\right\| = \left\|\sum_{p\in\mathbb{N}}(a_{np} - a_{n,p+1})x^p - a_{n1}\right\| \le$$

$$\le \|a_{n1}\| + \sum_{p\in\mathbb{N}} \|a_{np} - a_{n,p+1}\|\,\|x^p\| \le$$

$$\leq \left(\|a_{n1}\| + \sum_{p \in \mathbb{N}} \|a_{np} - a_{n,p+1}\| \right) \alpha$$

for every $n \in \mathbb{N}$ and so

$$\lim_{n \to \infty} (x - 1)x_n = 0 .$$

Since

$$(x^p - 1)x_n = \left(\sum_{k=0}^{p-1} x^k \right) (x - 1)x_n ,$$

$$\lim_{n \to \infty} (x^p - 1)x_n = 0$$

for any $p \in \mathbb{N}$.

There is a finite family $(\alpha_p)_{p \in I}$ in $\mathbb{R}_+$ with $I \subset \mathbb{N}$,

$$y = \sum_{p \in I} \alpha_p x^{p-1} , \quad \sum_{p \in I} \alpha_p = 1 .$$

It follows that

$$(y - 1)x_n = \sum_{p \in I} \alpha_p x^{p-1} x_n - \sum_{p \in I} \alpha_p x_n = \sum_{p \in I} \alpha_p (x^{p-1} - 1)x_n$$

for every $n \in \mathbb{N}$. Hence

$$\lim_{n \to \infty} (y - 1)x_n = 0 .$$

■

Lemma 5.2.4.2 *There is a* $\gamma \in \mathbb{R}_+$ *with*

$$(1 - t)^n + \sum_{p \in \mathbb{N}} \left| \binom{n}{p - 1} t^{p-1} (1 - t)^{n-p+1} - \binom{n}{p} t^p (1 - t)^{n-p} \right| \leq \frac{\gamma}{\sqrt{nt(1 - t)}}$$

for every $n \in \mathbb{N}$ *and* $t \in \left[\frac{1}{n+1}, \frac{n}{n+1} \right]$.

For every $p \in \mathbb{N}_n$ put

$$\alpha_p := \frac{\binom{n}{p} t^p (1 - t)^{n-p}}{\binom{n}{p-1} t^{p-1} (1 - t)^{n-p+1}} = \frac{n!(p - 1)!(n - p + 1)!t}{n!p!(n - p)!(1 - t)} = \frac{(n - p + 1)t}{p(1 - t)} .$$

Then

$$p \leq tn \Longrightarrow \alpha_p \geq \frac{(n - tn + 1)t}{tn(1 - t)} = 1 + \frac{1}{n(1 - t)} > 1 ,$$

$$p \geq tn+1 \Longrightarrow \alpha_p \leq \frac{(n-tn-1+1)t}{(tn+1)(1-t)} = \frac{nt}{nt+1} < 1\,.$$

Since

$$\alpha_1 = \frac{\binom{n}{1}t^1(1-t)^{n-1}}{\binom{n}{0}t^0(1-t)^n} = \frac{nt}{1-t} \geq \frac{n\frac{1}{n+1}}{1-\frac{1}{n+1}} = 1\,,$$

$$\alpha_n = \frac{\binom{n}{n}t^n(1-t)^0}{\binom{n}{n-1}t^{n-1}(1-t)^1} = \frac{t}{n(1-t)} \leq \frac{n\frac{1}{n+1}}{n(1-\frac{1}{n+1})} \leq 1\,,$$

there is a $k_n \in \mathbb{N}_{n-1}$ such that $|k_n - tn| < 1$ and

$$\mathbb{N}_n \cup \{0\} \longrightarrow \mathbb{R}\,, \quad p \longmapsto \binom{n}{p}t^p(1-t)^{n-p}$$

is an increasing function for $p \leq k_n$ and a decreasing function for $p \geq k_n$. Hence this function takes its supremum at $k := k_n \in \mathbb{N}_{n-1}$, such that

$$|k - tn| < 1\,.$$

By the Stirling inequalities (for $n > 1$),

$$(1-t)^n + \sum_{p\in\mathbb{N}} \left| \binom{n}{p-1}t^{p-1}(1-t)^{n-p+1} - \binom{n}{p}t^p(1-t)^{n-p} \right| =$$

$$= 2\binom{n}{k}t^k(1-t)^{n-k} = \frac{2n!t^k(1-t)^{n-k}}{k!(n-k)!} \leq 2\frac{\sqrt{2\pi n}n^n e^{\frac{1}{12}}e^k e^{n-k}t^k(1-t)^{n-k}}{e^n\sqrt{2\pi k}k^k\sqrt{2\pi(n-k)}(n-k)^{n-k}} =$$

$$= \frac{2t^k(1-t)^{n-k}e^{\frac{1}{12}}}{\sqrt{2\pi n\frac{k}{n}(1-\frac{k}{n})}(\frac{k}{n})^k(1-\frac{k}{n})^{n-k}} = \frac{\delta_n}{\sqrt{nt(1-t)}}\,,$$

where

$$\delta_n := \frac{2e^{\frac{1}{12}}}{\sqrt{2\pi}}\left(\frac{tn}{k}\right)^{k+\frac{1}{2}}\left(\frac{(1-t)n}{n-k}\right)^{n-k+\frac{1}{2}}$$

for every $n \in \mathbb{N}\backslash\{1\}$. Given $n \in \mathbb{N}\backslash\{1\}$,

$$\delta_n \leq \frac{2e^{\frac{1}{12}}}{\sqrt{2\pi}}\left(\frac{k+1}{k}\right)^{k+\frac{1}{2}}\left(\frac{n-k+1}{n-k}\right)^{n-k+\frac{1}{2}}\,,$$

so that

$$\gamma := \sup_{n\in\mathbb{N}}\delta_n \leq \frac{2e^{\frac{1}{12}}}{\sqrt{2\pi}}\sup_{n\in\mathbb{N}}\left(\frac{k_n+1}{k_n}\right)^{k_n+\frac{1}{2}}\sup_{n\in\mathbb{N}}\left(\frac{n-k_n+1}{n-k_n}\right)^{n-k_n+\frac{1}{2}} < \infty\,. \quad \blacksquare$$

Proposition 5.2.4.3 *Let E be a Hilbert space. Take $u \in \mathcal{L}(E)^{\#}$. Put*

$$F := \operatorname{Ker}(1-u)$$

and let $(\alpha_{np})_{n,p\in\mathbb{N}}$ be a family in $\mathbb{R}_+$, such that

$$\sum_{p\in\mathbb{N}} \alpha_{np} = 1$$

for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \left(\alpha_{n1} + \sum_{p\in\mathbb{N}} |\alpha_{np} - \alpha_{n,p+1}| \right) = 0 .$$

Then

$$u\pi_F = \pi_F u = \pi_F$$

and

$$\lim_{n\to\infty} \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} x = \pi_F x$$

for every $x \in E$. In particular,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} u^{p-1} x = \pi_F x \qquad \textbf{(Mean Ergodic Theorem)},$$

$$\lim_{n\to\infty} (t_n 1 + (1-t_n)u)^n x = \pi_F x$$

for every $x \in E$ and every sequence $(t_n)_{n\in\mathbb{N}}$ in $]0,1[$ with

$$\lim_{n\to\infty} n t_n (1-t_n) = \infty .$$

We have

$$\sum_{p\in\mathbb{N}} \left\| \alpha_{np} u^{p-1} \right\| \le \sum_{p\in\mathbb{N}} \alpha_{np} \|u^{p-1}\| \le \sum_{p\in\mathbb{N}} \alpha_{np} = 1$$

for every $n \in \mathbb{N}$. In particular, $(\alpha_{np} u^{p-1})_{p\in\mathbb{N}}$ is absolutely summable and

$$\left\| \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} \right\| \le 1 .$$

Take $x \in E$. Let A be the convex hull of $\{u^{p-1}x \,|\, p \in \mathbb{N}\}$ and take $y \in A$. There is a finite family $(\beta_p)_{p\in I}$ in $\mathbb{R}_+$ such that $I \subset \mathbb{N}$,

$$y = \sum_{p \in I} \beta_p u^{p-1} x\,, \quad \sum_{p \in I} \beta_p = 1\,.$$

Put

$$v := \sum_{p \in I} \beta_p u^{p-1}\,.$$

v belongs to the convex hull of $\{u^{p-1} \mid p \in \mathbb{N}\}$ and

$$\sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} x - \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} y =$$

$$= \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} x - \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} v x = \left(\sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} \right) (1 - v) x$$

for every $n \in \mathbb{N}$. By Proposition 5.2.4.1,

$$\lim_{n \to \infty} \left(\sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} x - \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} y \right) = 0\,.$$

Given $n \in \mathbb{N}$,

$$\left\| \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} x \right\| \leq \left\| \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} x - \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} y \right\| + \left\| \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} y \right\| \leq$$

$$\leq \left\| \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} x - \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} y \right\| + \|y\|\,.$$

Since

$$\sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} x \in \overline{A}\,,$$

it follows

$$d_A(0) \leq \liminf_{n \to \infty} \left\| \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} x \right\| \leq \limsup_{n \to \infty} \left\| \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} x \right\| \leq \|y\|\,.$$

Since y is arbitrary,

$$\lim_{n \to \infty} \left\| \sum_{p \in \mathbb{N}} \alpha_{np} u^{p-1} x \right\| = d_A(0)\,.$$

By Proposition 5.2.1.1 b), $\left(\sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} x\right)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Define

$$w : E \longrightarrow E\,, \quad x \longmapsto \lim_{n\to\infty} \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} x\,.$$

By Corollary 1.4.1.3, $w \in \mathcal{L}(E)$ and

$$\|w\| \leq \sup_{n\in\mathbb{N}} \left\| \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} \right\| \leq 1\,.$$

We have

$$uw = wu$$

and for $n \in \mathbb{N}$ and $x \in E$,

$$\left\| u \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} x - \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} x \right\| \leq \left\| (u-1) \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} \right\| \|x\|$$

and so

$$uwx - wx = \lim_{n\to\infty} u \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} x - \lim_{n\to\infty} \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} x =$$

$$= \lim_{n\to\infty} \left(u \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} x - \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} x \right) = 0$$

for every $x \in E$ by Proposition 5.2.4.1, i.e.

$$uw = w\,.$$

Hence

$$w \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} = \sum_{p\in\mathbb{N}} \alpha_{np} w u^{p-1} = \sum_{p\in\mathbb{N}} \alpha_{np} w = w$$

for every $n \in \mathbb{N}$ and

$$w^2 x = w \left(\lim_{n\to\infty} \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} x \right) = \lim_{n\to\infty} w \sum_{p\in\mathbb{N}} \alpha_{np} u^{p-1} x = wx$$

for every $x \in E$. Thus w is a projection in E.

Take $x \in F$. Then $ux = x$, and so $u^n x = x$ for every $n \in \mathbb{N}$. We deduce that $wx = x$, i.e. $x \in \operatorname{Im} w$, so that $F \subset \operatorname{Im} w$. Now take $x \in \operatorname{Im} w$. There is a $y \in E$ with $x = wy$. Then

$$x - ux = wy - uwy = wy - wy = 0 .$$

Thus $x \in F$ and so $\operatorname{Im} w \subset F$. Hence $\operatorname{Im} w = F$ and w is a projection of E onto F. By Proposition 5.2.3.5 e $\Rightarrow$ a,

$$w = \pi_F .$$

We now prove the final assertions. Put

$$\alpha_{np} := \begin{cases} \frac{1}{n} \text{ if } p \leq n \\ 0 \text{ if } p > n . \end{cases}$$

Then

$$\sum_{p \in \mathbb{N}} \alpha_{np} = 1 ,$$

$$\alpha_{n1} + \sum_{p \in \mathbb{N}} |\alpha_{np} - \alpha_{n,p+1}| = \frac{2}{n}$$

for every $n \in \mathbb{N}$, so that

$$\lim_{n \to \infty} \left(\alpha_{n1} + \sum_{p \in \mathbb{N}} |\alpha_{np} - \alpha_{n,p+1}| \right) = 0 .$$

Put

$$\alpha_{np} := \binom{n}{p-1} t_n^{p-1} (1 - t_n)^{n-p+1} .$$

Then

$$\sum_{p \in \mathbb{N}} \alpha_{np} = (t_n + (1 - t_n))^n = 1 ,$$

$$\lim_{n \to \infty} \left(\alpha_{n1} + \sum_{p \in \mathbb{N}} |\alpha_{np} - \alpha_{n,p+1}| \right) = 0$$

by Lemma 5.2.4.2. ■

Corollary 5.2.4.4 *Let E be a Hilbert space. Take $u \in \mathcal{L}(E)$ and $\alpha \in \mathbb{K}$ such that $|\alpha| = \|u\|$. Let F be a closed vector subspace of $\mathrm{Ker}\,(\alpha 1 - u)$. Then*

$$u\pi_F = \pi_F u = \alpha\pi_F\,.$$

In particular, F reduces u.

We may assume that $\alpha \neq 0$ and $F \neq \{0\}$. Put

$$v := \frac{1}{2\alpha}u + \frac{1}{2}\pi_F\,.$$

Then

$$\|v\| \leq \frac{1}{2|\alpha|}\|u\| + \frac{1}{2}\|\pi_F\| \leq 1$$

(Proposition 5.2.3.5 a $\Rightarrow$ e) and

$$vx = \frac{1}{2\alpha}ux + \frac{1}{2}\pi_F x = x$$

for any $x \in F$. Moreover, for $x \in \mathrm{Ker}\,(1 - v)$,

$$\|x\| = \|vx\| \leq \frac{1}{2|\alpha|}\|u\|\,\|x\| + \frac{1}{2}\|\pi_F x\| = \frac{1}{2}\|x\| + \frac{1}{2}\|\pi_F x\|\,,$$

$$\|x\| = \|\pi_F x\|\,,$$

$$\|x - \pi_F\|^2 = \|x\|^2 - \|\pi_F x\|^2 = 0\,,$$

$$x = \pi_F x \in F$$

(Proposition 5.2.3.1). Hence

$$\|v\| = 1\,, \quad F = \mathrm{Ker}\,(1 - v)\,.$$

By Proposition 5.2.4.3,

$$v\pi_F = \pi_F v = \pi_F$$

and this implies

$$u\pi_F = \pi_F u = \alpha\pi_F\,.$$

The final assertion follows from Corollary 5.2.3.12 b). ■

Remark. The relation

$$u\pi_F = \alpha\pi_F$$

holds even if $|\alpha| \neq \|u\|$ but it is, in general, not the case that

$$\pi_F u = \alpha\pi_F$$

if $|\alpha| \neq \|u\|$. For instance, let $E := \mathbb{K}^2$, let u have the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix},$$

and let $\alpha := 1$ and $F := \operatorname{Ker}(1 - u)$. Then π_F and $\pi_F u$ have the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

respectively.

5.2.5 The Fréchet–Riesz Theorem

Proposition 5.2.5.1 (0) *Let E be a pre-Hilbert space.*

a) $x \in E \Rightarrow \langle \cdot | x \rangle \in E'$, $\|\langle \cdot | x \rangle\| = \|x\|$.

b) *The map*

$$E \longrightarrow E', \quad x \longmapsto \langle \cdot | x \rangle$$

is conjugate linear, injective, and continuous.

a) $\langle \cdot | x \rangle$ is linear. By Schwarz' Inequality, $\langle \cdot | x \rangle$ is continuous and

$$\|\langle \cdot | x \rangle\| \leq \|x\| .$$

It follows from

$$\|x\|^2 = \langle x | x \rangle \leq \|\langle \cdot | x \rangle\| \, \|x\|$$

that

$$\|x\| \leq \|\langle \cdot | x \rangle\| ,$$

$$\|\langle \cdot | x \rangle\| = \|x\| .$$

b) The map

$$E \longrightarrow E', \quad x \longmapsto \langle \cdot | x \rangle$$

is conjugate linear. By a), it preserves the norm. Hence it is continuous and injective. ■

Theorem 5.2.5.2 (0) (Fréchet-Riesz, 1907) *Let E be a Hilbert space and take $x' \in E'$. Then there is a unique $x \in E$ with*

$$\langle \cdot | x \rangle = x' .$$

The uniqueness follows from Proposition 5.2.5.1 b). To prove the existence, we may assume that $x' \neq 0$. Put

$$F := \operatorname{Ker} x' .$$

F is a closed subspace of E with $F \neq E$. Hence $F^\perp \neq \{0\}$ (Corollary 5.2.3.10 b $\Rightarrow$ a). Let $y \in F^\perp \backslash \{0\}$. Put

$$y' := \langle \cdot | y \rangle .$$

Then

$$\operatorname{Ker} x' = F \subset \operatorname{Ker} y' .$$

Hence

$$y' = \alpha x'$$

for some $\alpha \in \mathbb{K}$ (Lemma 1.2.6.3). Since $y' \neq 0$, we have that $\alpha \neq 0$. Put

$$x := \frac{1}{\overline{\alpha}} y .$$

Then

$$\langle \cdot | x \rangle = \langle \cdot | \frac{1}{\overline{\alpha}} y \rangle = \frac{1}{\alpha} \langle \cdot | y \rangle = \frac{1}{\alpha} y' = x' . \qquad \blacksquare$$

Corollary 5.2.5.3 (0) *If E is a Hilbert space, then the map*

$$E \longrightarrow E', \quad x \longmapsto \langle \cdot | x \rangle$$

is an isometry of real Banach spaces.

The assertion follows immediately from Proposition 5.2.5.1 and Theorem 5.2.5.2. $\blacksquare$

Corollary 5.2.5.4 (Hellinger–Toeplitz, 1910) *Let E be a Hilbert space and $u : E \to E$ a linear map such that*

$$\langle ux | y \rangle = \langle x | uy \rangle$$

for all $x, y \in E$. Then u is continuous.

The assertion follows immediately from Corollary 1.4.2.20 and Corollary 5.2.5.3. $\blacksquare$

Corollary 5.2.5.5 (0) *All Hilbert spaces are reflexive.*

Let E be a Hilbert space and consider

$$u : E \longrightarrow E', \quad x \longmapsto \langle \cdot | x \rangle .$$

Take $x'' \in E''$. Then $\overline{x'' \circ u}$ is continuous and linear (Proposition 5.2.5.1). By the Fréchet–Riesz Theorem (Theorem 5.2.5.2), there is an $x \in E$ with

$$ux = \overline{x'' \circ u}\,.$$

Take $y' \in E'$ and $y \in E$ with

$$uy = y'$$

(Theorem 5.2.5.2). We have

$$\langle j_E x, y' \rangle = \langle x, y' \rangle = \langle x|y \rangle = \overline{\langle y|x \rangle} = \overline{\langle y, ux \rangle} =$$

$$= \overline{\langle y, \overline{x'' \circ u} \rangle} = x'' \circ u(y) = \langle x'', uy \rangle = \langle x'', y' \rangle\,.$$

Since y' is arbitrary,

$$j_E x = x''\,.$$

Hence j_E is surjective and E is reflexive. ■

Corollary 5.2.5.6 *Let E be a pre-Hilbert space, F a subspace of E, and $y' \in F'$. Then there is a unique $x' \in E'$ such that*

$$x'|F = y'\,, \quad \|x'\| = \|y'\|\,.$$

We may assume E to be complete (Corollary 5.1.1.7) and F to be closed. By the Fréchet-Riesz Theorem, there is a $y \in F$ such that

$$y' = \langle \cdot | y \rangle_F\,.$$

Define

$$x' : E \longrightarrow \mathbb{K}\,, \quad x \longmapsto \langle x|y \rangle\,.$$

Then

$$x'|F = y'\,, \quad \|x'\| = \|\langle \cdot|y \rangle_E\| = \|\langle \cdot|y \rangle_F\| = \|y'\|$$

(Proposition 5.2.5.1 a)).

Take $z' \in E'$ with

$$z'|F = y'\,, \quad \|z'\| = \|y'\|\,.$$

By the Fréchet-Riesz Theorem, there is a $z \in E$ such that

$$z' = \langle \cdot | z \rangle\,.$$

Then, given any $x \in F$,

$$\langle x|z-y\rangle = \langle x|z\rangle - \langle x|y\rangle = z'(x) - y'(x) = 0\,,$$

so that $z - y \in F^{\perp}$. By Proposition 5.2.3.1 (and Proposition 5.2.5.1 a)), we now see that

$$\|z-y\|^2 = \|z\|^2 - \|y\|^2 = \|z'\|^2 - \|y'\|^2 = 0\,,$$

$$z = y\,,$$

$$z' = x'\,. \qquad \blacksquare$$

Proposition 5.2.5.7 *Let T be a Hausdorff space, μ a positive Radon measure on T, and $\mathcal{F}$ a closed vector subspace of $L^2(\mu)$, such that $\mathcal{F}^{\#}$ is equicontinuous and the map*

$$\mathcal{F} \longrightarrow \mathbb{K}\,, \quad x \longmapsto x(t)$$

is continuous for every $t \in T$. If we consider $\mathcal{F}$ with the induced Hilbert space structure (Example 5.1.2.1), there there is a unique continuous function

$$k : T \times T \longrightarrow \mathbb{R}\,,$$

such that $k(\cdot, t) \in \mathcal{F}$ and

$$\langle x|k(\cdot,t)\rangle = x(t)$$

for every $t \in T$ and $x \in \mathcal{F}$. Moreover,

$$k(s,t) = \langle k(\cdot,t)|k(\cdot,s)\rangle\,,$$

$$k(t,s) = \overline{k(s,t)}$$

and

$$k(t,t) = \|k(\cdot,t)\|^2$$

for all $s, t \in T$ and the map

$$T \longrightarrow \mathcal{F}\,, \quad t \longmapsto k(\cdot,t)$$

is continuous. If $\mathcal{F} \neq \{0\}$, then for every $t \in T$, the function $\frac{k(\cdot,t)}{k(t,t)}$ has the smallest norm in the set $\{x \in \mathcal{F} \mid |x(t)| = 1\}$.

Take $t \in T$. By the Fréchet–Riesz Theorem (Theorem 5.2.5.2), there is a unique $k_t \in \mathcal{F}$ with

$$\langle x|k_t\rangle = x(t)$$

for every $x \in \mathcal{F}$. Consider

$$k : T \times T \longrightarrow \mathbb{K}\,, \quad (s,t) \longmapsto k_t(s)\,.$$

We have

$$\langle k(\cdot,t)\,|\,k(\cdot,s)\rangle = k(s,t)\,,$$

so that

$$\overline{k(t,s)} = k(s,t)\,,$$

$$k(t,t) = \|k(\cdot,t)\|^2$$

for all $s,t \in T$.

Take $t \in T$ and $\varepsilon > 0$. Since $\mathcal{F}^\#$ is equicontinuous, there is a neighbourhood U of t such that

$$|x(s) - x(t)| < \varepsilon$$

for every $x \in \mathcal{F}^\#$ and $s \in U$. Thus

$$\|k(\cdot,s) - k(\cdot,t)\| = \|\langle\cdot|k_s - k_t\rangle\| =$$

$$= \sup_{x\in\mathcal{F}^\#} |\langle x|k_s - k_t\rangle| = \sup_{s\in\mathcal{F}^\#} |x(s) - x(t)| \le \varepsilon$$

for every $s \in U$ (Proposition 5.2.5.1 a)). Hence the map

$$T \longrightarrow \mathcal{F}\,, \quad t \longmapsto k(\cdot,t)$$

is continuous.

Given $(s_0,t_0),(s,t) \in T \times T$,

$$|k(s,t) - k(s_0,t_0)| \le |k(s,t) - k(s_0,t)| + |k(s_0,t) - k(s_0,t_0)| =$$

$$= |\langle k(\cdot,t)|k(\cdot,s) - k(\cdot,s_0)\rangle| + |\langle k(\cdot,t) - k(\cdot,t_0)|k(\cdot,s_0)\rangle| \le$$

$$\leq \|k(\cdot,t)\|\,\|k(\cdot,s)-k(\cdot,s_0)\| + \|k(\cdot,t)-k(\cdot,t_0)\|\,\|k(\cdot,s_0)\|\,,$$

by Schwarz' Inequality. Hence k is continuous.

In order to prove the last assertion, let $x \in \mathcal{F}$ with $|x(t)| = 1$. Then

$$1 = |x(t)| = |\langle x, k(\cdot,t)\rangle| \leq \|x\|\,\|k(\cdot,t)\|\,,$$

so

$$\|x\| \geq \frac{1}{\|k(\cdot,t)\|} = \left\|\frac{k(\cdot,t)}{k(t,t)}\right\|\,. \qquad \blacksquare$$

Example 5.2.5.8 *Let T be an open set of some Euclidean space, λ the Lebesgue measure on T, and*

$$\mathcal{F} := \{x \in L^2(\lambda) \,|\, x \text{ is harmonic}\}$$

be endowed with the induced Hilbert space structure (Example 5.1.2.1, Example 3.1.2.9). Then there is a unique continuous function

$$k : T \times T \longrightarrow \mathbb{R}\,,$$

such that $k(\cdot,t) \in \mathcal{F}$ and

$$\langle x | k(\cdot,t)\rangle = x(t)$$

for every $t \in T$ and $x \in \mathcal{F}$. Moreover, given $s,t \in T$,

$$k(s,t) = \langle k(\cdot,t) \,|\, k(\cdot,s)\rangle\,,$$

$$k(t,s) = \overline{k(s,t)}\,,$$

and

$$k(t,t) = \|k(\cdot,t)\|^2\,,$$

and the map

$$T \longrightarrow \mathcal{F}\,, \quad t \longmapsto k(\cdot,t)$$

is continuous. If $\mathcal{F} \neq \{0\}$, then for each $t \in T$, the function $\frac{k(\cdot,t)}{k(t,t)}$ has the smallest norm in the set $\{x \in \mathcal{F} \,|\, x(t) = 1\}$.

The assertion follows immediately from Example 3.1.2.9 and Proposition 5.2.5.7. $\blacksquare$

Example 5.2.5.9 *Take* $\mathbb{K} = \mathbb{C}$. *Let* T *be an open set of* $\mathbb{C}$. *Let* λ *denote the Lebesgue measure on* T *and take*

$$\mathcal{A} := \{x \in L^2(\lambda) \,|\, x \text{ is analytic}\}$$

with the induced Hilbert space structure (Example 5.1.2.1, Example 3.1.2.10). Then there is a unique continuous function

$$k : T \times T \longrightarrow \mathbb{C},$$

such that $k(\cdot, t) \in \mathcal{A}$ *and*

$$\langle x | k(\cdot, t) \rangle = x(t)$$

for every $t \in T$ *and* $x \in \mathcal{A}$ *(*k *is called* ***the Bergman kernel of*** T*). Given* $s, t \in T$,

$$k(s,t) = \langle k(\cdot, t) \,|\, k(\cdot, s) \rangle,$$

$$k(t,s) = \overline{k(s,t)},$$

and

$$k(t,t) = \|k(\cdot, t)\|^2$$

and the map

$$T \longrightarrow \mathcal{A}, \quad t \longmapsto k(\cdot, t)$$

is continuous. If $\mathcal{A} \neq \{0\}$, *then for each* $t \in T$, *the function* $\frac{k(\cdot,t)}{k(t,t)}$ *has the smallest norm in the set* $\{x \in \mathcal{A} \,|\, x(t) = 1\}$.

The assertions follow immediately from Example 3.1.2.10 and Proposition 5.2.5.7. ■

Example 5.2.5.10 *Take* $\mathbb{K} = \mathbb{C}$. *Let* $T \neq \mathbb{C}$ *be a simply connected domain of* $\mathbb{C}$. *Let* λ *denote the Lebesgue measure on* T *and* k *the Bergman kernel of* T. *Take* $s \in T$. *Let* f *be a bijective conformal map of* T *on* $U_1^{\mathbb{C}}(0)$ *with* $f(s) = 0$. *Then*

$$f' \in L^2(\lambda), \quad \|f'\|_2 = \sqrt{\pi},$$

and

$$k(s,t) = \frac{f'(s)\overline{f'(t)}}{\pi}$$

for every $t \in T$.

Given $r \in]0,1[$, put

$$T_r := f^{-1}\left(\overline{U_r^{\mathbb{C}}(0)}\right),$$

$$\Gamma_r : [0,2\pi] \longrightarrow T, \quad \theta \longmapsto f^{-1}(re^{i\theta}).$$

Let x be an analytic function on T and take $r \in]0,1[$. $\frac{x}{f}$ is a meromorphic function on T analytic on $T\backslash\{s\}$. The residue of $\frac{x}{f}$ in s is

$$\lim_{\substack{T \ni t \to s \\ t \neq s}} \frac{(t-s)x(t)}{f(t)} = \frac{x(s)}{f'(s)}.$$

By the Theorem of Residues

$$\frac{x(s)}{f'(s)} = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{x(t)}{f(t)}dt = \frac{1}{2\pi i r^2}\int_{\Gamma_r} \overline{f(t)}x(t)dt$$

since $|f| = r$ on Γ_r. By Green's formula,

$$\frac{x(s)}{f'(s)} = \frac{1}{\pi r^2}\int_{T_r} \overline{f'}x d\lambda.$$

Putting $x := f'$ in the above equality,

$$\int_{T_r} |f'|^2 d\lambda = \pi r^2,$$

i.e. $f' \in L^2(\lambda)$ and $\|f'\|_2 = \sqrt{\pi}$. If $x \in L^2(\lambda)$, then by the above equality,

$$\frac{x(s)}{f'(s)} = \frac{1}{\pi}\langle x|f'\rangle.$$

Hence

$$k(\cdot,s) = \frac{1}{\pi}\overline{f'(s)}f',$$

and so

$$k(s,t) = \frac{f'(s)\overline{f'(t)}}{\pi}$$

for every $t \in T$, by Example 5.2.5.9. ■

Example 5.2.5.11 *The Bergman kernel of* $U_1^{\mathbb{C}}(0)$ *is*

$$U_1^{\mathbb{C}}(0) \times U_1^{\mathbb{C}}(0) \longrightarrow \mathbb{C}, \quad (s,t) \longmapsto \frac{1}{\pi(1-s\bar{t})^2}.$$

Let $s \in U_1^{\mathbb{C}}(0)$. The map

$$f : U_1^{\mathbb{C}}(0) \longrightarrow U_1^{\mathbb{C}}(0), \quad t \longmapsto \frac{t-s}{1-\overline{s}t}$$

is bijective, conformal, and vanishes at s. Moreover,

$$f'(t) = \frac{1-\overline{s}t+\overline{s}(t-s)}{(1-\overline{s}t)^2} = \frac{1-|s|^2}{(1-\overline{s}t)^2}$$

for every $t \in U_1^{\mathbb{C}}(0)$. Thus

$$\frac{f'(s)\overline{f'(t)}}{\pi} = \frac{(1-|s|^2)(1-|s|^2)}{\pi(1-|s|^2)^2(1-s\bar{t})^2} = \frac{1}{\pi(1-s\bar{t})^2}$$

for every $t \in T$ and the assertion follows from Example 5.2.5.10. ■

Example 5.2.5.12 *If* $T := \{\alpha \in \mathbb{C} \mid \operatorname{im}\alpha > 0\}$, *then*

$$T \times T \longrightarrow \mathbb{C}, \quad (s,t) \longmapsto -\frac{1}{\pi(s-\bar{t})^2}$$

is the Bergman kernel of T.

Take $s \in T$. The map

$$f : T \longrightarrow U_1^{\mathbb{C}}(0), \quad t \longmapsto \frac{t-s}{t-\overline{s}}$$

is bijective, conformal, and vanishes at s. Given $t \in T$,

$$f'(t) = \frac{t-\overline{s}-t+s}{(t-\overline{s})^2} = \frac{s-\overline{s}}{(t-\overline{s})^2}.$$

Hence

$$\frac{f'(s)\overline{f'(t)}}{\pi} = \frac{(s-\overline{s})(\overline{s}-s)}{\pi(s-\overline{s})^2(\bar{t}-s)^2} = -\frac{1}{\pi(s-\bar{t})^2}$$

for every $t \in T$ and the assertion follows from Example 5.2.5.10. ■

5.3 Adjoint Operators

The notion of the adjoint of an operator, the most important single concept in the theory of Hilbert spaces, is introduced in this section. The central tool is the Fréchet-Riesz Theorem from Section 5.2. When endowed with this adjunction, the algebra of operators on a Hilbert space forms a C^*-algebra. Important properties of selfadjoint and normal operators are established.

5.3.1 General Results

Proposition 5.3.1.1 (0) *Let E, F be normed spaces and f a continuous sesquilinear form on $E \times F$. Then the map*

$$F \longrightarrow E', \quad y \longmapsto f(\cdot, y)$$

is conjugate linear and continuous.

It is obvious that $f(\cdot, y) \in E'$ for every $y \in F$ and that the map

$$F \longrightarrow E', \quad y \longmapsto f(\cdot, y)$$

is conjugate linear. By Proposition 1.2.9.2 a $\Rightarrow$ c, there is an $\alpha \in \mathbb{R}_+$, with

$$|f(x, y)| \leq \alpha \|x\| \, \|y\|$$

for every pair $(x, y) \in E \times F$. It follows that

$$\|f(\cdot, y)\| \leq \alpha \|y\|$$

for every $y \in F$, i.e. the map

$$F \longrightarrow E', \quad y \longmapsto f(\cdot, y)$$

is continuous. ■

Corollary 5.3.1.2 (0) *Let E, F be Hilbert spaces and f a continuous sesquilinear form on $E \times F$. Then there is a unique map $g : F \to E$, such that*

$$f(x, y) = \langle x | g(y) \rangle$$

for every $(x, y) \in E \times F$. g is linear and continuous.

The uniqueness follows from Proposition 5.2.5.1 b). Define

$$u : E \longrightarrow E', \quad x \longmapsto \langle \cdot | x \rangle ,$$

$$v : F \longrightarrow E', \quad y \longmapsto f(\cdot, y) ,$$

$$g := u^{-1} \circ v$$

(Corollary 5.2.5.3). Then

$$f(x,y) = \langle x, vy \rangle = \langle x | u^{-1} v y \rangle = \langle x | g(y) \rangle$$

for every $(x,y) \in E \times F$ (Corollary 5.2.5.3), which proves the existence. Since u^{-1} and v are conjugate linear and continuous (Proposition 5.2.5.1 b), Proposition 5.3.1.1), g is linear and continuous. ■

Theorem 5.3.1.3 (Lax-Milgram, 1954) *Let E be a Hilbert space, f a sesquilinear form on E, and α, β strictly positive real numbers with*

$$|f(x,y)| \leq \alpha \|x\| \, \|y\| ,$$

$$|f(x,x)| \geq \beta \|x\|^2$$

for every $x, y \in E$. Then there is a unique $u : E \to E$ with

$$f(x,y) = \langle x | uy \rangle$$

for every $x, y \in E$. u is an isomorphism of Banach spaces and

$$\|u\| \leq \alpha , \quad \|u^{-1}\| \leq \frac{1}{\beta} .$$

The existence, uniqueness, continuity, and linearity of u all follow from Corollary 5.3.1.2. It follows from

$$\|uy\|^2 = \langle uy | uy \rangle = f(uy, y) \leq \alpha \|uy\| \, \|y\|$$

that

$$\|uy\| \leq \alpha \|y\|$$

for every $y \in E$, so that

$$\|u\| \leq \alpha .$$

It follows from

$$\beta\|x\|^2 \leq |f(x,x)| = |\langle x|ux\rangle| \leq \|x\|\,\|ux\|$$

that

$$\|ux\| \geq \beta\|x\|$$

for every $x \in E$ and

$$\beta\|x\|^2 = 0$$

for every $x \in (\operatorname{Im} u)^\perp$. Hence

$$(\operatorname{Im} u)^\perp = 0$$

and

$$\overline{\operatorname{Im} u} = E$$

(Corollary 5.2.3.10 b $\Rightarrow$ a). By Proposition 1.2.1.18 b),d), u is an isomorphism of Banach spaces and

$$\|u^{-1}\| \leq \frac{1}{\beta}.$$ ■

Theorem 5.3.1.4 (0) *Let E, F be Hilbert spaces and take $u \in \mathcal{L}(E,F)$. There is a unique map $u^* : F \to E$ such that*

$$\langle ux|y\rangle = \langle x|u^*y\rangle$$

for every $(x,y) \in E \times F$. Putting

$$v : E \longrightarrow E', \quad x \longmapsto \langle\cdot|x\rangle,$$

$$w : F \longrightarrow F', \quad y \longmapsto \langle\cdot|y\rangle,$$

then

$$u^* = v^{-1} \circ u' \circ w \in \mathcal{L}(E,F)$$

(Corollary 5.2.5.3) and

$$\|u^*\| = \|u\|, \quad \|u^* \circ u\| = \|u\|^2.$$

u^* is called ***the adjoint (operator) of*** *u.*

The map

$$E \times F \longrightarrow \mathbb{K}\,, \quad (x,y) \longmapsto \langle ux|y\rangle$$

is a continuous sesquilinear form. The existence, uniqueness, continuity, and linearity of u^* all follow from Corollary 5.3.1.2. Given any $(x,y) \in E \times F$,

$$\langle ux|y\rangle = \langle ux, wy\rangle = \langle x, u'wy\rangle = \langle x|v^{-1}u'wy\rangle\,,$$

so that

$$u^* = v^{-1} \circ u' \circ w\,,$$

$$\|u^*\| = \|u'\| = \|u\|$$

(Corollary 5.2.5.3, Theorem 1.3.4.2 b)), and

$$\|u^* \circ u\| \leq \|u^*\|\,\|u\| = \|u\|^2$$

(Corollary 1.2.1.5). We have

$$\|ux\|^2 = \langle ux|ux\rangle = \langle x|u^*ux\rangle \leq \|x\|\,\|u^*ux\| \leq \|u^* \circ u\|\,\|x\|^2$$

for every $x \in E$, and so

$$\|u\|^2 \leq \|u^* \circ u\|\,,$$

$$\|u^* \circ u\| = \|u\|^2\,. \qquad \blacksquare$$

Proposition 5.3.1.5 (0) *Let E, F be Hilbert spaces.*

a) $u \in \mathcal{L}(E,F) \Rightarrow u^{**} = u$.

b) *The map*

$$\mathcal{L}(E,F) \longrightarrow \mathcal{L}(F,E)\,, \quad u \longmapsto u^*$$

is conjugate linear.

a) Given $(x,y) \in E \times F$,

$$\langle u^*y|x\rangle = \overline{\langle x|u^*y\rangle} = \overline{\langle ux|y\rangle} = \langle y|ux\rangle\,.$$

Hence

$$u^{**} = u\,.$$

b) Take $u, v \in \mathcal{L}(E,F)$ and $\alpha, \beta \in \mathbb{K}$. Then

$$\langle(\alpha u + \beta v)x|y\rangle = \langle \alpha ux + \beta vx|y\rangle = \alpha\langle ux|y\rangle + \beta\langle vx|y\rangle =$$

$$= \alpha\langle x|u^*y\rangle + \beta\langle x|v^*y\rangle = \langle x|\overline{\alpha}u^*y + \overline{\beta}v^*y\rangle = \langle x|(\overline{\alpha}u^* + \overline{\beta}v^*)y\rangle$$

for every $(x,y) \in E \times F$. Hence

$$(\alpha u + \beta y)^* = \overline{\alpha}u^* + \overline{\beta}v^*\,. \qquad \blacksquare$$

Example 5.3.1.6 *If u_r and u_ℓ are the right and left shift on ℓ^2, then*

$$u_r^* = u_\ell\,, \quad u_\ell^* = u_r$$

(Example 5.1.2.3).

Given $x, y \in \ell^2$,

$$\langle u_r x|y\rangle = \sum_{n=1}^{\infty} x_n \overline{y}_{n+1} = \langle x|u_\ell y\rangle\,.$$

Hence

$$u_r^* = u_\ell\,, \quad u_\ell^* = u_r^{**} = u_r$$

(Proposition 5.3.1.5 a)). $\blacksquare$

Proposition 5.3.1.7 (0) *Let E, F, G be Hilbert spaces. Take $u \in \mathcal{L}(E,F)$ and $v \in \mathcal{L}(F,G)$. Then*

$$(v \circ u)^* = u^* \circ v^*\,.$$

Given $(x,y) \in E \times G$,

$$\langle(v \circ u)x|y\rangle = \langle vux|y\rangle = \langle ux|v^*y\rangle = \langle x|u^*v^*y\rangle = \langle x|(u^* \circ v^*)y\rangle$$

and so

$$(v \circ u)^* = u^* \circ v^*\,. \qquad \blacksquare$$

Proposition 5.3.1.8 (0) *Let E be a real pre-Hilbert space and $\overset{\circ}{E}$ the complexification of the underlying real vector space (Lemma 2.1.5.1 c)). Then the map*

$$\overset{\circ}{E} \times \overset{\circ}{E} \longrightarrow \mathbb{C}\,,$$

$$((a,x),(b,y)) \longmapsto \langle a|b\rangle + \langle x|y\rangle + i\langle x|b\rangle - i\langle a|y\rangle$$

is a scalar product. Its canonical norm is the Euclidean norm on the product $E \times E$. If E is complete, then $\overset{\circ}{E}$ is also complete. The complex pre-Hilbert space $\overset{\circ}{E}$ is called ***the complexification of*** *E.*

Given $(a,x),(b,y),(c,z) \in \overset{\circ}{E}$ and $\alpha,\beta \in \mathbb{R}$,

$$\langle (a,x)+(b,y)|(c,z)\rangle = \langle (a+b,x+y)|(c,z)\rangle =$$

$$= \langle a+b|c\rangle + \langle x+y|z\rangle + i\langle x+y|c\rangle - i\langle a+b|z\rangle =$$

$$= \langle a|c\rangle + \langle b|c\rangle + \langle x|z\rangle + \langle y|z\rangle + i\langle x|c\rangle + i\langle y|c\rangle - i\langle a|z\rangle - i\langle b|z\rangle =$$

$$= \langle (a,x)|(c,z)\rangle + \langle (b,y)|(c,z)\rangle\,,$$

$$\langle (\alpha+i\beta)(a,x)|(b,y)\rangle = \langle (\alpha a - \beta x, \beta a + \alpha x)|(b,y)\rangle =$$

$$= \alpha\langle a|b\rangle - \beta\langle x|b\rangle + \beta\langle a|y\rangle + \alpha\langle x|y\rangle +$$

$$i\beta\langle a|b\rangle + i\alpha\langle x|b\rangle - i\alpha\langle a|y\rangle + i\beta\langle x|y\rangle =$$

$$= (\alpha+i\beta)(\langle a|b\rangle + \langle x|y\rangle + i\langle x|b\rangle - i\langle a|y\rangle) = (\alpha+i\beta)\langle (a,x)|(b,y)\rangle\,,$$

$$\langle (a,x)|(b,y)\rangle = \langle a|b\rangle + \langle x|y\rangle + i\langle x|b\rangle - i\langle a|y\rangle =$$

$$= \langle b|a\rangle + \langle y|x\rangle + i\langle b|x\rangle - i\langle y|a\rangle = \overline{\langle (b,y)|(a,x)\rangle}\,,$$

$$\langle (a,x)|(a,x)\rangle = \langle a|a\rangle + \langle x|x\rangle + i\langle x|a\rangle - i\langle a|x\rangle = \|a\|^2 + \|x\|^2\,. \qquad \blacksquare$$

Example 5.3.1.9 *Take $n \in \mathbb{N}$ and $u \in \mathcal{L}(\mathbb{R}^n)$. Then the map*

$$\overset{\circ}{\widehat{\mathbb{R}^n}} \longrightarrow \mathbb{C}^n, \quad (x,y) \longmapsto (x_k + iy_k)_{k \in \mathbb{N}_n}$$

is an isometry of complex Hilbert spaces. If we use this isometry to identify the Hilbert spaces, then u and the map

$$\overset{\circ}{\widehat{\mathbb{R}^n}} \longrightarrow \overset{\circ}{\widehat{\mathbb{R}^n}}, \quad (x,y) \longmapsto (ux, uy)$$

have the same associated matrices. ■

Example 5.3.1.10 *Let $(T, \mathfrak{T}, \mu)$ be a measure space, $L^2(\mu, \mathbb{R})$ and $L^2(\mu,\mathbb{C})$ the corresponding Hilbert spaces in the real and complex case, respectively, and k a real function on $T \times T$ belonging to $L^2(\mu \otimes \mu)$. Put*

$$u : L^2(\mu, \mathbb{R}) \longrightarrow L^2(\mu, \mathbb{R}), \quad x \longmapsto k^\mu x$$

(Proposition 3.1.6.17 a)),

$$\overset{\circ}{u} : \overset{\circ}{\widehat{L^2(\mu, \mathbb{R})}} \longrightarrow \overset{\circ}{\widehat{L^2(\mu, \mathbb{R})}}, \quad (x,y) \longmapsto (ux, uy).$$

Then the map

$$\overset{\circ}{\widehat{L^2(\mu, \mathbb{R})}} \longrightarrow L^2(\mu,\mathbb{C}), \quad (x,y) \longmapsto x + iy$$

is an isometry of complex Hilbert spaces and

$$\overset{\circ}{u}x = k^\mu x$$

for all $x \in L^2(\mu,\mathbb{C})$. ■

Proposition 5.3.1.11 (0) *Let E, F be real Hilbert spaces, $\overset{\circ}{E}$, and $\overset{\circ}{F}$ the complexification of E and F, respectively, and $\overset{\circ}{\mathcal{L}}(E,F)$, $\overset{\circ}{\mathcal{L}}(F,E)$ and $\overset{\circ}{\mathcal{K}}(E,F)$ the complexification of the underlying real vector spaces of $\mathcal{L}(E,F)$, $\mathcal{L}(F,E)$ and $\mathcal{K}(E,F)$, respectively. For*

$$(u,v) \in \overset{\circ}{\mathcal{L}}(E,F) \quad (\textit{resp.} \ (u,v) \in \overset{\circ}{\mathcal{L}}(F,E)),$$

define

$$\widetilde{(u,v)} : \overset{\circ}{E} \longrightarrow \overset{\circ}{F}, \quad (x,y) \longmapsto (ux - vy, uy + vx)$$

$$(\textit{resp.} \ \widetilde{(u,v)} : \overset{\circ}{F} \longrightarrow \overset{\circ}{E}, \quad (x,y) \longmapsto (ux - vy, uy + vx)).$$

a) *If* $(u,v) \in \overset{\circ}{\mathcal{L}}(E,F)$ *then*

$$\widetilde{(u,v)} \in \mathcal{L}(\overset{\circ}{E},\overset{\circ}{F}),$$

$$\sup\{\|u\|,\|v\|\} \leq \|\widetilde{(u,v)}\| \leq \|u\| + \|v\|,$$

$$\widetilde{(u,v)}^* = \widetilde{(u^*,-v^*)},$$

$$(u,v) \in \overset{\circ}{\mathcal{K}}(E,F) \Longrightarrow \widetilde{(u,v)} \in \mathcal{K}(\overset{\circ}{E},\overset{\circ}{F}).$$

b) *The maps*

$$\overset{\circ}{\mathcal{L}}(E,F) \longrightarrow \mathcal{L}(\overset{\circ}{E},\overset{\circ}{F}), \quad (u,v) \longmapsto \widetilde{(u,v)},$$

$$\overset{\circ}{\mathcal{K}}(E,F) \longrightarrow \mathcal{K}(\overset{\circ}{E},\overset{\circ}{F}), \quad (u,v) \longmapsto \widetilde{(u,v)}$$

are bijective and linear.

a) By Lemma 2.1.5.3, $\widetilde{(u,v)}$ is linear. Choose $(x,y) \in \dot{E}$. By Proposition 5.3.1.8,

$$\|\widetilde{(u,v)}(x,y)\|^2 = \|(ux - vy, uy + vx)\|^2 = \|ux - vy\|^2 + \|uy + vx\|^2 \leq$$

$$\leq (\|ux\| + \|vy\|)^2 + (\|uy\| + \|vx\|)^2 \leq$$

$$\leq (\|u\|\,\|x\| + \|v\|\,\|y\|)^2 + (\|u\|\,\|y\| + \|v\|\,\|x\|)^2 =$$

$$= \|u\|^2\|x\|^2 + 2\|u\|\,\|v\|\,\|x\|\,\|y\| + \|v\|^2\|y\|^2 +$$

$$+\|u\|^2\|y\|^2 + 2\|u\|\,\|v\|\,\|x\|\,\|y\| + \|v\|^2\|x\|^2 \leq$$

$$\leq (\|u\|^2 + 2\|u\|\,\|v\| + \|v\|^2)(\|x\|^2 + \|y\|^2) = (\|u\| + \|v\|)^2\|(x,y)\|^2.$$

Hence $\widetilde{(u,v)}$ is continuous and

$$\|\widetilde{(u,v)}\| \leq \|u\| + \|v\|.$$

Let $x \in E$. By Proposition 5.3.1.8,

$$\|ux\|^2 + \|vx\|^2 = \|(ux, vx)\|^2 = \|\widetilde{(u,v)}(x,0)\|^2 \leq \|\widetilde{(u,v)}\|^2\|x\|^2 ,$$

so that

$$\sup\{\|u\|, \|v\|\} \leq \|\widetilde{(u,v)}\| .$$

Let $(a,b) \in \overset{\circ}{E}$ and $(c,d) \in \overset{\circ}{F}$. We have

$$\langle \widetilde{(u,v)}(a,b) \mid (c,d)\rangle = \langle (ua - vb, ub + va) \,|\, (c,d)\rangle =$$

$$= \langle ua - vb|c\rangle + \langle ub + va|d\rangle - i\langle ua - vb|d\rangle + i\langle ub + va|c\rangle =$$

$$= \langle a|u^*c\rangle - \langle b|v^*c\rangle + \langle b|u^*d\rangle + \langle a|v^*d\rangle -$$

$$-i\langle a|u^*d\rangle + i\langle b|v^*d\rangle + i\langle b|u^*c\rangle + i\langle a|v^*c\rangle =$$

$$= \langle a|u^*c + v^*d\rangle + \langle b|u^*d - v^*c\rangle - i\langle a|u^*d - v^*c\rangle + i\langle b|u^*c + v^*d\rangle =$$

$$= \langle (a,b)|(u^*c + v^*d, u^*d - v^*c)\rangle = \left\langle (a,b) \,\middle|\, \widetilde{(u^*, -v^*)}(c,d) \right\rangle ,$$

so that

$$\widetilde{(u,v)}^* = \widetilde{(u^*, -v^*)} .$$

b) follows from a) and Proposition 2.1.5.6. ■

Corollary 5.3.1.12 (0) *Let E be a Hilbert space and $\overset{\circ}{E}$, $\overset{\circ}{\mathcal{L}}(E)$ the complexifications of E and of the underlying involutive algebra of $\mathcal{L}(E)$ (Proposition 2.3.1.40), respectively. For $(u,v) \in \overset{\circ}{\mathcal{L}}(E)$, define*

$$\widetilde{(u,v)} : \overset{\circ}{E} \longrightarrow \overset{\circ}{E}, \quad (x,y) \longmapsto (ux - vy, uy + vx) .$$

Then the maps

$$\overset{\circ}{\mathcal{L}}(E) \longrightarrow \mathcal{L}(\overset{\circ}{E}), \quad (u,v) \longmapsto \widetilde{(u,v)},$$

$$\overset{\circ}{\mathcal{K}}(E) \longrightarrow \mathcal{K}(\overset{\circ}{E}), \quad (u,v) \longmapsto \widetilde{(u,v)}$$

are bijective involutive unital algebra homomorphisms (hence $\overset{\circ}{\mathcal{L}}(E)$ and $\overset{\circ}{\mathcal{K}}(E)$ may be identified via this map with $\mathcal{L}(\overset{\circ}{E})$ and $\mathcal{K}(\overset{\circ}{E})$, respectively).

By Proposition 5.3.1.11, the above maps are well-defined and are bijective, linear, and involutive. By Corollary 2.1.5.8, the map is a unital algebra homomorphism. ■

Theorem 5.3.1.13 (0) *If $E \neq \{0\}$ is a Hilbert space, then $\mathcal{L}(E)$ is a unital C^*-algebra with respect to the involution*

$$\mathcal{L}(E) \longrightarrow \mathcal{L}(E), \quad u \longmapsto u^* .$$

In the complex case, the assertion follows from Theorem 5.3.1.4, Proposition 5.3.1.5, and Proposition 5.3.1.7 (and Example 2.2.2.4). The real case follows from the complex one and from Corollary 5.3.1.12. ■

Remark. By Proposition 2.3.1.14 (and the above theorem)

$$\sigma(u^*) = \{\overline{\alpha} \,|\, \alpha \in \sigma(u)\}$$

for every $u \in \mathcal{L}(E)$. Example 5.3.1.6 shows that $\sigma_p(u^*)$ may be different from $\{\overline{\alpha} \,|\, \alpha \in \sigma_p(u)\}$. They nevertheless coincide if u is normal (Corollary 5.3.4.5) or compact (Corollary 5.3.2.7).

Corollary 5.3.1.14 *If E is a Hilbert space, then there is a unique order relation on $\mathcal{L}(E)'$ which renders it an ordered Banach space and*

$$\mathcal{L}(E)'_+ = \{u' \in \operatorname{Re}\mathcal{L}(E)' \,|\, u \in \mathcal{L}(E) \Longrightarrow u'(u^*u) \in \mathbb{R}_+\} .$$

In the complex case,

$$\mathcal{L}(E)'_+ = \{u' \in \mathcal{L}(E)' \,|\, u \in \mathcal{L}(E) \Longrightarrow u'(u^*u) \in \mathbb{R}_+\} .$$

The assertion follows immediately from Theorem 5.3.1.13 and Proposition 2.3.4.13 b),d). ■

Proposition 5.3.1.15 *Let F be a closed vector subspace of the Hilbert space E and $i : F \to E$ the inclusion map. Then*

$$i^*x = \pi_F x , \quad \pi_F^* y = iy$$

for every $x \in E$ and $y \in F$.

We have

$$\langle iz|x\rangle = \langle z|x\rangle = \langle z|\pi_F x\rangle$$

for every $z \in F$ (Proposition 5.2.3.1), so that

$$i^* x = \pi_F x$$

and

$$\langle \pi_F z | y \rangle = \langle z | y \rangle = \langle z | iy \rangle$$

for every $z \in E$ (Proposition 5.2.3.1). Thus

$$\pi_F^* y = iy . \qquad \blacksquare$$

Proposition 5.3.1.16 (6) *Let u be a normal operator on the complex Hilbert space $E \neq \{0\}$ and α an isolated point of $\sigma(u)$ (α is isolated if e.g. u is compact and $\alpha \neq 0$). Then the resolvent of u has a pole of order 1 at α and its residue is the orthogonal projection of E onto $\operatorname{Ker}(\alpha 1 - u)$.*

By Proposition 4.1.3.17 and Theorem 5.3.1.13, the resolvent of u has in α a pole of order 1, its residue is an orthogonal projection p in E, and there is an operator v on E, such that

$$(\alpha 1 - u)p = 0 , \quad v(\alpha 1 - u) = p - 1 .$$

If $x \in \operatorname{Ker}(\alpha 1 - u)$, then

$$px - x = v(\alpha 1 - u)x = 0 ,$$

$$x = px \in \operatorname{Im} p ,$$

and so

$$\operatorname{Ker}(\alpha 1 - u) \subset \operatorname{Im} p .$$

If $x \in \operatorname{Im} p$, then

$$(\alpha 1 - u)x = (\alpha 1 - u)px = 0 ,$$

$$x \in \operatorname{Ker}(\alpha 1 - u) ,$$

and so

$$\operatorname{Im} p \subset \operatorname{Ker}(\alpha 1 - u) .$$

If u is compact and $\alpha \neq 0$, then, by Theorem 3.1.5.1 b), α is an isolated point of $\sigma(u)$. $\blacksquare$

Proposition 5.3.1.17 *If E is a Hilbert space, then $\mathcal{K}(E)$ is an involutive closed ideal and so a hereditary C^*-subalgebra of $\mathcal{L}(E)$ (Theorem 5.3.1.13). If $\mathcal{F}$ is a Gelfand C^*-subalgebra of $\mathcal{K}(E)$, then $\sigma(\mathcal{F})$ is discrete.*

By Corollary 3.1.1.13, $\mathcal{K}(E)$ is a closed ideal of $\mathcal{L}(E)$ and so by Proposition 4.3.4.5 e), a hereditary C^*-subalgebra of $\mathcal{L}(E)$.

Take $u \in \mathcal{F}$. Then

$$\sigma_{\mathcal{F}}(u) = \sigma_{\mathcal{L}(E)}(u)$$

(Theorem 4.1.2.12) and so $\sigma_{\mathcal{F}}(u)\backslash\{0\}$ is discrete (Theorem 3.1.5.1 b)). By Proposition 4.1.2.26, $\sigma(\mathcal{F})$ is discrete. ■

Proposition 5.3.1.18 *If u is a positive operator on a Hilbert space, then*

$$\operatorname{Im} u^{\alpha} \subset \operatorname{Im} u$$

for every $\alpha > 1$. Moreover,

$$\langle ux|x\rangle = 0 \Longrightarrow ux = 0$$

for every vector x of the Hilbert space.

Since

$$u^{\alpha} = uu^{\alpha-1},$$

we have that

$$\operatorname{Im} u^{\alpha} \subset \operatorname{Im} u.$$

For the last assertion remark that

$$\|u^{\frac{1}{2}}x\|^2 = \langle u^{\frac{1}{2}}x|u^{\frac{1}{2}}x\rangle = \langle u^{\frac{1}{2}}u^{\frac{1}{2}}x|x\rangle = \langle ux|x\rangle = 0$$

so that

$$u^{\frac{1}{2}}x = 0,$$

$$ux = u^{\frac{1}{2}}u^{\frac{1}{2}}x = 0.$$ ■

Proposition 5.3.1.19 *If u is a self-normal operator on a Hilbert space, then every isolated point of $\sigma(u)$ is in $\sigma_p(u)$.*

Let α be an isolated point of $\sigma(u)$. Define

$$f : \sigma(u) \longrightarrow \mathbb{K}, \quad \beta \longmapsto \delta_{\alpha\beta},$$

$$j : \sigma(u) \longrightarrow \mathbb{K}, \quad \beta \longmapsto \beta,$$

$$v := f(u), \quad w := (j - \alpha f)(u).$$

Then

$$v^2 = v, \quad wv = 0, \quad u = \alpha v + w$$

and

$$1 \in f(\sigma(u)) = \sigma(f(u)) = \sigma(v)$$

(Corollary 4.1.3.5), so that

$$v \neq 0, \quad \operatorname{Im} v \neq \{0\}.$$

Given $x \in \operatorname{Im} v$,

$$vx = x,$$

$$wx = wvx = 0,$$

$$ux = \alpha vx + wx = \alpha x,$$

so that

$$\alpha \in \sigma_p(u). \qquad \blacksquare$$

Example 5.3.1.20 *The right (left) shift of ℓ^2 is an extreme point of $\mathcal{L}(\ell^2)^\#$ (Example 4.1.1.7), 1 is its right (left) carrier, and*

$$u : \ell^2 \longrightarrow \ell^2, \quad \xi \longmapsto \xi - \xi e_1$$

is its left (right) carrier.

Let x be the right shift of ℓ^2. Then x^* is the left shift of ℓ^2 (Example 5.3.1.6) and $x^*x = 1$, $xx^* = u$. By Theorem 4.3.3.20 b $\Rightarrow$ a & e, x is an extreme point of $\mathcal{L}(\ell^2)^\#$ and 1 and u are its right and left carrier, respectively. The assertion for the left shift of ℓ^2 follows. $\blacksquare$

Proposition 5.3.1.21 (0) *Let E, F be Hilbert spaces and u a bijective operator from E to F. u is an isometry iff $u^{-1} = u^*$.*

If u is an isometry then

$$\langle x|u^{-1}y\rangle = \langle ux|y\rangle$$

for every $(x, y) \in E \times F$ (Corollary 5.1.1.4). Thus

$$u^{-1} = u^*$$

(Theorem 5.3.1.4). Conversely, if

$$u^{-1} = u^*,$$

then

$$\|x\|^2 = \langle x|x\rangle = \langle x|u^{-1}ux\rangle = \langle x|u^*ux\rangle = \langle ux|ux\rangle = \|ux\|^2$$

for every $x \in E$ (Theorem 5.3.1.4). Thus u is an isometry. ■

Corollary 5.3.1.22 (0) *An operator on a Hilbert space is an isometry iff it is unitary. Its spectrum is then contained in $\{\alpha \in \mathbb{K} \mid |\alpha| = 1\}$.*

Let E be a Hilbert space and take $u \in \mathcal{L}(E)$. If u is an isometry, then by Proposition 5.3.1.21

$$u^{-1} = u^*.$$

Thus

$$u^*u = uu^* = 1,$$

showing that u is unitary. If u is unitary, then u is invertible and

$$u^* = u^{-1}.$$

By Proposition 5.3.1.21, u is an isometry.

The final assertion follows from Corollary 2.2.4.7. ■

5.3.2 Supplementary Results

Proposition 5.3.2.1 (0) *Let* $(S, \mathfrak{S}, \mu)$, $(T, \mathfrak{T}, \nu)$ *be measure spaces. Take* $k \in L^2(\mu \otimes \nu)$, *and let* u *be the operator*

$$L^2(\nu) \longrightarrow L^2(\mu), \quad x \longmapsto k^\nu x$$

(Proposition 3.1.6.17 a)). Then, given $y \in L^2(\mu)$

$$(u^* y)(t) = (k^{*\mu} y)(t) = \int \overline{k}(s,t) y(s) d\mu(s)$$

for ν *almost all* $t \in T$ *(Example 5.1.2.1).*

Take $x \in L^2(\nu)$. Then $y \otimes x \in L^2(\mu \otimes \nu)$ and so

$$(y \otimes x)k \in L^1(\mu \otimes \nu)$$

(Hölder). By Fubini's Theorem

$$\langle ux|y\rangle = \int (ux)(s)\overline{y(s)} d\mu(s) = \int \left(\int k(s,t)x(t) d\nu(t) \right) \overline{y(s)} d\mu(s) =$$

$$= \int x(t) \left(\int k(s,t)\overline{y(s)} d\mu(s) \right) d\nu(t) =$$

$$= \int x(t) \left(\overline{\int \overline{k(s,t)} y(s) d\mu(s)} \right) d\nu(t) = \langle x|k^{*\mu} y\rangle ,$$

which proves the assertion (Theorem 5.3.1.4). ■

Proposition 5.3.2.2 (0) *Let* E, F *be Hilbert spaces and take* $u \in \mathcal{L}(E, F)$. *Then the following are equivalent:*

a) u *is compact.*

b) u^* *is compact.*

c) $u^* \circ u$ *is compact.*

d) $u \circ u^*$ *is compact.*

a $\Rightarrow$ c & d and b $\Rightarrow$ c & d follow from Proposition 3.1.1.11.

c $\Rightarrow$ a. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in E. Put

$$\alpha := \sup_{n\in\mathbb{N}} \|x_n\| .$$

By Proposition 3.1.1.19 a $\Rightarrow$ b, there is a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, such that $(u^*uy_n)_{n\in\mathbb{N}}$ converges. Given $n, p \in \mathbb{N}$,

$$\|uy_n - uy_p\|^2 = \|u(y_n - y_p)\|^2 = \langle u(y_n - y_p) | u(y_n - y_p)\rangle =$$

$$= \langle y_n - y_p | u^*u(y_n - y_p)\rangle \leq \|y_n - y_p\| \, \|u^*u(y_n - y_p)\| \leq 2\alpha \|u^*u(y_n - y_p)\| .$$

Thus $(uy_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence convergent. By Proposition 3.1.1.19 b $\Rightarrow$ a, u is compact.

d $\Rightarrow$ b follows from c $\Rightarrow$ a and Proposition 5.3.1.5 a). ■

Proposition 5.3.2.3 (0) *Let $(E_\iota)_{\iota\in I}$, $(F_\iota)_{\iota\in I}$ be two families of Hilbert spaces and take $(u_\iota)_{\iota\in I} \in \prod_{\iota\in I} \mathcal{L}(E_\iota, F_\iota)$ with*

$$\sup_{\iota\in I} \|u_\iota\| < \infty .$$

Then

$$(\bigoplus u_\iota)^* = \bigoplus_{\iota\in I} u_\iota^*$$

(Proposition 5.1.3.3 b)).

We have

$$\sup_{\iota\in I} \|u_\iota^*\| = \sup_{\iota\in I} \|u_\iota\| < \infty .$$

Given $(x, y) \in \left(\bigoplus_{\iota\in I} E_\iota\right) \times \left(\bigoplus_{\iota\in I} F_\iota\right)$,

$$\left\langle x \middle| \left(\bigoplus_{\iota\in I} u_\iota^*\right) y \right\rangle = \sum_{\iota\in I} \langle x_\iota | u_\iota^* y_\iota\rangle = \sum_{\iota\in I} \langle u_\iota x_\iota | y_\iota\rangle = \left\langle \left(\bigoplus_{\iota\in I} u_\iota\right) x \middle| y \right\rangle$$

and the assertion now follows (Theorem 5.3.1.4). ■

Proposition 5.3.2.4 (0) *If E, F are Hilbert spaces and $u \in \mathcal{L}(E, F)$, then*

$$\operatorname{Ker} u = \operatorname{Ker}(u^* \circ u) = (\operatorname{Im} u^*)^\perp ,$$

$$\overline{\operatorname{Im} u} = \overline{\operatorname{Im}(u \circ u^*)} = (\operatorname{Ker} u^*)^\perp .$$

If $E = F$, $K := \overline{\operatorname{Im} u}$, $L := (\operatorname{Ker} u)^\perp$ then π_K and π_L are the left and the right carrier of u in $\mathcal{L}(E)$, respectively (Theorem 5.3.1.13).

It follows immediately from

$$(x, y) \in E \times F \Longrightarrow \langle x | u^* y \rangle = \langle ux | y \rangle$$

that

$$\operatorname{Ker} u = (\operatorname{Im} u^*)^\perp .$$

The inclusion

$$\operatorname{Ker} u \subset \operatorname{Ker} (u^* \circ u)$$

is trivial. Given $x \in \operatorname{Ker} (u^* \circ u)$,

$$\|ux\|^2 = \langle ux | ux \rangle = \langle x | u^* ux \rangle = 0 ,$$

so that

$$ux = 0 , \quad x \in \operatorname{Ker} u .$$

Hence

$$\operatorname{Ker} (u^* \circ u) \subset \operatorname{Ker} u , \quad \operatorname{Ker} u = \operatorname{Ker} (u^* \circ u) .$$

Now

$$(\operatorname{Ker} u^*)^\perp = (\operatorname{Im} u)^{\perp\perp} = \overline{\operatorname{Im} u}$$

(Proposition 5.3.1.5 a), Corollary 5.2.3.9) and

$$\overline{\operatorname{Im} (u \circ u^*)} = (\operatorname{Ker} (u \circ u^*))^\perp = (\operatorname{Ker} u^*)^\perp$$

(Propositions 5.3.1.5 a) and 5.3.1.7).

We prove now the last assertion. For every $x \in E$,

$$\pi_K ux = ux ,$$

so that

$$\pi_K u = u .$$

Take $v \in \mathcal{L}(E)$ with

$$vu = 0 .$$

Then

$$vux = 0$$

for all $x \in E$, so that, by continuity,

$$v|K = 0\,.$$

Hence

$$v\pi_K = 0$$

and π_K is the left carrier of x. Since

$$L = (\mathrm{Ker}\, u)^{\perp} = \overline{\mathrm{Im}\, u^*}\,,$$

it follows that π_L is the left carrier of u^*. By Proposition 4.3.3.3, π_L is the right carrier of u. ■

Corollary 5.3.2.5 (0) *Let E, F be Hilbert spaces and take $u \in \mathcal{L}(E,F)$ Suppose that for some $\alpha > 0$,*

$$\|ux\| \geq \alpha\|x\|\,, \quad \|u^*y\| \geq \alpha\|y\|$$

for all $(x,y) \in E \times F$. Then u is an isomorphism of Banach spaces and

$$\|u^{-1}\| \leq \frac{1}{\alpha}\,.$$

u^* is obviously injective. By Proposition 5.3.2.4,

$$\overline{\mathrm{Im}\, u} = (\mathrm{Ker}\, u^*)^{\perp} = \{0\}^{\perp} = E$$

and the assertion now follows from Proposition 1.2.1.18 b),d). ■

Proposition 5.3.2.6 (0) *Let E, F be Hilbert spaces and $u \in \mathcal{F}(E,F)$. Then u^* and $u^* \circ u$ are Fredholm operators and*

$$\mathrm{Dim\,Ker}\, u^* = \mathrm{Dim\,Coker}\, u\,, \quad \mathrm{Dim\,Ker}\, u = \mathrm{Dim\,Coker}\, u^*\,,$$

$$\mathrm{Ind}\, u^* = -\mathrm{Ind}\, u\,, \quad \mathrm{Ind}\,(u \circ u^*) = 0\,,$$

$$\mathrm{Im}\, u = \mathrm{Im}\,(u \circ u^*) = (\mathrm{Ker}\, u^*)^{\perp}\,.$$

If u has index 0, then

$$\mathrm{Dim\,Ker}\, u = \mathrm{Dim\,Ker}\, u^* = \mathrm{Dim\,Coker}\, u = \mathrm{Dim\,Coker}\, u^* < \infty\,.$$

In particular if $E = F$ then

$$\sigma^e(u^*) = \{\overline{\alpha} | \alpha \in \sigma^e(u)\}\,.$$

Since

$$\operatorname{Dim}\operatorname{Ker} u^* = \operatorname{Dim}\operatorname{Ker} u' = \operatorname{Dim}\operatorname{Coker} u$$

(Theorem 5.3.1.4, Proposition 3.1.3.4), we deduce that

$$\operatorname{Dim}\operatorname{Ker} u = \operatorname{Dim}\operatorname{Ker} u^{**} = \operatorname{Dim}\operatorname{Coker} u^*$$

(Proposition 5.3.1.5 a)). In particular, u^* is Fredholm and

$$\operatorname{Ind} u^* = -\operatorname{Ind} u\,.$$

By Proposition 3.1.3.7, $u \circ u^*$ is Fredholm and

$$\operatorname{Ind}(u \circ u^*) = 0\,.$$

Since $\operatorname{Im} u$ and $\operatorname{Im}(u \circ u^*)$ are closed,

$$\operatorname{Im} u = \operatorname{Im}(u \circ u^*) = (\operatorname{Ker} u^*)^\perp$$

by Proposition 5.3.2.4. The assertion for $\operatorname{Ind} u = 0$ follows.

In order to prove the final assertion take $\alpha \in \mathbb{K}$. By Proposition 3.1.3.25 a),

$$(\alpha \in \sigma^e(u)) \Longleftrightarrow (\alpha 1 - u \notin \mathcal{F}(E)) \Longleftrightarrow$$

$$\Longleftrightarrow (\overline{\alpha} 1 - u^* \notin \mathcal{F}(E)) \Longleftrightarrow (\overline{\alpha} \in \sigma^e(u^*))\,. \qquad \blacksquare$$

Corollary 5.3.2.7 *Let E be a Hilbert space. Take $u \in \mathcal{K}(E)$ and $\alpha \in \mathbb{K}$. If E is finite-dimensional or if $\alpha \neq 0$, then*

$$\operatorname{Dim}\operatorname{Ker}(\alpha 1 - u) = \operatorname{Dim}\operatorname{Ker}(\overline{\alpha} 1 - u^*) =$$

$$= \operatorname{Dim}\operatorname{Coker}(\alpha 1 - u) = \operatorname{Dim}\operatorname{Coker}(\overline{\alpha} 1 - u^*) < \infty\,,$$

$$\operatorname{Im}(\alpha 1 - u) = (\operatorname{Ker}(\overline{\alpha} 1 - u^*))^\perp\,,$$

$$\alpha \in \sigma_p(u) \Longleftrightarrow \overline{\alpha} \in \sigma_p(u^*)\,.$$

By the Corollaries 3.1.3.12 and 3.1.3.13,

$$\alpha 1 - u \in \mathcal{F}(E)\,, \quad \mathrm{Ind}(\alpha 1 - u) = 0\,.$$

By Proposition 5.3.2.6,

$$\mathrm{Dim\,Ker}\,(\alpha 1 - u) = \mathrm{Dim\,Ker}(\overline{\alpha} 1 - u^*) =$$

$$= \mathrm{Dim\,Coker}\,(\alpha 1 - u) = \mathrm{Dim\,Coker}(\overline{\alpha} 1 - u^*) < \infty\,,$$

$$\mathrm{Im}(\alpha 1 - u) = (\mathrm{Ker}\,(\overline{\alpha} 1 - u^*))^{\perp}\,.$$

It follows that

$$\alpha \in \sigma_p(u) \Longleftrightarrow \overline{\alpha} \in \sigma_p(u^*)\,. \qquad \blacksquare$$

Proposition 5.3.2.8 (0) *Let u be a projection in the Hilbert space E. Then the following are equivalent:*

a) *u is an orthogonal projection (in the sense of Corollary 5.2.3.2).*

b) *u is selfadjoint, i.e. u is an orthogonal projection of the C^*-algebra $\mathcal{L}(E)$ (Theorem 5.3.1.13 and Definition 4.1.2.18).*

c) *u is positive.*

d) *u is normal.*

a $\Rightarrow$ b. Take $x, y \in E$. Then

$$\langle ux | y - uy \rangle = 0 = \langle ux - x | uy \rangle$$

(Proposition 5.2.3.1 a $\Rightarrow$ b), so that

$$\langle ux | y \rangle = \langle ux | y - uy \rangle + \langle ux | uy \rangle = \langle ux | uy \rangle = \langle ux - x | uy \rangle + \langle x | uy \rangle = \langle x | uy \rangle\,,$$

i.e. u is selfadjoint (Theorem 5.3.1.4).

b $\Rightarrow$ a. Since

$$\|u\|^2 = \|u^* u\| = \|u^2\| = \|u\|$$

(Theorem 5.3.1.4),

$$\|u\| \leq 1\,.$$

By Proposition 5.2.3.5 e $\Rightarrow$ a, u is an orthogonal projection.

b $\Rightarrow$ c follows from Proposition 4.2.1.22.

c $\Rightarrow$ d is trivial.

d $\Rightarrow$ a. Now

$$(u^*u)^2 = u^*uu^*u = u^*u^*uu = u^*u\,,$$

i.e. u^*u is a projection. Since u^*u is selfadjoint, we deduce from b $\Rightarrow$ a, that u^*u is an orthogonal projection. Hence

$$\|u\|^2 = \|u^*u\| \leq 1$$

(Theorem 5.3.1.4, Proposition 5.2.3.5 a $\Rightarrow$ e), so that

$$\|u\| \leq 1$$

and u is an orthogonal projection (Proposition 5.2.3.5 e $\Rightarrow$ a). ■

Proposition 5.3.2.9 (0) *Let F be a closed vector subspace of the Hilbert space E. Take $u \in \mathcal{L}(E)$. Then F is u-invariant iff $F^\perp$ is u^* invariant. In particular, F reduces u iff F is u-invariant and u^*-invariant. In this case,*

$$v^*x = u^*x$$

for every $x \in F$, where

$$v : F \longrightarrow F\,, \quad x \longmapsto ux\,.$$

If $\mathcal{F}$ is an involutive set of $\mathcal{L}(H)$, then F is $\mathcal{F}$-invariant iff $\pi_F \in \mathcal{F}^c$.

The first assertion follows from

$$x, y \in E \Longrightarrow \langle ux|y\rangle = \langle x|u^*y\rangle$$

and $F = F^{\perp\perp}$ (Corollary 5.2.3.9). Then second assertion follows from

$$x, y \in F \Longrightarrow \langle vx|y\rangle = \langle x|u^*y\rangle\,.$$

The final assertion follows from the first one and Corollary 5.2.3.12 b). ■

Corollary 5.3.2.10 *Let E be a Hilbert space. Take $u \in \mathcal{L}(E)$. Let $(F_\iota)_{\iota\in I}$ be a family of closed vector subspaces of E such that each F_ι ($\iota \in I$) reduces u. Then both $\bigcap_{\iota\in I} F_\iota$ and the closed vector subspace of E generated by $\bigcup_{\iota\in I} F_\iota$ reduce u.* ■

Corollary 5.3.2.11 *Let E be a Hilbert space, u a normal (selfadjoint, unitary) operator on E, F a closed vector subspace of E which reduces u and*

$$v : F \longrightarrow F, \quad x \longmapsto ux .$$

Then v is normal (selfadjoint, unitary). ■

Definition 5.3.2.12 (0) *Let E be a pre-Hilbert space and F a vector space. Given $(x, y) \in E \times F$, define*

$$\langle \cdot | x \rangle y : E \longrightarrow F, \quad z \longmapsto \langle z | x \rangle y .$$

If F is normed, then $\langle \cdot | x \rangle y \in \mathcal{L}(E, F)$ and

$$\| \langle \cdot | x \rangle y \| = \|x\| \, \|y\|$$

for every $(x, y) \in E \times F$.

Proposition 5.3.2.13 (0) *Let E, F, G be Hilbert spaces.*

a) *The map*

$$F \times E \longrightarrow \mathcal{L}(E, F), \quad (y, x) \longmapsto \langle \cdot | x \rangle y$$

is sesquilinear.

b) $(x, y) \in E \times F \Rightarrow (\langle \cdot | x \rangle y)^* = \langle \cdot | y \rangle x$.

c) $(x, y) \in E \times F$, $v \in \mathcal{L}(F, G) \Rightarrow v \circ (\langle \cdot | x \rangle y) = \langle \cdot | x \rangle vy$.

d) $u \in \mathcal{L}(E, F)$, $(y, z) \in F \times G \Rightarrow (\langle \cdot | y \rangle z) \circ u = \langle \cdot | u^* y \rangle z$.

e) $(x, y_1) \in E \times F$, $(y_2, z) \in F \times G \Rightarrow$

$\Rightarrow (\langle \cdot | y_2 \rangle z) \circ (\langle \cdot | x \rangle y_1) = \langle y_1 | y_2 \rangle \langle \cdot | x \rangle z$.

f) *If $(x_1, y_1), (x_2, y_2) \in E \times F$ such that*

$$\langle \cdot | x_1 \rangle y_1 = \langle \cdot | x_2 \rangle y_2 \neq 0 ,$$

then

$$x_1 = \alpha x_2 , \quad y_2 = \overline{\alpha} y_1 ,$$

where

$$\alpha := \frac{\langle x_1 | x_2 \rangle}{\|x_2\|^2} .$$

a) is easy to see.

b) Given $(a,b) \in E \times F$,

$$\langle(\langle\cdot|x\rangle y)a|b\rangle = \langle\langle a|x\rangle y|b\rangle = \langle a|x\rangle\langle y|b\rangle = \langle a|\langle b|y\rangle x\rangle = \langle a|(\langle\cdot|y\rangle x)b\rangle\ .$$

c) Given $a \in E$,

$$v \circ (\langle\cdot|x\rangle y)a = v\langle a|x\rangle y = \langle a|x\rangle vy = (\langle\cdot|x\rangle vy)a\,.$$

d) Given $a \in E$,

$$((\langle\cdot|y\rangle z) \circ u)a = (\langle\cdot|y\rangle z)ua = \langle ua|y\rangle z = \langle a|u^*y\rangle z = (\langle\cdot|u^*y\rangle z)a\,.$$

e) By c) and a)

$$(\langle\cdot|y_2\rangle z) \circ (\langle\cdot|x\rangle y_1) = \langle\cdot|x\rangle(\langle y_1|y_2\rangle z) = \langle y_1|y_2\rangle\langle\cdot|x\rangle z\,.$$

f) We have

$$\langle x_2|x_1\rangle y_1 = \|x_2\|^2 y_2\,,$$

so

$$y_2 = \overline{\alpha}y_1$$

and

$$\langle\cdot|x_1\rangle y_1 = \overline{\alpha}\langle\cdot|x_2\rangle y_1 = \langle\cdot|\alpha x_2\rangle y_1\,.$$

It follows

$$x_1 = \alpha x_2\,. \qquad \blacksquare$$

Remark. If we put

$$x_n := \frac{1}{\sqrt{n}}\langle\cdot|e_n\rangle e_1 \in \mathcal{L}(\ell^2)$$

for every $n \in \mathbb{N}$ (Example 5.1.2.3), then $(x_n^* x_n)_{n\in\mathbb{N}}$ is summable in $\mathcal{L}(\ell^2)$, but $(x_n x_n^*)_{n\in\mathbb{N}}$ is not. Indeed, by b) and e),

$$x_n^* x_n = \frac{1}{n}\langle\cdot|e_1\rangle e_n \circ \langle\cdot|e_n\rangle e_1 = \frac{1}{n}\langle\cdot|e_n\rangle e_n\,,$$

$$x_n x_n^* = \frac{1}{n}\langle\cdot|e_n\rangle e_1 \circ \langle\cdot|e_1\rangle e_n = \frac{1}{n}\langle\cdot|e_1\rangle e_1$$

for every $n \in \mathbb{N}$. Take $A \in \mathfrak{P}_f(\mathbb{N})$. For $\xi \in \ell^2$,

$$\left\| \left(\sum_{n \in A} x_n^* x_n \right) \xi \right\|^2 = \left\| \sum_{n in A} \frac{1}{n} \xi_n e_n \right\|^2 = \sum_{n \in A} \frac{1}{n^2} |\xi_n|^2 \leq \|\xi\|^2 \sum_{n \in A} \frac{1}{n^2}$$

(Pythagoras' Theorem) and

$$\left\| \left(\sum_{n \in A} x_n x_n^* \right) e_1 \right\| \geq \sum_{n \in A} \frac{1}{n},$$

so that

$$\left\| \sum_{n \in A} x_n^* x_n \right\| \leq \sum_{n \in A} \frac{1}{n^2},$$

$$\left\| \sum_{n \in A} x_n x_n^* \right\| \geq \sum_{n \in A} \frac{1}{n}.$$

By Proposition 1.1.6.6, $(x_n^* x_n)_{n \in \mathbb{N}}$ is summable and $(x_n x_n^*)_{n \in \mathbb{N}}$ is not summable.

Corollary 5.3.2.14 (0) *If E is a Hilbert space, then*

$$\{\langle \cdot | x \rangle x \,|\, x \in E\}^c = \mathbb{K} 1_E .$$

Take

$$u \in \{\langle \cdot | x \rangle x \,|\, x \in E\}^c .$$

Define

$$f : E \longrightarrow \mathbb{K}, \quad x \longmapsto \begin{cases} \frac{\langle ux | x \rangle}{\|x\|^2} & if \ x \neq 0 \\ 0 & if \ x = 0 . \end{cases}$$

If $x \in E$, then by Proposition 5.3.2.13 c),d),f),

$$\langle \cdot | u^* x \rangle x = (\langle \cdot | x \rangle x) \circ u = u \circ (\langle \cdot | x \rangle x) = \langle \cdot | x \rangle ux ,$$

$$ux = f(x)x .$$

It follows

$$f(x)x + f(y)y = ux + uy = u(x + y) =$$

$$= (f(x + y))(x + y) = f(x + y)x + f(x + y)y ,$$

$$f(x) = f(x + y) = f(y)$$

for all $x, y \in E$. Hence f is constant and so $u \in \mathbb{K} 1_E$. ■

Proposition 5.3.2.15 (0) *Let E be a Hilbert space, F a Banach space, $\mathfrak{G}$ an upward directed set of closed subspaces of E the union of which is dense in E, and $\mathfrak{F}$ the upper section filter of $\mathfrak{G}$. If $u \in \mathcal{K}(E,F)$, then*

$$\lim_{G,\mathfrak{F}} u \circ \pi_G = u .$$

Let v be the isometry of real Banach spaces

$$E \longrightarrow E' , \quad x \longmapsto \langle \cdot | x \rangle$$

(Corollary 5.2.5.3). We have $u' \in \mathcal{K}(F',E')$ (Theorem 3.1.1.22 a $\Rightarrow$ b), $v^{-1} \circ u'$ is compact (Proposition 3.1.1.11),

$$(u \circ \pi_G)' = \pi_G' \circ u' = v \circ \pi_G^{\bullet} \circ v^{-1} \circ u' = v \circ \pi_G \circ v^{-1} \circ u'$$

(Corollary 1.3.4.5, Theorem 5.3.1.4, Proposition 5.3.2.8 a $\Rightarrow$ b), and

$$\|u \circ \pi_G - u\| = \|(u \circ \pi_G - u)'\| =$$

$$= \|v \circ \pi_G \circ v^{-1} \circ u' - u'\| = \|\pi_G \circ v^{-1} \circ u' - v^{-1} \circ u'\|$$

for every $G \in \mathfrak{G}$ (Theorem 1.3.4.2 b)). By Proposition 5.2.3.3,

$$\lim_{G,\mathfrak{F}} \|u \circ \pi_G - u\| = \lim_{G,\mathfrak{F}} \|\pi_G \circ (v^{-1} \circ u') - (v^{-1} \circ u')\| = 0 ,$$

$$\lim_{G,\mathfrak{F}} u \circ \pi_G = u . \qquad \blacksquare$$

Corollary 5.3.2.16 (0) *Let E be a Hilbert space, $\mathfrak{F}$ an upward directed set of finite-dimensional vector subspaces of E, the union of which is dense in E, $\mathfrak{G}$ the upper section filter of $\mathfrak{F}$, and define*

$$\pi : \mathfrak{F} \longrightarrow \mathcal{K}(E) , \quad F \longmapsto \pi_F .$$

Then $\pi(\mathfrak{G})$ is an approximate unit of $\mathcal{K}(E)$ (Proposition 5.3.1.17).

The assertion follows immediately from Propositions 5.2.3.3 and 5.3.2.15 (and Corollary 5.2.3.2). $\blacksquare$

Corollary 5.3.2.17 (0) *Let E,F be Hilbert spaces, $\mathfrak{E}$ (resp. $\mathfrak{F}$) an upward directed set of finite-dimensional vector subspaces of E (resp. F) the union of which is dense in E (resp. in F), and take $u \in \mathcal{K}(E,F)$. Then for every $\varepsilon > 0$, there are $E_0 \in \mathfrak{E}$ and $F_0 \in \mathfrak{F}$ such that*

$$\|\pi_H \circ u \circ \pi_G - u\| < \varepsilon$$

for all $G \in \mathfrak{E}$ and $H \in \mathfrak{F}$ with $E_0 \subset G$, $F_0 \subset H$.

By Corollary 5.3.2.15, there are $E_0 \in \mathfrak{E}$ and $F_0 \in \mathfrak{F}$ such that

$$\|u \circ \pi_G - u\| < \frac{\varepsilon}{2}, \quad \|u^* \circ \pi_H - u^*\| < \frac{\varepsilon}{2}$$

for all $G \in \mathfrak{E}$ and $H \in \mathfrak{F}$ with $E_0 \subset G$, $F_0 \subset H$. Take $G \in \mathfrak{E}$ and $H \in \mathfrak{F}$ with $E_0 \subset G$, $F_0 \subset H$. Then (Theorem 5.3.1.4, Proposition 5.3.1.7, Corollary 5.2.3.2)

$$\|\pi_H \circ u \circ \pi_G - u\| \leq \|\pi_H \circ u \circ \pi_G - u \circ \pi_G\| + \|u \circ \pi_G - u\| \leq$$

$$\leq \|\pi_H \circ u - u\| \, \|\pi_G\| + \|u \circ \pi_G - u\| \leq$$

$$\leq \|u^* - u^* \circ \pi_H\| + \|u \circ \pi_G - u\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \blacksquare$$

Example 5.3.2.18 *If u_r, u_ℓ denote the right an left shift of ℓ^2, respectively, and q denotes the quotient map*

$$\mathcal{L}(\ell^2) \longrightarrow \mathcal{L}(\ell^2)/\mathcal{K}(\ell^2),$$

then qu_r, qu_ℓ are unitary and

$$(qu_r)^* = qu_\ell,$$

but there is no $v \in \mathcal{L}(\ell^2)/\mathcal{K}(\ell^2)$ with

$$e^v = qu_r \quad or \quad e^v = qu_\ell.$$

By the last assertion of Proposition 5.3.2.2, q is a homomorphism of involutive unital algebras (Proposition 5.3.1.17) and by Example 5.3.1.6,

$$u_r^* = u_\ell.$$

By Example 3.1.2.16,

$$(qu_r)^{-1} = qu_\ell.$$

Hence qu_r and qu_ℓ are unitary.

Assume there is a $v \in \mathcal{L}(\ell^2)/\mathcal{K}(\ell^2)$ with

$$e^v = qu_r.$$

Take $w \in \mathcal{L}(\ell^2)$ with

$$qw = v.$$

Then

$$qe^w = e^{qw} = e^v = qu_r .$$

Hence, by Proposition 3.1.3.21 c),

$$\operatorname{Ind} u_r = \operatorname{Ind} e^w = 0$$

since e^w is invertible. This is a contradiction, since

$$\operatorname{Ind} u_r = -1 . \qquad \blacksquare$$

Definition 5.3.2.19 (0) *Given the Hilbert space E, take $x \in E$ and $\mathcal{F} \subset \mathcal{L}(E)$. Then $\mathcal{F}$ is said* ***to act irreducibly on*** *E if $\mathcal{F}\backslash\{0\} \neq \emptyset$ and if 0 and E are the only $\mathcal{F}$-invariant closed vector subspaces of E. $\mathcal{F}$ is said* ***to act non-degenerately on*** *E if*

$$\{uy | (u, y) \in \mathcal{F} \times E\}^{\perp\perp} = E .$$

x is called ***cyclic for*** *$\mathcal{F}$, if*

$$\{ux \,|\, u \in \mathcal{F}\}^{\perp\perp} = E .$$

x is called ***separating for*** *$\mathcal{F}$ if*

$$u \in \mathcal{F},\ ux = 0 \Longrightarrow u = 0 .$$

It is obvious that if there is a cyclic element for $\mathcal{F}$, then $\mathcal{F}$ acts non-degenerately on E.

Proposition 5.3.2.20 (0) *Let E be a Hilbert space and take $\mathcal{F} \subset \mathcal{L}(E)$.*

a) *If every $x \in E\backslash\{0\}$ is cyclic for $\mathcal{F}$, then $\mathcal{F}$ acts irreducibly on E.*

b) *If $\mathcal{F}$ is a subalgebra of $\mathcal{L}(E)$ acting irreducibly on E, then every $x \in E \backslash\{0\}$ is cyclic for $\mathcal{F}$. If p is an orthogonal projection of E belonging to $\mathcal{F}$ such that $p\mathcal{F}p$ is one-dimensional, then there is an $x \in E$ such that*

$$\|x\| = 1 , \quad p = \langle \cdot | x \rangle x .$$

a) It is obvious that $\mathcal{F}\backslash\{0\} \neq \emptyset$. Let F be an $\mathcal{F}$–invariant closed vector subspace of E. Assume that $F \neq \{0\}$ and take $x \in F\backslash\{0\}$. By Corollary 5.2.3.9,

$$E = \{ux \,|\, u \in \mathcal{F}\}^{\perp\perp} \subset F^{\perp\perp} = F\,.$$

Hence $F = E$ and $\mathcal{F}$ acts irreducibly on E.

b) We put

$$F := \{ux \,|\, u \in \mathcal{F}\}\,.$$

Then F is an $\mathcal{F}$–invariant vector subspace of E and so $\overline{F}$ is also $\mathcal{F}$-invariant. Assume $F = \{0\}$. Then $\mathbb{K}x$ is $\mathcal{F}$–invariant, so that $\mathbb{K}x = E$. Take $u \in \mathcal{F}\backslash\{0\}$. Then $ux \neq 0$ and this contradicts the assumption $F = \{0\}$. Hence $\overline{F} = E$. By Corollary 5.2.3.9,

$$\{ux \,|\, u \in \mathcal{F}\}^{\perp\perp} = \overline{F} = E\,.$$

x is thus cyclic for $\mathcal{F}$.

Assume $\operatorname{Im} p$ is not one–dimensional. Then there are $x, y \in \operatorname{Im} p$ such that

$$\|x\| = \|y\| = 1\,, \quad x \perp y\,.$$

Since x is cyclic for $\mathcal{F}$, there is a $u \in \mathcal{F}$ such that

$$\|ux - y\| < 1\,.$$

Since $p\mathcal{F}p$ is one–dimensional, there is an $\alpha \in \mathbb{K}$ such that

$$pup = \alpha p\,.$$

Thus

$$1 \leq \|\alpha x - y\| = \|pupx - py\| \leq \|ux - y\| < 1\,,$$

which is a contradiction. Hence $\operatorname{Im} p$ is one–dimensional. Take $x \in \operatorname{Im} p$ with $\|x\| = 1$. Then

$$\operatorname{Ker} p = (\operatorname{Im} p)^{\perp} = \operatorname{Ker} \langle \cdot | x \rangle x$$

(Proposition 5.2.3.5 a $\Rightarrow$ d) and

$$px = x = \langle x | x \rangle x = (\langle \cdot | x \rangle x)x\,.$$

Hence

$$p = \langle \cdot | x \rangle x\,. \qquad \blacksquare$$

Proposition 5.3.2.21 (0) *Let E be a Hilbert space and take $\mathcal{F} \subset \mathcal{L}(E)$. Then the following are equivalent:*

a) *$\mathcal{F}$ acts non-degenerately on E.*

b) *Given $x \in E\backslash\{0\}$, there is a $u \in \mathcal{F}$ with*

$$u^*x \neq 0\,.$$

If these conditions are fulfilled, $\mathcal{F}$ is a subalgebra of $\mathcal{L}(H)$, and $\mathfrak{F}$ is an approximate unit of $\mathcal{F}$, then

$$\lim_{u,\mathfrak{F}} ux = x$$

for every $x \in E$.

a $\Rightarrow$ b. Assume that

$$u^*x = 0$$

for every $u \in \mathcal{F}$. Then

$$\langle x|uy\rangle = \langle u^*x|y\rangle = 0$$

for every $(u, y) \in \mathcal{F} \times E$. Hence

$$x \in \{uy\,|\,(u,y) \in \mathcal{F} \times E\}^{\perp} = \{0\}\,,$$

$$x = 0\,.$$

b $\Rightarrow$ a. Take

$$x \in \{uy\,|\,(u,y) \in \mathcal{F} \times E\}^{\perp}\,.$$

Then

$$\|u^*x\|^2 = \langle u^*x|u^*x\rangle = \langle x|uu^*x\rangle = 0$$

for every $u \in \mathcal{F}$, so that $x = 0$. Thus

$$\{uy\,|\,(u,y) \in \mathcal{F} \times E\}^{\perp} = \{0\}\,,$$

$$\{uy\,|\,(u,y) \in \mathcal{F} \times E\}^{\perp\perp} = E\,.$$

Hence $\mathcal{F}$ acts non-denerately on E.

We now prove the last assertion. Given $(v, x) \in \mathcal{F} \times E$,

$$\lim_{u,\mathfrak{F}} uvx = vx .$$

Hence if F is the vector subspace of E, generated by $\{vx \,|\, (v, x) \in \mathcal{F} \times E\}$, then

$$\lim_{u,\mathfrak{F}} ux = x$$

for every $x \in F$. By a) and Corollary 5.2.3.9, F is dense in E. Hence

$$\lim_{u,\mathfrak{F}} ux = x$$

for every $x \in E$. ■

Proposition 5.3.2.22 (0) *Let E be a Hilbert space and take $\mathcal{F} \subset \mathcal{L}(E)$.*

a) *If x is cyclic for $\mathcal{F}$, then x separates $\mathcal{F}^c$.*

b) *If x separates $\Pr \mathcal{F}^c$ and $\mathcal{F}$ is a quasiunital involutive subalgebra of $\mathcal{L}(E)$ acting non-degenerately on E, then x is cyclic for $\mathcal{F}$.*

a) Take $u \in \mathcal{F}^c$ with

$$ux = 0 .$$

Then

$$uvx = vux = 0$$

for every $v \in \mathcal{F}$. Hence if F denotes the vector subspace of E generated by $\{vx \,|\, v \in \mathcal{F}\}$, then

$$F \subset \operatorname{Ker} u .$$

By Corollary 5.2.3.9,

$$E = \{vx \,|\, v \in \mathcal{F}\}^{\perp\perp} = \overline{F} \subset \operatorname{Ker} u ,$$

$$u = 0 .$$

Hence x separates $\mathcal{F}^c$.

b) Define

$$F := \{ux \,|\, u \in \mathcal{F}\}^{\perp\perp}.$$

F is the closed vector space generated by $\{ux \,|\, u \in \mathcal{F}\}$ (Corollary 5.2.3.9). It is therefore $\mathcal{F}$-invariant. By Corollary 5.3.2.9, $\pi_F \in \mathcal{F}^c$. Let $\mathfrak{F}$ be an approximate unit of $\mathcal{F}$. By Proposition 5.3.2.21,

$$x = \lim_{u,\mathfrak{F}} ux \in F.$$

Hence

$$\pi_F x = x,$$

$$(1 - \pi_F)x = 0.$$

Since $1 - \pi_F \in \Pr \mathcal{F}^c$ and x separates $\Pr \mathcal{F}^c$, it follows that

$$1 - \pi_F = 0,$$

$$\pi_F = 1,$$

$$F = E.$$

Hence x is cyclic for $\mathcal{F}$. ■

Proposition 5.3.2.23 (0) *Let E be a separable Hilbert space and $\mathcal{F}$ a commutative involutive subalgebra of $\mathcal{L}(E)$. If $\mathcal{F}$ acts non-degenerately on E, then $\mathcal{F}$ has a separating vector.*

Let $\mathfrak{A}$ be the set of subsets A of E such that

$$\{ux \,|\, u \in \mathcal{F}\} \neq \{0\}$$

for every $x \in A$ and

$$\{ux \,|\, u \in \mathcal{F}\} \perp \{uy \,|\, u \in \mathcal{F}\},$$

for distinct $x, y \in A$. By Zorn's Lemma, $\mathfrak{A}$ contains a maximal element A. Take

$$y \in \{ux \,|\, (u, x) \in \mathcal{F} \times A\}^{\perp}.$$

Take $u, v \in \mathcal{F}$ and $x \in A$. Then

$$\langle ux \,|\, vy \rangle = \langle v^* ux \,|\, y \rangle = 0 \,.$$

Hence

$$\{ux \,|\, u \in \mathcal{F}\} \perp \{uy \,|\, u \in \mathcal{F}\} \,.$$

Since A is maximal in $\mathfrak{A}$,

$$\{uy \,|\, u \in \mathcal{F}\} = \{0\} \,.$$

Hence

$$y \in \bigcap_{u \in \mathcal{F}} \operatorname{Ker} u \,,$$

$$\{ux \,|\, (u,x) \in \mathcal{F} \times A\}^{\perp} \subset \bigcap_{u \in \mathcal{F}} \operatorname{Ker} u \,,$$

$$\{ux \,|\, (u,x) \in \mathcal{F} \times E\} = \bigcup_{u \in \mathcal{F}} \operatorname{Im} u \subset \bigcup_{u \in \mathcal{F}} (\operatorname{Im} u^*)^{\perp\perp} =$$

$$= \bigcup_{u \in \mathcal{F}} (\operatorname{Ker} u)^{\perp} \subset \{ux \,|\, (u,x) \in \mathcal{F} \times A\}^{\perp\perp}$$

(Proposition 5.3.2.4),

$$E = \{ux | (u,x) \in \mathcal{F} \times E\}^{\perp\perp} = \{ux \,|\, (u,x) \in \mathcal{F} \times A\}^{\perp\perp} \,.$$

Since E is separable, A is countable. Let $\varphi : A \to \mathbb{N}$ be an injective map and put

$$y := \sum_{x \in A} \frac{1}{2^{\varphi(x)} \|x\|} x \,.$$

Take $v \in \mathcal{F}$ with

$$vy = 0 \,.$$

Since the elements of the family $(vx)_{x \in A}$ are pairwise orthogonal,

$$vx = 0$$

for every $x \in A$ (Pythagoras' Theorem). Hence

$$vux = uvx = 0$$

for every $(u,x) \in \mathcal{F} \times A$. It follows that

$$v = 0$$

(Corollary 5.2.3.9). Thus y separates $\mathcal{F}$. ■

Proposition 5.3.2.24 (0) *Let E be a Hilbert space, F a closed vector subspace of E, and $\mathcal{F}$ a subalgebra of $\mathcal{L}(E)$ acting irreducibly on E. Given $u \in \mathcal{F}$, define*

$$\widetilde{u} : F \longrightarrow F, \quad x \longmapsto \pi_F u x .$$

Then $\{\widetilde{u} \,|\, u \in \mathcal{F}\}$ acts irreducibly on F.

Take $x, y \in F$ with $x \neq 0$. By Proposition 5.3.2.20 b), x is cyclic. Hence there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that

$$\lim_{n\to\infty} u_n x = y$$

(Corollary 5.2.3.9). Thus

$$\lim_{n\to\infty} \widetilde{u}_n x = y .$$

Hence x is cyclic for $\{\widetilde{u} \,|\, u \in \mathcal{F}\}$. By Proposition 5.3.2.20 a), $\{\widetilde{u} \,|\, u \in \mathcal{F}\}$ acts irreducibly on F. ■

Definition 5.3.2.25 *Let E, F be Hilbert spaces. An operator $u : E \to F$ is called a* ***partial isometry*** *if*

$$\|ux\| = \|x\| .$$

for all $x \in (\operatorname{Ker} u)^\perp$.

Proposition 5.3.2.26 *Let E, F be Hilbert spaces and take $u \in \mathcal{L}(E, F)$. Then the following are equivalent:*

a) *u is a partial isometry.*

b) *u^* is a partial isometry.*

c) *$u^* \circ u$ is an orthogonal projection.*

d) *$u \circ u^*$ is an orthogonal projection.*

If these conditions are fulfilled, then

$$(\operatorname{Ker} u)^\perp = \operatorname{Im} u^*, \quad (\operatorname{Ker} u^*)^\perp = \operatorname{Im} u ,$$

and $u^* \circ u$ (resp. $u \circ u^*$) is the orthogonal projection of E on $\operatorname{Im} u^*$ (of F on $\operatorname{Im} u$).

a $\Rightarrow$ b & c, and the last assertion. By the hypothesis, the map

$$v : (\operatorname{Ker} u)^{\perp} \longrightarrow \operatorname{Im} u\,, \quad x \longmapsto ux$$

is an isometry. Hence $\operatorname{Im} u$ is closed and by Proposition 5.3.2.4,

$$\operatorname{Im} u = (\operatorname{Ker} u^{*})^{\perp}\,.$$

Take $x, y \in (\operatorname{Ker} u)^{\perp}$. Then

$$\langle ux \,|\, uy\rangle = \langle x|y\rangle$$

(Corollary 5.1.1.8 a $\Rightarrow$ b), so that

$$\langle x|u^{*}uy\rangle = \langle x|y\rangle\,.$$

Hence the map

$$\operatorname{Im} u \longrightarrow (\operatorname{Ker} u)^{\perp}\,, \quad z \longmapsto u^{*}z$$

is the inverse of v. It follows that u^{*} is also a partial isometry and therefore

$$\operatorname{Im} u^{*} = (\operatorname{Ker} u)^{\perp}\,.$$

Moreover, $u^{*} \circ u$ (resp. $u \circ u^{*}$) is the orthogonal projection of E on $\operatorname{Im} u^{*}$ (of F on $\operatorname{Im} u$).

c $\Rightarrow$ a. We have

$$\operatorname{Ker} u = \operatorname{Ker}(u^{*} \circ u)\,,$$

$$(\operatorname{Ker} u)^{\perp} = (\operatorname{Ker}(u^{*}\circ u))^{\perp} = \operatorname{Im}(u^{*}\circ u)$$

(Proposition 5.3.2.4) and

$$\|u\|_{\mathfrak{l}}^{2} = \|u^{*}\|^{2} = \|u^{*} \circ u\| \leq 1$$

(Theorem 5.3.1.4). Take $x \in (\operatorname{Ker} u)^{\perp}$. Then

$$\|x\| = \|u^{*}ux\| \leq \|ux\| \leq \|x\|\,,$$

so that

$$\|ux\| = \|x\|$$

and u is a partial isometry.

b $\Rightarrow$ a & d and d $\Rightarrow$ b follow from a $\Rightarrow$ b & c and c $\Rightarrow$ a. ■

Proposition 5.3.2.27 *Let* $(H_\iota)_{\iota\in I}$, $(K_\iota)_{\iota\in I}$ *be two families of Hilbert spaces and* E *the* C^**-direct sum of the family* $(\mathcal{K}(H_\iota, K_\iota))_{\iota\in I}$. *Put*

$$H := \bigoplus_{\iota\in I} H_\iota\,, \quad K := \bigoplus_{\iota\in I} K_\iota$$

and for each $u \in E$, *define*

$$\widetilde{u} := \bigoplus_{\iota\in I} u_\iota\,.$$

Then

$$\widetilde{u} \in \mathcal{K}(H, K)$$

and the map

$$E \longrightarrow \mathcal{K}(H, K)\,, \quad u \longmapsto \widetilde{u}$$

is an injective homomorphism of involutive vector spaces (algebras if $(H_\iota)_{\iota\in I} = (K_\iota)_{\iota\in I}$*).*

The first assertion follows from Proposition 3.1.2.17. It is obvious that the map

$$E \longrightarrow \mathcal{K}(H, K)\,, \quad u \longmapsto \widetilde{u}$$

is injective. By Proposition 5.1.3.4 and Proposition 5.3.2.3, this map is a homomorphism of involutive algebras. ■

Proposition 5.3.2.28 *The following are equivalent for every Hilbert space* E*:*

a) E *is finite-dimensional.*

b) $\mathcal{K}(E)$ *is a* W^**-algebra.*

c) $\mathcal{K}(E)$ *is unital.*

a $\Rightarrow$ b is trivial.

b $\Rightarrow$ c follows from Theorem 4.4.1.8 h).

c $\Rightarrow$ a. Let p be the unit of $\mathcal{K}(E)$ and put $F := \operatorname{Im} p$. Then $p \in \operatorname{Pr}\mathcal{K}(E)$ so that $p = \pi_F$ (Proposition 5.3.2.8 b $\Rightarrow$ a). Take $x \in F^\perp$. By Proposition 5.3.2.13 c),

$$\langle\cdot|x\rangle x = p\circ\langle\cdot|x\rangle x = \langle\cdot|x\rangle px = 0$$

so that

$$\|x\|^2 x = \langle x|x\rangle x = 0\,,$$

$$x = 0\,.$$

Hence $F^\perp = \{0\}$,

$$F = F^{\perp\perp} = E\,,$$

and

$$p = \pi_E = 1_E\,.$$

By Proposition 3.1.1.10, E is finite-dimensional. ■

Proposition 5.3.2.29 *Let E be an infinite-dimensional Hilbert space, $\mathcal{F}$ a closed ideal of $\mathcal{L}(E)$, and $\mathcal{G}$ a C^*-subalgebra of $\mathcal{L}(E)$. Then $\mathcal{F}+\mathcal{G}$ is the C^*-subalgebra of $\mathcal{L}(E)$ generated by $\mathcal{F}\cup\mathcal{G}$ and the C^*- algebras $(\mathcal{F}+\mathcal{G})/\mathcal{F}$ and $\mathcal{G}/(\mathcal{F}\cap\mathcal{G})$ are canonically isometric. If in addition $\mathcal{F}\cap\mathcal{G} = \{0\}$ then*

$$\|v\| \leq \|u+v\|$$

for all $(u,v) \in \mathcal{F}\times\mathcal{G}$.

By Theorem 5.3.1.13, $\mathcal{L}(E)$ is a C^*-algebra and the assertion follows from Corollary 4.2.6.7. ■

5.3.3 Selfadjoint Operators

Proposition 5.3.3.1 (0) *Let E, F be Hilbert spaces and $\mathcal{S}$ the involutive vector space of continuous sesqilinear forms on E (Proposition 2.3.3.3 c)). Given $u \in \mathcal{L}(E)$, define*

$$\widetilde{u} : E \times E \longrightarrow \mathbb{K}, \quad (x, y) \longmapsto \langle ux|y\rangle .$$

a) *$\widetilde{u} \in \mathcal{S}$ for every $u \in \mathcal{L}(E)$.*

b) *The map*

$$\mathcal{L}(E) \longrightarrow \mathcal{S}, \quad u \longmapsto \widetilde{u}$$

is an isomorphism of involutive vector spaces.

c) *$\widetilde{u^* \circ u}$ is positive for every $u \in \mathcal{L}(E, F)$.*

a) Now

$$|\widetilde{u}(x, y)| = |\langle ux|y\rangle| \leq \|ux\| \, \|y\| \leq \|u\| \, \|x\| \, \|y\|$$

and so, by Proposition 1.2.9.2 c $\Rightarrow$ a, $\widetilde{u}$ is continuous.

b) Consider

$$\varphi : \mathcal{L}(E) \longrightarrow \mathcal{S}, \quad u \longmapsto \widetilde{u} .$$

It is obvious that φ is linear. By Corollary 5.3.1.2 and Theorem 5.3.1.4, φ is bijective. Take $u \in \mathcal{L}(E)$. Then

$$\widetilde{u^*}(x, y) = \langle u^* x|y\rangle = \langle x|u^{**} y\rangle = \langle x|uy\rangle = \overline{\langle uy|x\rangle} = \overline{\widetilde{u}(y, x)} = \widetilde{u}^*(x, y)$$

for all $x, y \in E$ (Theorem 5.3.1.4, Proposition 5.3.1.5 a)). Thus

$$\widetilde{u^*} = \widetilde{u}^*$$

and φ is involutive.

c) Given $x \in E$,

$$\widetilde{u^* \circ u}(x, x) = \langle (u^* \circ u)x|x\rangle = \langle u^* ux|x\rangle = \langle ux|ux\rangle \in \mathbb{R}_+ .$$

Thus by b), $\widetilde{u^* \circ u}$ is positive. ■

Corollary 5.3.3.2 *An operator u on a real Hilbert space E is selfadjoint iff*

$$\langle ux|y\rangle = \frac{1}{4} \left(\langle u(x + y) \, | \, x + y\rangle - \langle u(x - y) \, | \, x - y\rangle \right)$$

for all $x, y \in E$.

This follows immediately from Proposition 5.3.3.1 b) and Proposition 2.3.3.7. ■

Corollary 5.3.3.3 (0) *An operator u on a complex Hilbert space E is selfadjoint iff*

$$\langle ux|x\rangle \in \mathbb{R}$$

for every $x \in E$.

This follows immediately from Proposition 5.3.3.1 b) and Proposition 2.3.3.8 a $\Leftrightarrow$ b. ■

Corollary 5.3.3.4 (0) *Let u be an operator on a Hilbert space E such that*

$$\langle ux|x\rangle = 0$$

for every $x \in E$. If u is selfadjoint or if $\mathbb{K} = \mathbb{C}$, then $u = 0$.

By Proposition 5.3.3.1 b) and Proposition 5.1.1.9,

$$\|ux\|^2 = \langle ux|ux\rangle = 0$$

for every $x \in E$ and so $u = 0$. ■

Remark. Let u be the operator on $\mathbb{R}^2$, defined by the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$\langle ux|x\rangle = 0$$

for every $x \in \mathbb{R}^2$. This example shows that the above Corollary is false without some hypothesis such as the selfadjointness of u or $\mathbb{K} = \mathbb{C}$.

Proposition 5.3.3.5 (0) *If u is a selfadjoint operator on the Hilbert space E, then*

$$\|u\| = \sup_{x \in E^\#} |\langle ux|x\rangle| = \inf\{\alpha \in \mathbb{R}_+ \,|\, x \in E \Longrightarrow |\langle ux|x\rangle| \leq \alpha\|x\|^2\}.$$

Define

$$\beta := \sup_{x \in E^{\#}} |\langle ux|x\rangle|\,,$$

$$\gamma := \inf\{\alpha \in \mathbb{R}_+ \,|\, x \in E \Longrightarrow |\langle ux|x\rangle| \leq \alpha\|x\|^2\}\,.$$

Take $\alpha \in \mathbb{R}_+$ with

$$x \in E \Longrightarrow |\langle ux|x\rangle| \leq \alpha\|x\|^2\,.$$

Given $x \in E$,

$$\|ux\|^2 = |\langle ux|ux\rangle| \leq \alpha\|x\|\,\|ux\|$$

(Proposition 5.3.3.1 b), Proposition 5.1.1.9) and so

$$\|ux\| \leq \alpha\|x\|\,.$$

It follows that

$$\|u\| \leq \alpha$$

and

$$\|u\| \leq \gamma\,.$$

Take $x \in E\backslash\{0\}$. Then

$$|\langle ux|x\rangle| = \|x\|^2 \left|\left\langle u\frac{1}{\|x\|}x \,\middle|\, \frac{1}{\|x\|}x\right\rangle\right| \leq \beta\|x\|^2\,,$$

so that

$$\gamma \leq \beta\,.$$

Finally,

$$|\langle ux|x\rangle| \leq \|ux\|\,\|x\| \leq \|u\|\,\|x\|^2$$

for every $x \in E$, so that

$$\beta \leq \|u\|\,.$$

Thus

$$\|u\| = \beta = \gamma\,.$$

■

Theorem 5.3.3.6 (0) *Let u be a selfadjoint operator on the Hilbert space $E \neq \{0\}$ and put*

$$\alpha := \inf\{\langle ux|x\rangle \,|\, x \in E, \|x\| = 1\},$$

$$\beta := \sup\{\langle ux|x\rangle \,|\, x \in E, \|x\| = 1\},$$

(Corollary 5.3.3.3). Then

$$\alpha, \beta \in \sigma(u) \subset [\alpha, \beta].$$

Take $\gamma \in]-\infty, \alpha[$. Then

$$0 < \alpha - \gamma \leq \langle ux|x\rangle - \gamma\langle x|x\rangle = \langle (u - \gamma 1)x|x\rangle \leq \|(u - \gamma 1)x\|$$

for every $x \in E$ with $\|x\| = 1$. By Corollary 5.3.2.5, $\gamma \notin \sigma(u)$. Hence

$$\sigma(u) \subset [\alpha, \infty[.$$

It follows that

$$-\sigma(u) = \sigma(-u) \subset [-\beta, \infty[$$

and

$$\sigma(u) \subset [\alpha, \beta].$$

Put

$$\gamma := \inf \sigma(u).$$

Then

$$\sigma(u - \gamma 1) = \sigma(u) - \gamma \subset \mathbb{R}_+,$$

i.e. $u - \gamma 1$ is positive. Thus

$$\langle ux|x\rangle = \langle (u - \gamma 1)x|x\rangle + \gamma = \langle (u - \gamma 1)^{1/2}x \,|\, (u - \gamma 1)^{1/2}x\rangle + \gamma \geq \gamma$$

for every $x \in E$, $\|x\| = 1$, and so $\alpha \geq \gamma$. Hence

$$\alpha = \gamma \in \sigma(u)$$

and

$$\beta \in \sigma(u). \qquad \blacksquare$$

Corollary 5.3.3.7 (0) *The operator u on the Hilbert space E is positive iff it is selfadjoint and*

$$\langle ux|x\rangle \in \mathbb{R}_+$$

for every $x \in E$. In the complex case, we may omit the adjective "selfadjoint". In particular, $\langle\cdot|x\rangle x$ and the map

$$\mathcal{L}(E) \longrightarrow \mathbb{K}, \quad u \longmapsto \langle ux|x\rangle$$

are positive for every $x \in E$.

The first assertion follows from Theorem 5.3.3.6 and the second from Corollary 5.3.3.3. The final assertion follows from the first one, Corollary 4.2.2.10, and Proposition 5.3.2.13 b). ■

Remark. The example given in the Remark to Corollary 5.3.3.4 shows that we cannot drop "selfadjoint" in the real case.

Corollary 5.3.3.8 (0) *If F and G are closed vector subspaces of a Hilbert space E, then $F \subset G$ iff $\pi_F \leq \pi_G$. In particular, if H is the closed vector subspace of E generated by $F \cup G$ (by $F \cap G$), then π_H is the supremum (infimum) of $\{\pi_F, \pi_G\}$ in $\Pr\mathcal{L}(E)$.*

This follows from Corollary 5.3.3.7 and Proposition 5.2.3.13 a $\Leftrightarrow$ e (and Proposition 5.3.2.8 a $\Leftrightarrow$ b). ■

Corollary 5.3.3.9 (0) *Let E, F be Hilbert spaces and take $v \in \mathcal{L}(E, F)$. Then $v^* \circ u \circ v$ is positive whenever u is a positive operator on F. In particular, $v^* \circ v$ is positive.*

$v^* \circ u \circ v$ is selfadjoint and by Corollary 5.3.3.7,

$$\langle v^* uvx|x\rangle = \langle uvx|vx\rangle \in \mathbb{R}_+$$

for every $x \in E$. By Corollary 5.3.3.7 again, $v^* \circ u \circ v$ is positive. ■

Proposition 5.3.3.10 (0) *Let E, F be Hilbert spaces. Take $u \in \mathcal{L}(E, F)$, and $\alpha > 0$.*

a) $\operatorname{Ker}(u^* \circ u)^\alpha = \operatorname{Ker} u$.

b) *u compact $\Leftrightarrow$ $(u^* \circ u)^\alpha$ is compact.*

By Corollary 5.3.3.9, $u^* \circ u$ is positive. Thus $(u^*u)^\alpha$ is well-defined.

a) Step 1 $v \in \mathrm{Re}\,\mathcal{L}(E)$, $n \in \mathbb{N} \Rightarrow \mathrm{Ker}\, v^{2^n} = \mathrm{Ker}\, v$

By Proposition 5.3.2.4,

$$\mathrm{Ker}\, v^2 = \mathrm{Ker}\, v\,,$$

and so the assertion follows by complete induction.

Step 2 $v \in \mathcal{L}(E)_+$, $0 < \beta < \gamma \Rightarrow \mathrm{Ker}\, v^\beta \subset \mathrm{Ker}\, v^\gamma$

The assertion follows from

$$v^\gamma = v^{\gamma-\beta} v^\beta\,.$$

Step 3 $v \in \mathcal{L}(E)_+ \Rightarrow \mathrm{Ker}\, v^\alpha = \mathrm{Ker}\, v$

Take $n \in \mathbb{N}$ such that

$$2^{-n} < \alpha < 2^n\,.$$

By Steps 1 and 2,

$$\mathrm{Ker}\, v \subset \mathrm{Ker}\, v^{\alpha 2^n} = \mathrm{Ker}\,(v^\alpha)^{2^n} = \mathrm{Ker}\, v^\alpha \subset \mathrm{Ker}\, v^{2^n} = \mathrm{Ker}\, v\,,$$

and so

$$\mathrm{Ker}\, v^\alpha = \mathrm{Ker}\, v$$

Step 4 $\mathrm{Ker}\,(u^*u)^\alpha = \mathrm{Ker}\, u$

By Proposition 5.3.2.4 and Step 3,

$$\mathrm{Ker}\,(u^* \circ u)^\alpha = \mathrm{Ker}\,(u^* \circ u) = \mathrm{Ker}\, u\,.$$

b) Step 1 $v \in \mathrm{Re}\,\mathcal{L}(E)$, $n \in \mathbb{N} \Rightarrow (v \text{ compact} \Leftrightarrow v^{2^n} \text{ compact})$

By Proposition 5.3.2.2 a $\Leftrightarrow$ c,

$$v \text{ compact} \iff v^2 \text{ compact}\,.$$

The assertion thus follows by complete induction.

Step 2 $v \in \mathcal{L}(E)_+$, $0 < \beta < \gamma$, v^β compact $\Rightarrow v^\gamma$ compact

The assertion follows from

$$v^\gamma = v^{\gamma-\beta} v^\beta$$

and Proposition 3.1.1.11.

Step 3 $v \in \mathcal{L}(E)_+ \Rightarrow (v \text{ compact} \Leftrightarrow v^\alpha \text{ compact})$

Let $n \in \mathbb{N}$ with

$$2^{-n} < \alpha < 2^n .$$

By Steps 1 and 2,

$$v \text{ compact} \Longrightarrow v^{\alpha 2^n} \text{ compact} \Longleftrightarrow (v^\alpha)^{2^n} \text{ compact} \Longleftrightarrow$$

$$\Longleftrightarrow v^\alpha \text{ compact} \Longrightarrow v^{2^n} \text{ compact} \Longleftrightarrow v \text{ compact} .$$

Thus

$$v \text{ compact} \Longleftrightarrow v^\alpha \text{ compact} .$$

Step 4 u compact $\Leftrightarrow (u^* \circ u)^\alpha$ compact

By Proposition 5.3.2.2 a $\Leftrightarrow$ c and Step 3,

$$u \text{ compact} \Longleftrightarrow u^* \circ u \text{ compact} \Longleftrightarrow (u^* \circ u)^\alpha \text{ compact} . \qquad \blacksquare$$

Corollary 5.3.3.11 (0) *If u is a normal operator on a Hilbert space, then, for every $n \in \mathbb{N}$*

$$\operatorname{Ker} u^n = \operatorname{Ker} u , \quad u \textit{ compact} \Longleftrightarrow u^n \textit{ compact} .$$

Since

$$(u^* u)^n = (u^*)^n u^n = (u^n)^* u^n ,$$

it follows by Proposition 5.3.3.10, that

$$\operatorname{Ker} u = \operatorname{Ker} (u^* u)^n = \operatorname{Ker} ((u^n)^* u^n) = \operatorname{Ker} u^n ,$$

$$u \text{ compact} \Longleftrightarrow (u^* u)^n \text{ compact} \Longleftrightarrow (u^n)^* u^n \text{ compact} \Longleftrightarrow$$

$$\Longleftrightarrow u^n \text{ compact} . \qquad \blacksquare$$

Remark. We cannot drop the adjective "normal" in the above corollary, as the following example shows:

$$u : \ell^2 \longrightarrow \ell^2 , \quad x \longmapsto (0, x_1, 0, x_3, 0, x_5 \ldots) .$$

Then u is not compact but $u^2 = 0$.

Corollary 5.3.3.12 (0) *Let E be a Hilbert space. Take $u \in \mathcal{L}(E)$ and define*

$$f : E \times E \longrightarrow \mathbb{K} , \quad (x, y) \longmapsto \langle ux | y \rangle .$$

a) *f is Hermitian iff u is selfadjoint.*

b) *f is positive iff u is positive. In this case*

$$\{x \in E \,|\, f(x, x) = 0\} = \operatorname{Ker} u .$$

c) *f is a scalar product iff u is injective and positive.*

a) follows from Proposition 5.3.3.1 a),b).

b) By a) and Corollary 5.3.3.7, f is positive iff u is positive. Take $x \in E$. Then

$$f(x, x) = \langle ux | x \rangle = \left\langle u^{\frac{1}{2}} x \,\middle|\, u^{\frac{1}{2}} x \right\rangle = \| u^{\frac{1}{2}} x \|^2 .$$

Thus, by Corollary 5.3.3.11,

$$f(x, x) = 0 \Longleftrightarrow x \in \operatorname{Ker} u^{\frac{1}{2}} \Longleftrightarrow x \in \operatorname{Ker} u .$$

c) follows from b). ■

Proposition 5.3.3.13 (0) *Let E, F, G be Hilbert spaces. Take $u \in \mathcal{L}(E, F)$ and let p (resp. q) be the orthogonal projection of E onto $(\operatorname{Ker} u)^{\perp}$ (of F on $\overline{\operatorname{Im} u}$).*

a) *p (resp. q) is the smallest orthogonal projection r in E (in F) such that*

$$u \circ r = u \quad (r \circ u = u) .$$

b) *Given $v \in \mathcal{L}(G, E)$ with $u \circ v = 0$, we have that $p \circ v = 0$.*

c) *Given $v \in \mathcal{L}(F, G)$ with $v \circ u = 0$, we have that $v \circ q = 0$.*

a) $1-p$ (resp. $1-q$) is the orthogonal projection of E onto $\operatorname{Ker} u$ (of F on $\operatorname{Im} u^{\perp}$) and so

$$u \circ (1-p) = 0\,, \quad ((1-q) \circ u = 0)\,,$$

$$u \circ p = u\,, \quad (q \circ u = u)\,.$$

Now

$$u \circ (1-r) = 0\,, \quad ((1-r) \circ u = 0)\,,$$

so that

$$\operatorname{Ker} r = \operatorname{Im}(1-r) \subset \operatorname{Ker} u\,, \quad (\operatorname{Im} u \subset \operatorname{Ker}(1-r) = \operatorname{Im} r)\,,$$

$$\operatorname{Im} p = (\operatorname{Ker} u)^{\perp} \subset (\operatorname{Ker} r)^{\perp} = \operatorname{Im} r\,, \quad (\operatorname{Im} q = \overline{\operatorname{Im} u} \subset \operatorname{Im} r)$$

$$p \leq r\,, \quad (q \leq r)$$

(Corollary 5.3.3.8).

b) Since

$$\operatorname{Im} v \subset \operatorname{Ker} u = \operatorname{Ker} p\,,$$

we have that

$$p \circ v = 0\,.$$

c) Since

$$\operatorname{Im} q = \overline{\operatorname{Im} u} \subset \operatorname{Ker} v\,,$$

we have that

$$v \circ q = 0\,.$$

■

Remark. If $E = F = G$, then the above proposition says that p and q are the right and left carrier of u, respectively.

Theorem 5.3.3.14 (0) *Let E be a Hilbert space, $\mathcal{F}$ a nonempty upward directed set of $\mathcal{L}(E)$, and $\mathfrak{F}$ its upper section filter. Then the following are equivalent:*

a) $\mathcal{F}$ *is bounded above.*

b) $\mathcal{F}$ *possesses a supremum in* $\mathcal{L}(E)$.

If these conditions are fulfilled, then:

c) $\left\langle \left(\bigvee_{u \in \mathcal{F}} u \right) x \middle| y \right\rangle = \lim_{u,\mathfrak{F}} \langle ux|y \rangle$ *for all* $x, y \in E$.

In particular, $\mathcal{L}(E)$ *and* $\mathcal{K}(E)$ *are* $\mathbb{C}$*-order complete (Corollary 5.3.1.12).*

a $\Rightarrow$ b & c. Replacing $\mathcal{F}$ by $\{u - u_0 \,|\, u \in \mathcal{F},\, u \geq u_0\}$ for some $u_0 \in \mathcal{F}$, if necessary, we may assume that $\mathcal{F} \subset \mathcal{L}(E)_+$. Let v be an upper bound for $\mathcal{F}$. Then

$$0 \leq \lim_{u,\mathfrak{F}} \langle ux|x \rangle = \sup_{u \in \mathcal{F}} \langle ux|x \rangle \leq \langle vx|x \rangle \leq \|vx\| \, \|x\| \leq \|v\| \, \|x\|^2$$

for every $x \in E$ (Corollary 5.3.3.7). Define

$$f : E \times E \longrightarrow \mathbb{K}, \quad (x, y) \longmapsto \lim_{u,\mathfrak{F}} \langle ux|y \rangle$$

(Propositions 5.3.3.1 a),b), 2.3.3.7 a $\Rightarrow$ b, and 2.3.3.8 a $\Rightarrow$ c & d). Then f is a positive sesquilinear form on E (Proposition 5.3.3.1 a),b)) and

$$|f(x, y)| \leq \sqrt{\widetilde{f}(x)\widetilde{f}(y)} \leq \|v\| \, \|x\| \, \|y\|$$

for all $x, y \in E$ (Schwarz's Inequality). Hence f is continuous (Proposition 1.2.9.2 c $\Rightarrow$ a). By Corollary 5.3.1.2, there is a $w \in \mathcal{L}(E)$ such that

$$f(x, y) = \langle x|wy \rangle$$

for all $x, y \in E$. Thus $w \in \mathcal{L}(E)_+$ (Corollary 5.3.3.12 b)) and

$$\langle ux|x \rangle \leq \langle wx|x \rangle \leq \langle vx|x \rangle,$$

whenever $u \in \mathcal{F}$ and $x \in E$. Hence

$$u \leq w \leq v$$

for every $u \in \mathcal{F}$ (Corollary 5.3.3.7). w is thus the supremum of $\mathcal{F}$ in $\mathcal{L}(E)$. Hence $\mathcal{L}(E)$ is $\mathbb{C}$-order complete. By Proposition 5.3.1.17, $\mathcal{K}(E)$ is also $\mathbb{C}$-order complete.

b $\Rightarrow$ a *is trivial.* ■

Corollary 5.3.3.15 *If E, F are complex Hilbert spaces, then*

$$\{u \in \mathcal{L}(E,F) \,|\, u \text{ is an isomorphism (isometry)}\}$$

is path-connected.

Define

$$\mathcal{U} := \{u \in \mathcal{L}(E) \,|\, u \text{ is invertible (unitary)}\}$$

$$\mathcal{V} := \{u \in \mathcal{L}(E,F) \,|\, u \text{ is an isomorphism (isometry)}\}\,.$$

Take $u, v \in \mathcal{V}$. Then $u^{-1} \circ v \in \mathcal{U}$ (Corollary 5.3.1.22). By Corollary 4.3.2.9 (Corollary 4.3.2.7), and Theorem 5.3.3.14, there is a continuous map

$$f : [0,1] \longrightarrow \mathcal{U}\,,$$

with

$$f(0) = 1_E\,, \quad f(1) = u^{-1} \circ v\,.$$

Then

$$[0,1] \longrightarrow \mathcal{V}\,, \quad \alpha \longmapsto uf(\alpha)$$

is a continuous path connecting u with v. ■

Remark. The corollary does not hold in the real case. In the finite-dimensional case, the set of isomorphisms (isometries) has two connected components.

Corollary 5.3.3.16 (Atkinson, 1951) *If E, F are complex Hilbert spaces, then for each $n \in \mathbb{Z}$,*

$$\{u \in \mathcal{F}(E,F) \,|\, \operatorname{Ind} u = n\}$$

is path-connected.

Case 1 $n = 0$

Take $u, v \in \mathcal{F}(E,F)$ with

$$\operatorname{Ind} u = \operatorname{Ind} v = 0\,.$$

By Corollary 3.1.3.15, there are isomorphisms $u_0, v_0 : E \to F$ and continuous paths ξ, η in $\mathcal{F}(E,F)$ connecting u and u_0 and v and v_0, respectively. By Corollary 5.3.3.15, there is a continuous path ζ in $\mathcal{F}(E,F)$ connecting u_0 and v_0. Then $\eta^{-1}\zeta\xi$ is a continuous path in $\mathcal{F}(E,F)$ connecting u and v. By Theorem 3.1.3.9, $\eta^{-1}\zeta\xi$ is a continuous path in $\{w \in \mathcal{F}(E,F) | \operatorname{Ind} w = 0\}$.

Case 2 $n \neq 0$

Take $u, v \in \mathcal{F}(E,F)$ with

$$\operatorname{Ind} u = \operatorname{Ind} v .$$

By Proposition 5.3.2.6,

$$u^* \in \mathcal{F}(F,E), \quad \operatorname{Ind} u^* = -\operatorname{Ind} u .$$

Hence $u^* \circ v \in \mathcal{F}(E), \quad u \circ u^* \in \mathcal{F}(F),$

$$\operatorname{Ind}(u^* \circ v) = \operatorname{Ind}(u \circ u^*) = 0$$

(Proposition 3.1.3.7). By Case 1, there are continuous maps

$$f : [0,1] \longrightarrow \mathcal{F}(E), \quad g : [1,2] \longrightarrow \mathcal{F}(F)$$

with

$$f(0) = 1_E, \quad f(1) = u^* \circ v,$$

$$g(1) = u \circ u^*, \quad g(2) = 1_F .$$

Then

$$[0,2] \longrightarrow \mathcal{F}(E,F), \quad \alpha \longmapsto \begin{cases} u \circ f(\alpha) & \text{if } \alpha \in [0,1] \\ g(\alpha) \circ v & \text{if } \alpha \in [1,2] \end{cases}$$

is a continuous path in $\mathcal{F}(E,F)$ (Proposition 3.1.3.7) connecting u and v. By Theorem 3.1.3.9, this path lies in $\{w \in \mathcal{F}(E,F) | \operatorname{Ind} w = n\}$. ■

Corollary 5.3.3.17 *Every positive operator on a Hilbert space E is the supremum of an upward directed set in $\mathcal{K}(E)_+$.*

Let F be a closed vector subspace of E and $\mathfrak{G}$ the set of finite-dimensional vector subspaces of F. Then $\{\pi_G | G \in \mathfrak{G}\}$ is an upward directed set of $\mathcal{K}(E)_+$

(Corollary 5.3.3.8) and by Proposition 5.2.1.4 and Theorem 5.3.3.14 c), π_F is its supremum.

Now take $u \in \mathcal{L}(E)_+$. By Corollary 4.3.2.11 (and Theorem 5.3.3.14), there is a sequence $(p_n)_{n\in\mathbb{N}}$ of orthogonal projections in E such that

$$\left(\sum_{k=1}^{n} \frac{1}{2^k} p_k\right)_{n\in\mathbb{N}}$$

is increasing and

$$u = \bigvee_{n\in\mathbb{N}} \left(\sum_{k=1}^{n} \frac{1}{2^k} p_k\right) .$$

Take $n \in \mathbb{N}$. By the above considerations, there is an upward directed set $\mathcal{F}_n$ of $\mathcal{K}(E)_+$ with supremum p_n. Put

$$\mathcal{F} := \left\{ \sum_{k=1}^{n} \frac{1}{2^k} u_k \;\middle|\; n \in \mathbb{N},\ (u_k)_{k\in\mathbb{N}_n} \in \prod_{k\in\mathbb{N}_n} \mathcal{F}_k \right\} .$$

Thus $\mathcal{F}$ is an upward directed set of $\mathcal{K}(E)_+$ and it is easy to see that u is its supremum in $\mathcal{L}(E)$. ■

Proposition 5.3.3.18 *Let E be a Hilbert space and take $u, v \in \mathcal{L}(E)$. If*

$$uu^* \leq vv^* ,$$

then there is a $w \in \mathcal{L}(E)$ such that

$$u = vw , \quad \|w\| \leq 1 .$$

If $x \in E$, then

$$\|u^*x\|^2 = \langle u^*x | u^*x\rangle = \langle uu^*x | x\rangle \leq \langle vv^*x | x\rangle = \|v^*x\|^2$$

(Corollary 5.3.3.7). Define

$$w_0 : \operatorname{Im} v^* \longrightarrow E , \quad v^*x \longmapsto u^*x .$$

By the above, w_0 is well-defined. Then $w_0 \in \mathcal{L}(\operatorname{Im} v^*, E)$ and

$$\|w_0\| \leq 1 .$$

By Proposition 5.2.3.18, there is a $w_1 \in \mathcal{L}(E)$ such that

$$w_1 | \operatorname{Im} v^* = w_0 , \quad \|w_1\| = \|w_0\| .$$

Put

$$w := w_1^* .$$

Then

$$\|w\| = \|w_1^*\| = \|w_1\| = \|w_0\| \leq 1 .$$

Take $x, y \in E$. Then

$$\langle ux|y\rangle = \langle x|u^*y\rangle = \langle x|w_0v^*y\rangle = \langle x|w_1v^*y\rangle = \langle x|w^*v^*y\rangle = \langle vwx|y\rangle .$$

Thus

$$ux = vwx ,$$

so that

$$u = vw . \qquad \blacksquare$$

Proposition 5.3.3.19 *Let E be a real Hilbert space. Take $u, v \in \mathcal{L}(E)$ and define*

$$\widetilde{(u,v)} : \overset{\circ}{E} \longrightarrow \overset{\circ}{E} , \quad (x,y) \longmapsto (ux - vy, uy + vx) .$$

Then $\widetilde{(u,v)}$ is positive iff it is selfadjoint and

$$\langle ux|x\rangle + \langle uy|y\rangle + 2\langle vx|y\rangle \in \mathbb{R}_+$$

for all $(x,y) \in \overset{\circ}{E}$.

Assume $\widetilde{(u,v)}$ is selfadjoint. By Proposition 2.3.1.39,

$$u = u^* , \quad v = -v^* .$$

Then

$$\langle \widetilde{(u,v)}(x,y) \,|\, (x,y)\rangle = \langle (ux - vy, uy + vx) \,|\, (x,y)\rangle =$$

$$= \langle ux - vy|x\rangle + \langle uy + vx|y\rangle - i\langle ux - vy|y\rangle + i\langle uy + vx|x\rangle =$$

$$= \langle ux|x\rangle + \langle uy|y\rangle + 2\langle vx|y\rangle$$

for all $(x,y) \in \overset{\circ}{E}$ (Corollary 5.3.3.7) and the assertion follows from Corollary 5.3.3.7. $\blacksquare$

Definition 5.3.3.20 (0) (F. Riesz, 1918) *Let E be a Banach space. Take $u \in \mathcal{L}(E)$ and $\alpha \in \mathbb{K}$. **The algebraic eigenspace of u corresponding to α** is defined to be*

$$\bigcup_{n \in \mathbb{N}} \operatorname{Ker} (\alpha 1 - u)^n .$$

*Its dimension is **the algebraic multiplicity of α with respect to u**, and its elements are **the algebraic eigenvectors of u corresponding to α**.*

Since the eigenspace is contained in the algebraic eigenspace, the algebraic multiplicity is at least the multiplicity. But if the multiplicity of α is 0, then the algebraic multiplicity of α is also 0. If u is compact and $\alpha \neq 0$, then $E_a\left(1 - \frac{1}{\alpha} u\right)$ (Definition 3.1.3.18) is the algebraic eigenspace of u corresponding to α and the algebraic multiplicity of α with respect to u is finite (Theorem 3.1.3.17 d)).

The algebraic eigenvector is also called "root vector", "null-element", "generalized eigenvector", "adjoint vector".

Proposition 5.3.3.21 (0) *The eigenspaces and the algebraic eigenspaces of normal operators on a Hilbert space coincide.*

Let u be a normal operator on a Hilbert space and take $\alpha \in \mathbb{K}$. Then $\alpha 1 - u$ is normal, so that

$$\bigcup_{n \in \mathbb{N}} \operatorname{Ker} (\alpha 1 - u)^n = \operatorname{Ker} (\alpha 1 - u) ,$$

by Corollary 5.3.3.11. ■

5.3.4 Normal Operators

Proposition 5.3.4.1 (0) *Let u be an operator on the Hilbert space E. u is normal iff*

$$\|ux\| = \|u^*x\|$$

for every $x \in E$. In this case

$$\operatorname{Ker} u = \operatorname{Ker} u^*$$

and

$$E = \overline{\operatorname{Im} u} \oplus \operatorname{Ker} u\,.$$

In particular, any normal Fredholm operator has index 0.

Given $x \in E$,

$$\|ux\|^2 - \|u^*x\|^2 = \langle ux|ux\rangle - \langle u^*x|u^*x\rangle =$$

$$= \langle u^*ux|x\rangle - \langle uu^*x|x\rangle = \langle (u^*u - uu^*)x|x\rangle\,.$$

Since $u^*u - uu^*$ is selfadjoint, it follows from Corollary 5.3.3.4 that the following are equivalent:

$$u \text{ is normal}\,,$$

$$u^*u - uu^* = 0\,,$$

$$x \in E \Longrightarrow \langle (u^*u - uu^*)x|x\rangle = 0\,,$$

$$x \in E \Longrightarrow \|ux\| = \|u^*x\|\,.$$

It follows immediately that

$$\operatorname{Ker} u = \operatorname{Ker} u^*\,.$$

By Proposition 5.3.2.4,

$$\operatorname{Ker} u^* = (\operatorname{Im} u)^\perp\,,$$

so that

$$E = (\operatorname{Im} u)^{\perp} \oplus (\operatorname{Im} u)^{\perp\perp} = \operatorname{Ker} u \oplus \overline{\operatorname{Im} u}$$

(Proposition 5.2.3.7, Corollary 5.2.3.9).

If u is a normal Fredholm operator, then by Proposition 5.3.2.6,

$$\operatorname{Ind} u = \operatorname{Dim} \operatorname{Ker} u - \operatorname{Dim} \operatorname{Coker} u =$$

$$= \operatorname{Dim} \operatorname{Ker} u - \operatorname{Dim} \operatorname{Ker} u^{*} = 0 . \qquad \blacksquare$$

Corollary 5.3.4.2 (0) *Let $E \neq \{0\}$ be a Hilbert space and u a normal operator on E. Put*

$$\alpha := \inf \{ \|ux\| \,|\, x \in E , \|x\| = 1 \} .$$

u is an isomorphism of Banach spaces iff $\alpha > 0$ and in this case

$$\|u^{-1}\| = \frac{1}{\alpha} , \quad \sigma(u) \subset \{ \beta \in \mathbb{K} \,|\, |\beta| \geq \alpha \} .$$

If u is an isomorphism of Banach spaces, then

$$\|x\| = \|u^{-1}ux\| \leq \|u^{-1}\| \, \|ux\|$$

for every $x \in E$. Hence

$$\alpha \geq \frac{1}{\|u^{-1}\|} > 0 .$$

Conversely, if $\alpha > 0$, then

$$\|u^{*}x\| = \|ux\| \geq \alpha \|x\|$$

for every $x \in E$ (Proposition 5.3.4.1). Thus u is an isomorphism of Banach spaces and

$$\|u^{-1}\| \leq \frac{1}{\alpha}$$

(Corollary 5.3.2.5). By the above

$$\|u^{-1}\| = \frac{1}{\alpha} .$$

Take $\beta \in U_{\alpha}^{\mathbb{K}}(0)$. Then

$$\|u - (u - \beta 1)\| = |\beta| < |\alpha| = \frac{1}{\|u^{-1}\|} .$$

By Corollary 2.2.4.4, $u - \beta 1$ is invertbile, so that $\beta \notin \sigma(u)$. $\blacksquare$

Corollary 5.3.4.3 (0) *Let u be a normal operator on the Hilbert space $E \neq \{0\}$. Take $\alpha \in \mathbb{K}$. Then the following are equivalent:*

a) $\alpha \in \sigma(u)$.

b) $\inf\{\|\alpha x - ux\| \mid x \in E, \|x\| = 1\} = 0$.

$\alpha 1 - u$ is a normal operator on E and the assertion follows from Corollary 5.3.4.2. ■

Corollary 5.3.4.4 (0) *Let u be a normal operator on the Hilbert space $E \neq \{0\}$. Take $\alpha \in \mathbb{K}$ and let*

$$F = \operatorname{Ker}(\alpha 1 - u).$$

a) $F = \operatorname{Ker}(\overline{\alpha} 1 - u^*)$.

b) *F reduces u.*

c) *If u is compact and $\alpha \neq 0$, then*

$$E = F \oplus \operatorname{Im}(\alpha 1 - u).$$

a) $\alpha 1 - u$ is normal, so that the assertion follows from Proposition 5.3.4.1.

b) Take $x \in F$. By a),

$$ux = \alpha x \in F, \quad u^* x = \overline{\alpha} x \in F.$$

Hence

$$u(F) \subset F, \quad u^*(F) \subset F.$$

By Proposition 5.3.2.9, F reduces u.

c) $\alpha 1 - u$ is a Freldholm operator (Corollary 3.1.3.12) and so $\operatorname{Im}(\alpha 1 - u)$ is closed. The assertion now follows from Proposition 5.3.4.1. ■

Corollary 5.3.4.5 (0) *If u is a normal operator on the Hilbert space $E \neq \{0\}$, then*

$$\sigma_p(u^*) = \{\overline{\alpha} | \alpha \in \sigma_p(u)\}$$

and

$$\operatorname{Ker}(\alpha 1 - u) \perp \operatorname{Ker}(\beta 1 - u)$$

for distinct $\alpha, \beta \in \mathbb{K}$.

The first assertion follows from Corollary 5.3.4.4 a). Take $x \in \mathrm{Ker}\,(\alpha 1 - u)$, $y \in \mathrm{Ker}\,(\beta 1 - u)$. Then

$$\alpha\langle x|y\rangle = \langle \alpha x|y\rangle = \langle ux|y\rangle = \langle x|u^*y\rangle = \langle x|\overline{\beta}y\rangle = \beta\langle x|y\rangle\,,$$

(Corollary 5.3.4.4 a)), so that

$$\langle x|y\rangle = 0$$

and

$$\mathrm{Ker}\,(\alpha 1 - u) \perp \mathrm{Ker}\,(\beta 1 - u) \qquad \blacksquare$$

Proposition 5.3.4.6 *Let u be a normal operator on the Hilbert space $E \neq \{0\}$. u is unitary iff*

$$\|ux\| = \|x\|$$

for every $x \in E$.

By Corollary 5.3.1.22, u is unitary iff it is an isometry, and by Corollary 5.3.4.2, this is equivalent to

$$x \in E \Longrightarrow \|ux\| = \|x\|\,. \qquad \blacksquare$$

Proposition 5.3.4.7 (0) *Let u be a self-normal operator on the Hilbert space $E \neq \{0\}$, such that $\sigma(u)\backslash\{0\}$ is discrete. Given $\alpha \in \mathbb{K}$, let π_α be the orthogonal projection of E onto $\mathrm{Ker}\,(\alpha 1 - u)$.*

a) $u = \sum\limits_{\alpha \in \sigma(u)} \alpha \pi_\alpha$ ***(the spectral decomposition of*** u ***)***.

b) $\alpha, \beta \in \mathbb{K} \Rightarrow \pi_\alpha \pi_\beta = \delta_{\alpha\beta}\pi_\alpha$.

c) $\sigma(u)$ *finite* $\Rightarrow \sum\limits_{\alpha \in \sigma(u)} \pi_\alpha = 1_E$.

d) *If*

$$f_\alpha : \sigma(u) \longrightarrow \mathbb{K}\,, \quad \beta \longmapsto \delta_{\alpha\beta}\,,$$

then

$$f_\alpha(u) = \pi_\alpha$$

for every $\alpha \in \mathbb{K}\backslash\{0\}$. In particular,

$$\sigma(u)\backslash\{0\} = \{\alpha \in \mathbb{K}\backslash\{0\}\,|\,\pi_\alpha \neq 0\}\,.$$

e) u *and* $\{\pi_\alpha \,|\, \alpha \in \sigma(u)\backslash\{0\}\}$ *generate the same closed unital subalgebra of* E. *It is finite-dimensional iff* $\sigma(x)$ *is finite.*

f) *Given* $f \in \mathcal{C}(\sigma(x))$, *with* $f(0) = 0$ *whenever* $\sigma(x)$ *is infinite,*

$$f(u) = \sum_{\alpha \in \sigma(u)} f(\alpha)\pi_\alpha .$$

g) $\{\pi_\alpha | \alpha \in \mathbb{K}\} \subset \{u\}^{cc}$.

a) - f). Let A denote the set of isolated points of $\sigma(x)$. By Theorem 4.1.4.6 a $\Rightarrow$ b & c, and Theorem 5.3.1.13 (and Proposition 5.3.2.8 b $\Rightarrow$ a), there is a family $(u_\alpha)_{\alpha \in A}$ of orthogonal projections in E such that

$$u = \sum_{\alpha \in A} \alpha u_\alpha ,$$

$$\alpha, \beta \in A \Longrightarrow u_\alpha u_\beta = \delta_{\alpha\beta} u_\alpha ,$$

$$\sigma(u) \text{ finite } \Longrightarrow \sum_{\alpha \in A} u_\alpha = 1 ,$$

$$\alpha \in \sigma(x)\backslash\{0\} \Longrightarrow u_\alpha \neq 0 .$$

Take $\alpha \in A$ and $x \in \operatorname{Im} u_\alpha$. We get, successively, that

$$u_\alpha x = x ,$$

$$\beta \in A\backslash\{\alpha\} \Longrightarrow u_\beta x = u_\beta u_\alpha x = 0 ,$$

$$ux = \sum_{\beta \in A} \beta u_\beta x = \alpha u_\alpha x = \alpha x ,$$

$$x \in \operatorname{Ker}(\alpha 1 - u) ,$$

$$\operatorname{Im} u_\alpha \subset \operatorname{Ker}(\alpha 1 - u) ,$$

$$u_\alpha = u_\alpha \pi_\alpha$$

(Proposition 5.2.3.13 a $\Rightarrow$ c)).

Let α, β be distinct elements of A. Then, by Corollary 5.3.4.5,

$$\mathrm{Ker}\,(\alpha 1 - u) \perp \mathrm{Ker}\,(\beta 1 - u)\,.$$

Thus, by Corollary 5.2.3.15 a $\Rightarrow$ b,

$$\pi_\alpha \pi_\beta = 0\,.$$

We deduce that

$$u_\alpha \pi_\beta = u_\alpha \pi_\alpha \pi_\beta = 0\,.$$

Take $\alpha \in A\backslash\{0\}$ and $x \in \mathrm{Ker}\,(\alpha 1 - u)$. We get, successively, that

$$\pi_\alpha x = x\,,$$

$$\alpha x = ux = \sum_{\beta \in A\backslash\{0\}} \beta u_\beta x = \sum_{\beta \in A\backslash\{0\}} \beta u_\beta \pi_\alpha x = \alpha u_\alpha x\,,$$

$$x = u_\alpha x \in \mathrm{Im}\, u_\alpha\,,$$

$$\mathrm{Ker}\,(\alpha 1 - u) \subset \mathrm{Im}\, u_\alpha\,,$$

$$\mathrm{Ker}\,(\alpha 1 - u) = \mathrm{Im}\, u_\alpha\,,$$

$$\pi_\alpha = u_\alpha\,.$$

All the assertions follow now from Theorem 4.1.4.6.

g) $\{u\}^{cc}$ is a unital C^*-subalgebra of $\mathcal{L}(E)$ (Corollary 4.1.4.2 b)), so that

$$\{\pi_\alpha \,|\, \alpha \in \mathbb{K}\backslash\{0\}\} \subset \{u\}^{cc}$$

by e). It remains to prove $\pi_0 \in \{u\}^{cc}$. Take $v \in \{u\}^c$ and $x \in E$ and put

$$A := \mathbb{K}\backslash\{0\}\,.$$

By b), for every $B \in \mathfrak{P}_f(A)$, $\sum\limits_{\alpha \in B} \pi_\alpha$ is an orthogonal projection so that

$$\left(1 - \sum_{\alpha \in B} \pi_\alpha\right) \sum_{\alpha \in B} \pi_\alpha = 0$$

$$\|x\|^2 = \sum_{x\in B} \|\pi_\alpha x\|^2 + \left\| x - \sum_{\alpha\in B} \pi_\alpha x \right\|^2 \geq \sum_{\alpha\in B} \|\pi_\alpha x\|^2$$

(Pythagoras' Theorem). Hence $(\pi_\alpha x)_{\alpha\in A}$ is summable in E (Corollary 5.2.2.4). By a) and b), we obtain successively,

$$u\left(x - \sum_{\alpha\in A} \pi_\alpha x\right) = ux - \sum_{\alpha\in A} u\pi_\alpha x = ux - \sum_{\alpha\in A} \alpha\pi_\alpha x = 0\,,$$

$$x - \sum_{\alpha\in A} \pi_\alpha x \in \operatorname{Ker} u\,,$$

$$x - \sum_{\alpha\in A} \pi_\alpha x = \pi_0\left(x - \sum_{\alpha\in A} \pi_\alpha x\right) = \pi_0 x - \sum_{\alpha\in A} \pi_0\pi_\alpha x = \pi_0 x\,,$$

$$v\pi_0 x = vx - \sum_{\alpha\in A} v\pi_\alpha x = vx - \sum_{\alpha\in A} \pi_\alpha vx = \pi_0 vx\,,$$

$$v\pi_0 = \pi_0 v\,,$$

$$\pi_0 \in \{u\}^{cc}\,. \qquad \blacksquare$$

Remark. π_0 does not belong in general to the unital C^*-subalgebra of $\mathcal{L}(E)$ generated by u. Indeed, let

$$u : \ell^2 \longrightarrow \ell^2\,, \quad x \longmapsto \left(x_1, 0, \frac{1}{3}x_3, 0, \frac{1}{5}x_5, \ldots\right)$$

and put

$$p_n : \ell^2 \longrightarrow \ell^2\,, \quad (x_k)_{k\in\mathbb{N}} \longmapsto (\delta_{nk}x_k)_{k\in\mathbb{N}}$$

for every $n \in \mathbb{N}$. Then

$$\operatorname{Ker} u = \{x \in \ell^2 \,|\, n \in \mathbb{N} \Longrightarrow x_{2n} = 0\}\,,$$

$$\left\{\alpha 1 + \sum_{n\in\mathbb{N}} \alpha_n p_{2n-1} \middle| \alpha \in \mathbb{K}, (\alpha_n)_{n\in\mathbb{N}} \in c_0\right\}$$

is the involutive unital closed subalgebra of $\mathcal{L}(E)$ generated by u, and π_0 does not belong to it.

5.4 Representations

A representation of an involutive Banach algebra E consists of a homomorphism of involutive algebras from E into the C^*-algebra of operators on a Hilbert space. We show that in the case of a C^*-algebra even faithful — that is to say norm-preserving — representations can be found. We shall return to the problem of representations later in the book.

5.4.1 Cyclic Representation

Definition 5.4.1.1 (0) *Let E be a (unital) involutive Banach algebra. A **(unital) representation of** E is a pair (H,φ), where H is a Hilbert space and φ is a (unital) involutive algebra homomorphism $E \to \mathcal{L}(H)$. (H,φ) is called **irreducible** if $\varphi(E)$ acts irreducibly on H. (H,φ) is called **non-degenerate** if $\varphi(E)$ acts non-degenerately on H. (H,φ) is called **faithful** (**the** 0-**representation**) if φ is injective (if $\varphi(E) = \{0\}$)). An element of H is called **cyclic for** (H,φ) if it is cyclic for $\varphi(E)$. An element of H is called **separating for** (H,φ) if it is separating for $\varphi(E)$. Two representations (H,φ) and (K,ψ) of E are called **equivalent** if there is an isometry $u : H \to K$ such that*

$$\varphi x = u^* \circ (\psi x) \circ u$$

*for every $x \in E$ (it is an equivalence relation). In this case u is called an **equivalence** of the two representations (H,φ) and (K,ψ).*

If E is a C^*-algebra, then a representation (H,φ) of E is faithful iff it preserves the norms (Theorem 4.2.6.6). If a C^*-algebra is simple then any representation of it is either faithful or the 0-representation.

Theorem 5.4.1.2 (0) (Gelfand-Naimark, 1943, Segal, 1947) ***The GNS-construction)*** *Let $x' \neq 0$ be a continuous positive linear form on an involutive (quasiunital) normed algebra E and f the positive sesquilinear form*

$$E \times E \longrightarrow \mathbb{K}\,, \quad (x,y) \longmapsto x'(y^*x)$$

(Proposition 2.3.4.6 b)). Put

$$F := \{x \in E \,|\, \widetilde{f}(x) = 0\} = N_{x'}\,.$$

Let $u : E \to E/F$ be the quotient map (Proposition 5.1.2.9 b)),

$$E/F \times E/F \longrightarrow \mathbb{K}\,, \quad (X,Y) \longmapsto \langle X|Y\rangle$$

the associated scalar product (Proposition 5.1.2.9 d)), and $\widetilde{E/F}$ *be completion of* E/F *(Corollary 5.1.1.7).* $\widetilde{E/F}$ *is called* ***the Hilbert space associated to*** x'.

a) $x \in E \Rightarrow \widetilde{f}(x) = \|ux\|^2 = x'(x^*x)$.

b) $u \in \mathcal{L}(E, E/F)\,,\ \|u\| \leq \|x'\|^{\frac{1}{2}} \quad \left(\|u\| = \|x'\|^{\frac{1}{2}}\right)$.

c) *E separable* $\Rightarrow$ *$\widetilde{E/F}$ separable.*

d) $x, y \in E \Rightarrow \widetilde{f}(xy) \leq \widetilde{f}(y)\|x^*x\|$.

e) $(x,y) \in E \times F \Rightarrow xy \in F$.

f) $x \in E\,,\ Y \in E/F\,,\ y_1, y_2 \in Y \Rightarrow u(xy_1) = u(xy_2)$.

Put

$$x \top Y := u(xy_1)\,.$$

g) *For every $x \in E$, there is a unique operator $\widetilde{x}$ on $\widetilde{E/F}$ such that for every $Y \in E/F$,*

$$\widetilde{x}Y = x \top Y\,.$$

Furthermore,

$$\|\widetilde{x}\| \leq \|x^*x\|^{\frac{1}{2}} \leq \|x\| \ \ (\text{and } x'(x^*x) \leq \|\widetilde{x}\|^2\|x'\|)\,.$$

h) *The map*

$$\varphi : E \longrightarrow \mathcal{L}(\widetilde{E/F})\,, \quad x \longmapsto \widetilde{x}$$

is a continuous homomorphism of involutive algebras with norm at most 1 *(equal to* 1*). φ is called* ***the algebra homomorphism associated to*** *x' and $(\widetilde{E/F}, \varphi)$ is called* ***the representation of*** *E* ***associated to*** *x'.*

i) *If E is unital, then the algebra homomorphism associated to x' is also unital.*

j) *If x' is an algebra homomorphism, then*

$$F = \operatorname{Ker} x'$$

and E/F is one-dimensional.

k) *If E is quasiunital, then there is a unique $\xi \in \widetilde{E/F}$ such that*

$$x'(x) = \langle ux|\xi\rangle$$

for all $x \in E$. ξ is cyclic for $(\widetilde{E/F}, \varphi)$,

$$\|\xi\| = \|x'\|^{\frac{1}{2}}$$

and

$$\tilde{x}\xi = ux\,, \quad x'(x) = \langle \tilde{x}\xi|\xi\rangle$$

for all $x \in E$. If $\mathfrak{F}$ is an approximate unit of E, then $u(\mathfrak{F})$ converges weakly to ξ. If E is unital then $\xi = u1$. ξ is called ***the cyclic vector associated to*** *x' and the triple $(\widetilde{E/F}, \varphi, \xi)$ is called* ***the GNS-triple of E associated to*** *x'. Two GNS-triples (H_1, φ_1, ξ_1), (H_2, φ_2, ξ_2) of E are called* ***equivalent*** *if there is an equivalence v of the representations (H_1, φ_1) and (H_2, φ_2) such that $v\xi_1 = \xi_2$. In this case v is called an* ***equivalence*** *of the two GNS-triples (H_1, φ_1, ξ_1), (H_2, φ_2, ξ_2) of E.*

a) We have

$$\tilde{f}(x) = f(x,x) = \langle ux|ux\rangle = \|ux\|^2\,,$$

$$\tilde{f}(x) = f(x,x) = x'(x^*x)\,.$$

b) If $x \in E$, then, by a),

$$\|ux\|^2 = x'(x^*x) \leq \|x'\|\,\|x^*x\| \leq$$

$$\leq \|x'\|\,\|x^*\|\,\|x\| = \|x'\|\,\|x\|^2\,,$$

so that $u \in \mathcal{L}(E, E/F)$ and $\|u\| \leq \|x'\|^{\frac{1}{2}}$. If $\mathfrak{F}$ is an approximate unit of E, then, by a),

$$\|u\|^2 \geq \lim_{x,\mathfrak{F}} \|ux\|^2 = \lim_{x,\mathfrak{F}} x'(x^*x) = \|x'\|$$

(Proposition 2.3.4.10 a)), so

$$\|u'\| = \|x'\|^{\frac{1}{2}}$$

if E is quasiunital.

c) follows from the fact that E/F is separable and dense in $\widetilde{E/F}$.

d) By a) and Corollary 2.3.4.8,

$$\widetilde{f}(xy) = x'(y^*x^*xy) \leq x'(y^*y)\|x^*x\| = \widetilde{f}(y)\|x^*x\| .$$

e) By d),

$$|\widetilde{f}(xy)| \leq \widetilde{f}(y)\|x^*x\| = 0 ,$$

so $xy \in F$.

f) By e),

$$xy_1 - xy_2 = x(y_1 - y_2) \in F .$$

g) Uniqueness is obvious. Define

$$v : E/F \longrightarrow E/F , \quad Y \longmapsto x \top Y .$$

v is linear. If $Y \in E/F$ and $y \in Y$, then, by a) and d),

$$\|vY\|^2 = \|x \top Y\|^2 = \|u(xy)\|^2 = \widetilde{f}(xy) \leq \widetilde{f}(y)\|x^*x\| = \|x^*x\| \, \|Y\|^2 .$$

Hence v is continuous and

$$\|v\| \leq \|x^*x\|^{\frac{1}{2}} .$$

Let $\widetilde{x}$ be the continuous extension of v. Then

$$\|\widetilde{x}\| = \|v\| \leq \|x^*x\|^{\frac{1}{2}} \leq \|x\| .$$

If $y \in E^{\#}$, then, by b),

$$\|\widetilde{x}\|^2\|x'\| \geq \|\widetilde{x}\|^2\|u\|^2 \geq \|\widetilde{x}\|^2\|uy\|^2 \geq \|\widetilde{x}(uy)\|^2 =$$

$$= \|x \top uy\|^2 = \|u(xy)\|^2 .$$

Hence, if $\mathfrak{F}$ is an approximate unit of E, then, by a),

$$x'(x^*x) = \|ux\|^2 = \lim_{y,\mathfrak{F}} \|u(xy)\|^2 \leq \|\widetilde{x}\|^2\|x'\| .$$

h) It is easy to see that φ is linear. Take $x, y \in E$, $A, B \in E/F$, and $a \in A$, $b \in B$. Then

$$\widetilde{xy}A = (xy) \top A = u(xya) = \widetilde{x}(y \top A) = \widetilde{x}\widetilde{y}A ,$$

$$\langle A|\widetilde{x^*}B\rangle = \langle A|x^*\top B\rangle = \langle A|u(x^*b)\rangle = x'(b^*xa) =$$

$$= \overline{x'(a^*x^*b)} = \overline{\langle B|u(xa)\rangle} = \langle u(xa)|B\rangle = \langle \widetilde{x}A|B\rangle .$$

Hence

$$\widetilde{xy} = \widetilde{x}\widetilde{y}, \quad \widetilde{x^*} = \widetilde{x}^*,$$

which shows that φ is a homomorphism of involutive algebras. By g), φ is continuous with norm at most 1. If $\mathfrak{F}$ is an approximate unit of E, then, by a),g), and Proposition 2.3.4.10 a),

$$\|x'\| = \lim_{x,\mathfrak{F}} x'(x^*x) \leq \|x'\| \lim_{x,\mathfrak{F}} \|\widetilde{x}\|^2 .$$

Hence, if E is quasiunital, φ has norm 1.

i) Take $X \in E/F$ and $x \in X$. Then

$$\widetilde{1}_E X = u(1_E x) = ux = X ,$$

so

$$\widetilde{1}_E = 1_{\widetilde{E/F}} .$$

j) We have

$$\widetilde{f}(x) = x'(x^*x) = x'(x^*)x'(x) = \overline{x'(x)}x'(x) = |x'(x)|^2 .$$

Thus

$$F = \operatorname{Ker} x'$$

and E/F is one-dimensional.

k) Take $x, y \in E$ and let $\mathfrak{F}$ be an approximate unit of E. Then

$$|x'(y^*x)|^2 \leq x'(y^*y)x'(x^*x) = x'(y^*y)\|ux\|^2$$

(a) and Proposition 2.3.4.6 c)). Thus

$$|x'(x)|^2 = \lim_{y,\mathfrak{F}} |x'(y^*x)|^2 \leq \|x'\| \, \|ux\|^2$$

(Proposition 2.3.4.9, Proposition 2.3.4.10 a)). Let y' be the factorization of x' through E/F. Then

$$|y'(ux)|^2 = |x'(x)|^2 \leq \|x'\| \, \|ux\|^2$$

for every $x \in E$, so that y' is continuous and

$$\|y'\| \leq \|x'\|^{\frac{1}{2}}.$$

We extend y' continuously to $\widetilde{E/F}$. By b),

$$x'(x^*x) = y'(u(x^*x)) \leq \|y'\|\,\|u(x^*x)\| \leq \|y'\|\,\|u\|\,\|x^*x\| = \|y'\|\,\|x'\|^{\frac{1}{2}}\|x^*x\|$$

for every $x \in E$. Hence

$$\|x'\| = \lim_{x,\mathfrak{F}} x'(x^*x) \leq \|y'\|\,\|x'\|^{\frac{1}{2}}$$

(Proposition 2.3.4.10 a)),

$$\|y'\| = \|x'\|^{\frac{1}{2}}.$$

By Corollary 5.2.5.3, there is a $\xi \in \widetilde{E/F}$, such that

$$y' = \langle \cdot | \xi \rangle, \quad \|\xi\| = \|y'\| = \|x'\|^{\frac{1}{2}}.$$

Then

$$x'(x) = y'(ux) = \langle ux|\xi\rangle$$

for all $x \in E$. The uniqueness of ξ is obvious.

Take $x, y \in E$. By f),

$$\langle uy|ux\rangle = x'(x^*y) = \langle u(x^*y)|\xi\rangle = \langle \widetilde{x^*}uy|\xi\rangle = \langle \widetilde{x}^*uy|\xi\rangle = \langle uy|\widetilde{x}\xi\rangle.$$

Since y is arbitrary,

$$\widetilde{x}\xi = ux.$$

Hence

$$\{\widetilde{x}\xi \,|\, x \in E\} = E/F$$

i.e., ξ is cyclic for $(\widetilde{E/F}, \varphi)$.

For every $x \in E$,

$$\langle ux|\xi\rangle = x'(x) = \lim_{y,\mathfrak{F}} x'(y^*x) = \lim_{y,\mathfrak{F}} \langle ux|uy\rangle.$$

Hence $u(\mathfrak{F})$ converges weakly to ξ. If E is unital, then by i),

$$u1 = \widetilde{1}\xi = \xi. \qquad \blacksquare$$

Proposition 5.4.1.3 *Let E be an involutive Banach algebra, (H,φ) a representation of E, and $\xi \in H$.*

a) *The map*

$$x' : E \longrightarrow \mathbb{K}, \quad x \longmapsto \langle(\varphi x)\xi|\xi\rangle$$

belongs to E'_+ and

$$x'(x^*x) = \|(\varphi x)\xi\|^2$$

for every $x \in E$. If $\mathfrak{F}$ is an approximate unit of E and the representation is non-degenerate then

$$\lim_{x,\mathfrak{F}}(\varphi x)\eta = \eta$$

for every $\eta \in H$ and $\|x'\| = \|\xi\|^2$.

b) *Assume E quasiunital and ξ cyclic for (H,φ) and let (K,ψ,η) be the GNS-triple of E associated to the above x'. Then there is a unique equivalence u of the representations (H,φ) and (K,ψ) such that*

$$u\xi = \eta.$$

In particular, (H,φ,ξ) may be identified with (K,ψ,η).

a) For every $x \in E$,

$$x'(x^*) = \langle((\varphi x^*)\xi|\xi\rangle = \langle((\varphi x)^*\xi|\xi\rangle =$$

$$= \langle\xi|(\varphi x)\xi\rangle = \overline{\langle(\varphi x)\xi|\xi\rangle} = \overline{x'(x)},$$

$$x'(x^*x) = \langle\varphi(x^*x)\xi|\xi\rangle = \langle((\varphi x)^*(\varphi x)\xi|\xi\rangle =$$

$$= \langle(\varphi x)\xi|(\varphi x)\xi\rangle = \|(\varphi x)\xi\|^2 \in \mathbb{R}_+.$$

Since φ is continuous, x' is also continuous.

For $y \in E$ and $\eta \in H$,

$$(\varphi y)\eta = \lim_{x,\mathfrak{F}}(\varphi(xy))\eta = \lim_{x,\mathfrak{F}}(\varphi x)(\varphi y)\eta.$$

Since the representation is non-degenerate,

$$\lim_{x,\mathfrak{F}}(\varphi x)\eta = \eta\,.$$

It follows

$$\|x'\| = \lim_{x,\mathfrak{F}} x'(x^*x) = \lim_{x,\mathfrak{F}} \|(\varphi x)\xi\|^2 = \|\xi\|^2$$

(Proposition 2.3.4.10 a)).

b) By a) and Theorem 5.4.1.2 k),

$$\|(\varphi x)\xi\|^2 = x'(x^*x) = \|(\psi x)\eta\|^2$$

for every $x \in E$. Hence the map

$$\{(\varphi x)\xi \mid x \in E\} \longrightarrow \{(\psi x)\eta \mid x \in E\}\,, \quad (\varphi x)\xi \longmapsto (\psi x)\eta$$

is well-defined and it is an isometry. Since ξ and η are cyclic for the representations (H, φ) and (K, ψ), respectively, the above map may be extended to an isometry $u : H \to K$. Take $x, y \in E$. Then

$$u(\varphi x)(\varphi y)\xi = u\varphi(xy)\xi = \psi(xy)\eta = (\psi x)(\psi y)\eta = (\psi x)u(\varphi y)\xi\,.$$

Since ξ is cyclic for (H, φ), it follows

$$u \circ (\varphi x) = (\psi x) \circ u\,.$$

Hence

$$(\psi x)u\xi = u(\varphi x)\xi = (\psi x)\eta\,,$$

$$(\psi x)(u\xi - \eta) = 0\,.$$

Since $\psi(E)$ acts non-degenerately on K, it follows

$$u\xi - \eta = 0\,, \quad u\xi = \eta$$

(Proposition 5.3.2.21 a $\Rightarrow$ b).

Let v be an equivalence of the representation (H, φ) and (K, ψ) with

$$v\xi = \eta\,.$$

For every $x \in E$,

$$u(\varphi x)\xi = (\psi x)u\xi = (\psi x)v\xi = v(\varphi x)\xi\,.$$

Since ξ is cyclic for (H, φ), $u = v$. ■

Corollary 5.4.1.4 *Let E be an involutive quasiunital Banach algebra. Take $x \in E$, $x', y' \in E'_+$. Let (H, φ, ξ) and (K, ψ, η) be the GNS-triples of E associated to x' and y', respectively.*

a) *If there is a $u \in \mathcal{L}(H, K)$ such that*

$$\varphi y = u^* \circ (\psi y) \circ u$$

for every $y \in E$ and

$$(\psi x)\eta = u\xi\,,$$

then

$$x' = xy'x^*\,.$$

b) *The following are equivalent:*

b$_1$) *There is an isometry $u : H \to K$ such that*

$$\varphi y = u^* \circ (\psi y) \circ u$$

for every $y \in E$ and

$$(\psi x)\eta = u\xi\,;$$

b$_2$) *$(\psi x)\eta$ is cyclic for (K, ψ) and*

$$x' = xy'x^*\,.$$

a) By Theorem 5.4.1.2 k), if $y \in E$, then

$$\langle y, x' \rangle = \langle (\varphi y)\xi | \xi \rangle = \langle u^*(\psi y)u\xi | \xi \rangle = \langle (\psi y)u\xi | u\xi \rangle =$$

$$= \langle (\psi y)(\psi x)\eta \,|\, (\psi x)\eta \rangle = \langle (\psi x)^*(\psi y)(\psi x)\eta | \eta \rangle =$$

$$= \langle \psi(x^*yx)\eta | \eta \rangle = \langle x^*yx, y' \rangle = \langle y, xy'x^* \rangle\,.$$

Hence

$$x' = xy'x^*\,.$$

$b_1 \Rightarrow b_2$. Since $u\xi$ is obviously cyclic for (K, ψ), the assertion follows from a).

$b_2 \Rightarrow b_1$. By Theorem 5.4.1.2 k), if $y \in E$, then

$$\langle (\varphi y)\xi | \xi \rangle = \langle y, x' \rangle = \langle y, xy'x^* \rangle = \langle x^*yx, y' \rangle =$$

$$= \langle (\psi(x^*yx))\eta | \eta \rangle = \langle (\psi x)^*(\psi y)(\psi x)\eta | \eta \rangle = \langle (\psi y)(\psi x)\eta | (\psi x)\eta \rangle .$$

b_1) now follows from Proposition 5.4.1.3 b). ■

Corollary 5.4.1.5 *Let E be an involutive quasiunital complex Banach algebra, F the C^*-hull of E, and $u : E \to F$ the canonical map. Then*

$$y' \circ u \in E'_+ , \quad \|y'\| = \|y' \circ u\|$$

for every $y' \in F'_+$, and the map

$$F'_+ \longrightarrow E'_+ , \quad y' \longmapsto y' \circ u$$

is bijective.

Take $x' \in E'_+$ and let (H, φ, ξ) be the GNS-triple of E associated to x'. By Proposition 4.1.1.22 f), there is an involutive algebra homomorphism $\psi : F \to \mathcal{L}(H)$, such that

$$\varphi = \psi \circ u .$$

Define

$$y' : F \longrightarrow \mathbb{C} , \quad y \longmapsto \langle (\psi y)\xi | \xi \rangle .$$

By Proposition 5.4.1.3 a) and Theorem 5.4.1.2 k), $y' \in F'_+$,

$$\|y'\| = \|\xi\|^2 = \|x'\| ,$$

and

$$y' \circ u(x) = \langle (\psi \circ u(x))\xi | \xi \rangle = \langle (\varphi x)\xi | \xi \rangle = x'(x)$$

for every $x \in E$. Hence

$$y' \circ u = x' .$$

■

Proposition 5.4.1.6 (0) *Let E be a quasiunital involutive Banach algebra. Take $x' \in E'_+$ and let (H, φ, ξ) be the GNS-triple of E associated to x'.*

a) *If $u \in \mathcal{L}(H)^{\#}_+ \cap \varphi(E)^c$, then the map*

$$y' : E \longrightarrow \mathbb{K}, \quad x \longmapsto \langle (\varphi x)u\xi|\xi\rangle$$

is a positive linear form less than x'.

b) *If y' is a positive linear form on E less than x', then there is a unique $u \in \mathcal{L}(H)^{\#}_+ \cap \varphi(E)^c$ such that*

$$y'(x) = \langle (\varphi x)u\xi|\xi\rangle$$

for every $x \in E$.

a) For any $x \in E$,

$$y'(x^*) = \langle (\varphi x^*)u\xi|\xi\rangle = \langle u\varphi(x^*)\xi|\xi\rangle = \langle u\varphi(x)^*\xi|\xi\rangle =$$

$$= \langle \xi|\varphi(x)u\xi\rangle = \overline{\langle \varphi(x)u\xi|\xi\rangle} = \overline{y'(x)},$$

$$y'(x^*x) = \langle (\varphi x^*x)u\xi|\xi\rangle = \langle (\varphi x)^*(\varphi x)u\xi|\xi\rangle = \langle (\varphi x)u\xi|(\varphi x)\xi\rangle =$$

$$= \langle u(\varphi x)\xi|(\varphi x)\xi\rangle \in \mathbb{R}_+.$$

(Corollary 5.3.3.7). Hence y' is positive. Since

$$\langle x, x' - y'\rangle = \langle x, x'\rangle - \langle x, y'\rangle =$$

$$= \langle (\varphi x)\xi|\xi\rangle - \langle (\varphi x)u\xi|\xi\rangle = \langle (\varphi x)(1-u)\xi|\xi\rangle$$

for every $x \in E$, it follows from the above result that $x' - y'$ is positive. Hence $y' \leq x'$.

b) Let $q : E \to E/N_{x'}$ be the quotient map and define

$$g : E \times E \longrightarrow \mathbb{K}, \quad (x, y) \longrightarrow y'(y^*x).$$

g is a positive sesquilinear form (Proposition 2.3.4.6 b)). By Proposition 2.3.4.6 c) and Theorem 5.4.1.2 a),

$$|g(x,y)|^2 = |y'(y^*x)|^2 \leq y'(y^*y)y'(x^*x) \leq x'(y^*y)x'(x^*x) = \|qx\|^2\|qy\|^2$$

for all $x, y \in E$. Hence there is a unique positive sesquilinear form h on $E/N_{x'}$ such that

$$h(qx, qy) = g(x,y)$$

for all $x, y \in E$. By the above inequality,

$$|h(X,Y)| \leq \|X\|\,\|Y\|$$

for all $X, Y \in E/F$. By Proposition 1.2.9.2 c $\Rightarrow$ d, we may extend h continuously to $H := \widetilde{E/N_{x'}}$. By Corollary 5.3.1.2, there is a $u \in \mathcal{L}(H)$ such that

$$h(X,Y) = \langle X|uY\rangle$$

for all $X, Y \in H$. We have

$$\langle X|u^*Y\rangle = \langle uX|Y\rangle = \langle \overline{Y|uX}\rangle =$$

$$= \overline{h(Y,X)} = h(X,Y) = \langle X|uY\rangle\,,$$

for all $X, Y \in H$. Thus u is selfadjoint. Since

$$\langle uX|X\rangle = h(X,X) \in [0,1]$$

for every $X \in H^{\#}$,

$$\sigma(u) \subset [0,1]$$

by Theorem 5.3.3.6. Hence $u \in \mathcal{L}(H)^{\#}_{+}$. We have

$$\langle uqx|qy\rangle = \langle qx|uqy\rangle = h(qx,qy) = g(x,y) = y'(y^*x)$$

for all $x, y \in E$.

Put $\widetilde{x} := \varphi x$ for every $x \in E$ and take $x, y, z \in E$. Then

$$\langle \widetilde{x}uqy|qz\rangle = \langle uqy|\widetilde{x}^*qz\rangle = \langle uqy|\widetilde{x^*}qz\rangle =$$

$$= \langle uqy|q(x^*z)\rangle = y'(z^*xy\rangle = \langle uq(xy)|qz\rangle = \langle u\widetilde{x}qy|qz\rangle\,.$$

Hence

$$\widetilde{x}u = u\widetilde{x}\,,$$

$$u \in \varphi(E)^c\,.$$

Take $x, y \in E$ and let $\mathfrak{F}$ be an approximate unit of E. Then

$$y'(yx) = \langle uqx|qy^*\rangle = \langle u\widetilde{x}\xi|\widetilde{y^*}\xi\rangle = \langle u\widetilde{x}\xi|\widetilde{y}^*\xi\rangle =$$

$$= \langle \widetilde{y}u\widetilde{x}\xi|\xi\rangle = \langle u\widetilde{y}\widetilde{x}\xi|\xi\rangle = \langle u\widetilde{yx}\xi|\xi\rangle\,,$$

$$y'(x) = \lim_{y,\mathfrak{F}} y'(yx) = \lim_{y,\mathfrak{F}} \langle u\widetilde{yx}\xi|\xi\rangle = \langle u\widetilde{x}\xi|\xi\rangle = \langle u(\varphi x)\xi|\xi\rangle\,.$$

Now we show that u is unique. Take $v \in \varphi(E)^c$ such that

$$y'(x) = \langle (\varphi x)v\xi|\xi\rangle$$

for every $x \in E$. Since

$$\langle uqx|qy\rangle = y'(y^*x) = \langle \widetilde{y^*x}v\xi|\xi\rangle = \langle \widetilde{y}^*\widetilde{x}v\xi|\xi\rangle =$$

$$= \langle v\widetilde{x}\xi|\widetilde{y}\xi\rangle = \langle vqx|qy\rangle$$

for all $x, y \in E$, we have that

$$v = u\,. \qquad \blacksquare$$

Proposition 5.4.1.7 (0) *Let E be an involutive Banach algebra, $E \neq \{0\}$, (H,φ) and (K,ψ) two representations of E, and u, v two equivalences of (H,φ) and (K,ψ). If $\varphi(E) \supset \mathcal{K}(H)$, then there is an $\alpha \in \mathbb{K}$ with*

$$u = \alpha v\,, \quad |\alpha| = 1\,.$$

For every $x \in E$,

$$u \circ (\varphi x) \circ u^* = \psi x = v \circ (\varphi x) \circ v^*\,,$$

so that

$$v^* \circ u \circ (\varphi x) = (\varphi x) \circ v^* \circ u\,.$$

Hence

$$v^* \circ u \in \varphi(E)^c \subset \mathcal{K}(H)^c\,.$$

By Corollary 5.3.2.14, there is an $\alpha \in \mathbb{K}$ with

$$v^* \circ u = \alpha 1 ,$$

$$u = \alpha v .$$

It follows

$$v \circ (\varphi x) \circ v^* = |\alpha|^2 v \circ (\varphi x) \circ v^*$$

for every $x \in E$, so that

$$|\alpha| = 1 . \qquad \blacksquare$$

Proposition 5.4.1.8 (0) *We use the notation of Theorem 5.4.1.2 and assume E is a real C^*-algebra. We denote by $\overset{\circ}{E}$ the complexification of E, by $\overset{\circ}{\widetilde{E/N_{x'}}}$ the complexification of the pre-Hilbert space $E/N_{x'}$ (Proposition 5.3.1.8), by*

$$\overset{\circ}{u} : \overset{\circ}{E} \longrightarrow \overset{\circ}{E}/N_{(x',0)}$$

the quotient map (Corollary 4.3.6.2 $c_3 \Rightarrow c_1$), and by (H, φ, ξ_0) and (K, ψ, η_0) the GNS-triples associated to x' and $(x', 0)$, respectively.

a) $N_{(x',0)} = N_{x'} \times N_{x'}$.

b) *Let $(\xi, \eta) \in \overset{\circ}{\widetilde{E/N_{x'}}}$ and take $x_1, x_2, y_1, y_2 \in E$ such that*

$$ux_1 = ux_2 = \xi , \quad uy_1 = uy_2 = \eta .$$

Then

$$(x_1, y_1) - (x_2, y_2) \in N_{(x',0)} .$$

Put

$$\widetilde{(\xi, \eta)} := \overset{\circ}{u}\,(x_1, y_1) .$$

c) *The map*

$$\rho : \overset{\circ}{\widetilde{E/N_{x'}}} \longrightarrow \overset{\circ}{E}/N_{(x',0)} , \quad (\xi, \eta) \longmapsto \widetilde{(\xi, \eta)}$$

is an isomorphism of complex pre-Hilbert spaces. Hence its extension

$$\widetilde{\rho} : \overset{\circ}{H} \longrightarrow K$$

is an isomorphism of Hilbert spaces.

d) $(x,y) \in \overset{\circ}{E} \Rightarrow \psi((x,y)) \circ \tilde{\rho} = \tilde{\rho} \circ (\varphi x, \varphi y)$.

e) $\eta_0 = \tilde{\rho}((\xi_0, 0))$.

a) For $(x,y) \in \overset{\circ}{E}$,

$$\langle (x,y)^*(x,y), (x',0) \rangle = \langle (x^*, -y^*)(x,y), (x',0) \rangle =$$

$$= \langle (x^*x + y^*y, x^*y - y^*x), (x',0) \rangle =$$

$$= \langle x^*x + y^*y, x' \rangle + i \langle x^*y - y^*x, x' \rangle = \langle x^*x + y^*y, x' \rangle .$$

Hence

$$(x,y) \in N_{(x',0)} \iff x', y' \in N_{x'},$$

i.e.

$$N_{(x',0)} = N_{x'} \times N_{x'} .$$

b) By a),

$$(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2) \in N_{x'} \times N_{x'} = N_{(x',0)} .$$

c) The surjectivity of the map is obvious. For $(x_1, y_1), (x_2, y_2) \in \overset{\circ}{E}$,

$$\langle \overset{\circ}{u}(x_1, y_1) \mid \overset{\circ}{u}(x_2, y_2) \rangle = \langle (x_2, y_2)^*(x_1, y_1), (x', 0) \rangle =$$

$$= \langle (x_2^*, -y_2^*)(x_1, y_1), (x', 0) \rangle = \langle (x_2^* x_1 + y_2^* y_1, x_2^* y_1 - y_2^* x_1), (x', 0) \rangle =$$

$$= \langle x_2^* x_1, x' \rangle + \langle y_2^* y_1, x' \rangle + i \langle x_2^* y_1, x' \rangle - i \langle y_2^* x_1, x' \rangle =$$

$$= \langle u x_1 | u x_2 \rangle + \langle u y_1 | u y_2 \rangle + i \langle u y_1 | u x_2 \rangle - i \langle u x_1 | u y_2 \rangle = \langle (u x_1, u y_1) \mid (u x_2, u y_2) \rangle ,$$

which proves the other assertions.

d) For $(x_0, y_0) \in \overset{\circ}{E}$,

$$\psi((x,y)) \tilde{\rho}(u x_0, u y_0) = \psi((x,y)) \widetilde{(u x_0, u y_0)} = \psi((x,y)) \overset{\circ}{u}((x_0, y_0)) = \overset{\circ}{u}((x,y)(x_0, y_0)) =$$

$$\overset{\circ}{u}((x x_0 - y y_0, x y_0 + y x_0)) = \widetilde{(u(x x_0 - y y_0), u(x y_0 + y x_0))} =$$

$$= \tilde{\rho}((u(xx_0 - yy_0), u(xy_0 + yx_0))) =$$

$$= \tilde{\rho}(((\varphi x)ux_0 - (\varphi y)uy_0\,,\ (\varphi x)uy_0 + (\varphi y)ux_0)) =$$

$$= \tilde{\rho}((\varphi x, \varphi y)(ux_0, uy_0))\,.$$

Hence

$$\psi((x,y))\circ\tilde{\rho} = \tilde{\rho}\circ(\varphi x, \varphi y)\,.$$

c) By c), for $(x,y) \in \mathring{E}$,

$$\langle \mathring{u}\,(x,y) \mid (\xi_0, 0)\rangle = \langle \widetilde{(ux, uy)} \mid (\xi_0, 0)\rangle =$$

$$= \langle \tilde{\rho}((ux, uy)) \mid (\xi_0, 0)\rangle = \langle ux|\xi_0\rangle + i\langle uy|\xi_0\rangle =$$

$$= \langle x, x'\rangle + i\langle y, x'\rangle = \langle (x,y), (x', 0)\rangle.$$

Hence

$$\eta_0 = \tilde{\rho}((\xi_0, 0))\,.$$

■

5.4.2 General Representations

Proposition 5.4.2.1 (0) *Let E be a (unital) involutive Banach algebra and $(H_\iota, \varphi_\iota)_{\iota\in I}$ a family of (unital) representations of E. Define*

$$H := \bigoplus_{\iota\in I} H_\iota ,$$

$$\varphi : E \longrightarrow \mathcal{L}(H) , \quad x \longmapsto \bigoplus_{\iota\in I} \varphi_\iota x .$$

Then (H,φ) is a (unital) representation of E; it is called the ***Hilbert sum of the family*** *$(H_\iota, \varphi_\iota)_{\iota\in I}$. If each (H_ι, φ_ι) is non-degenerate, then (H,φ) is non-degenerate.*

By Corollary 4.1.1.20, $\|\varphi_\iota\| \leq 1$ for every $\iota \in I$. Thus φ is well-defined. By the Propositions 5.1.3.3 and 5.1.3.4, φ is a (unital) algebra homomorphism and by Proposition 5.3.2.3, φ is involutive. The last assertion is obvious. ■

Corollary 5.4.2.2 (0) *Let E be an involutive quasiunital (unital) Banach algebra and $(x'_\iota)_{\iota\in I}$ a nonempty family in $\tau(E)$. Given $\iota \in I$, let (H_ι, φ_ι) denote the representation associated to x'_ι and put*

$$H := \bigoplus_{\iota\in I} H_\iota \qquad \textbf{\textit{(the Hilbert space associated to } } (x'_\iota)_{\iota\in I} \textbf{\textit{)}}$$

$$\varphi : E \longrightarrow \mathcal{L}(H) , \quad x \longmapsto \bigoplus_{\iota\in I} \varphi_\iota x \qquad \textbf{\textit{(the algebra homomorphism associated to } } (x'_\iota)_{\iota\in I} \textbf{\textit{)}} .$$

*Then (H,φ) is a non-degenerate (unital) representation of E (**the representation of E associated to** $(x'_\iota)_{\iota\in I}$), φ has norm 1, and*

$$\|\varphi x\|^2 \geq \sup_{\iota\in I} x'_\iota(x^*x)$$

for every $x \in E$. If H_ι is separable whenever $\iota \in I$ (this occurs if E is separable) and if I is countable, then H is separable.

By Proposition 5.4.2.1, (H,φ) is a non-degenerate (unital) representation of E and by Theorem 5.4.1.2 g),h), φ has norm 1 and

$$\|\varphi x\|^2 \geq \sup_{\iota\in I} x'_\iota(x^*x)$$

for every $x \in E$. The last assertion follows from Proposition 5.1.3.1 d) (and Theorem 5.4.1.2 c)). ■

Definition 5.4.2.3 (0) *Let E be an involutive quasiunital Banach algebra. For every nonempty subset A of $\tau(E)$, the representation of E (the Hilbert space, the algebra homomorphism) associated to $(x')_{x' \in A}$ will be called* ***the representation of E (the Hilbert space, the algebra homomorphism) associated to A. The universal representation of*** *E is the representation of E associated to $\tau(E)$.*

Corollary 5.4.2.4 (0) *Let E be an involutive quasiunital Banach algebra and (H, φ) its universal representation. Then for each $x' \in E'_+$, there is a $\xi \in H$ such that*

$$\|\xi\| = \|x'\|^{\frac{1}{2}}$$

and

$$x'(x) = \langle (\varphi x)\xi | \xi \rangle$$

for every $x \in E$.

The assertion follows immediately from Theorem 5.4.1.2 k). ■

Corollary 5.4.2.5 (0) (Gelfand–Naimark, 1943) *Let E be a C^*-algebra. If A is a subset of $\tau(E)$ such that $\tau_0(E) \subset \overline{A}$, then the representation of E associated to A is faithful. If E is separable, then we may take A to be countable and, in this case, the Hilbert space associated to A is separable.*

E is quasiunital (Theorem 4.2.8.2). Let (H, φ) the representation of E associated to A. By Corollary 4.2.8.5 d) and Corollary 5.4.2.2,

$$\|x\|^2 = \|x^*x\| = \sup_{x' \in \tau_0(E)} x'(x^*x) = \sup_{x' \in A} x'(x^*x) \leq \|\varphi x\|^2 \leq \|x\|^2$$

for every $x \in E$.

If E is separable, then $\tau(E)$ has a countable base (Proposition 2.3.5.9 b)), so we may take A to be countable. By Corollary 5.4.2.2, H is separable is this case. ■

Remark. A C^*-algebra E may have a faithful representation (H, φ) with H separable even if E itself is not separable as the Example $\mathcal{L}(\ell^2)$ shows (Proposition 5.5.2.11 a)). See also the Remark of Example 5.4.3.7.

Proposition 5.4.2.6 (0) *Let E be a real C^*-algebra, $\overset{\circ}{E}$ its complexification, and (H,φ) the universal representation of $\overset{\circ}{E}$. Denote by $\widetilde{H}$ the underlying real vector space of H endowed with the scalar product*

$$\widetilde{H} \times \widetilde{H} \longrightarrow \mathbb{R}\,, \quad (\xi,\eta) \longmapsto \langle\langle \xi|\eta\rangle\rangle := \operatorname{re}\langle \xi|\eta\rangle\,.$$

a) *$\widetilde{H}$ is a Hilbert space and its canonical norm coincides with the canonical norm of H.*

b) *If we identify E with the subset $E\times\{0\}$ of $\overset{\circ}{E}$ then $(\widetilde{H},\varphi|E)$ is a faithful representation of E called* ***the complex universal representation of*** *E.*

c) *Let $x' \in E'$ and let $((\xi_\iota,\eta_\iota))_{\iota\in I}$ be a finite family in $H\times H$ such that*

$$\sum_{\iota\in I}\langle\varphi((x,y))\xi_\iota|\eta_\iota\rangle = x'(x) + ix'(y)$$

for every $(x,y) \in \overset{\circ}{E}$. Then

$$x'(x) = \sum_{\iota\in I}\langle\langle(\varphi x)\xi_\iota|\eta_\iota\rangle\rangle$$

for every $x \in E$.

d) *For every $x' \in \mathcal{E}'_+$ there is a $\xi \in H$ such that*

$$\|\xi\| = \|x'\|^{\frac{1}{2}}\,, \quad x'(x) = \langle\langle(\varphi x)\xi|\xi\rangle\rangle$$

for every $x \in E$.

e) *For every $x' \in E'$ there is a family $((\xi_k,\eta_k))_{k\in\mathbb{N}_4}$ in $H\times H$ such that*

$$x'(x) = \sum_{k=1}^{4}\langle\langle(\varphi x)\xi_k|\eta_k\rangle\rangle$$

for every $x \in E$.

a) is easy to see.

b) follows from Corollary 5.4.2.5 (and Proposition 4.1.1.27 c),d), Theorem 4.2.8.2).

c) For every $x \in E$,

$$\sum_{\iota\in I}\langle\varphi((x,0))\xi_\iota|\eta_\iota\rangle = x'(x) \in \mathbb{R}$$

so that

$$\sum_{\iota\in I}\langle\langle(\varphi x)\xi_\iota|\eta_\iota\rangle\rangle = \operatorname{re}\sum_{\iota\in I}\langle\varphi((x,0))\xi_\iota|\eta_\iota\rangle = x'(x)\,.$$

d) Put

$$y' : \overset{\circ}{E} \longrightarrow \mathbb{C}\,, \quad (x,y) \longmapsto x'(x) + ix'(y)\,.$$

By Proposition 4.1.1.27 c) (and Theorem 4.2.8.2), $y' \in (\overset{\circ}{E})'_+$ and

$$\|y'\| = \|x'\|\,.$$

By Corollary 5.4.2.4, there is a $\xi \in H$ such that $\|\xi\| = \|y'\|^{\frac{1}{2}}$ and

$$y'((x,y)) = \langle\varphi((x,y))\xi|\xi\rangle\,.$$

By c),

$$x'(x) = \langle\langle(\varphi x)\xi|\xi\rangle\rangle$$

for every $x \in E$.

e) Put

$$y' : \overset{\circ}{E} \longrightarrow \mathbb{C}\,, \quad (x,y) \longmapsto x'(x) + ix'(y)\,.$$

Then $y' \in (\overset{\circ}{E})'$ so that there is a family $(y'_k)_{k\in\mathbb{N}_4}$ in $(\overset{\circ}{E})'_+$ such that

$$y' = y'_1 - y'_2 + iy'_3 - iy'_4$$

(Corollary 4.1.2.7 d)). By Corollary 5.4.2.4 (and Theorem 4.2.8.2), for every $k \in \mathbb{N}_4$ there is a $\xi_k \in H$ such that

$$y'_k((x,y)) = \langle\varphi((x,y))\xi_k|\xi_k\rangle$$

for every $(x,y) \in \overset{\circ}{E}$. Put

$$\eta_1 := \xi_1\,, \quad \eta_2 := -\xi_2\,, \quad \eta_3 := -i\xi_3\,, \quad \eta_4 := i\xi_4\,.$$

Then

$$y'((x,y)) = \sum_{k=1}^{4}\langle\varphi((x,y))\xi_k|\eta_k\rangle$$

for every $(x,y) \in \overset{\circ}{E}$. By c),

$$x'(x) = \sum_{k=1}^{4}\langle\langle(\varphi x)\xi_k|\eta_k\rangle\rangle$$

for every $x \in E$. ■

Proposition 5.4.2.7 (0) *Let E be an involutive quasiunital Banach algebra, A a nonempty subset of $\tau(E)$, and F the closed vector subspace of E' generated by*

$$\{yx'z \,|\, y, z \in E\,,\ x' \in A\}\,.$$

If E_F is Hausdorff, then the representation of E associated to A is faithful.

Let φ be the algebra homomorphism associated to A and take $x \in E\backslash\{0\}$. Since E_F is Hausdorff, there are $y, z \in E$ and $x' \in A$ with

$$\langle x, yx'z\rangle \neq 0\,.$$

Let ψ be the algebra homomorphism associated to x' and let q be the quotient map $E \to E/N_{x'}$. Then

$$0 \neq |\langle x, yx'z\rangle| = |\langle zxy, x'\rangle| \leq x'((zxy)^*(zxy))^{\frac{1}{2}} =$$

$$= \|q(zxy)\| = \|\psi(zx)qy\| = \|(\psi z)(\psi x)qy\| \leq$$

$$\leq \|\psi z\|\,\|\psi x\|\,\|qy\| \leq \|\varphi x\|\,\|\varphi z\|\,\|qy\|$$

(Proposition 2.3.4.10 a),b), Theorem 5.4.1.2 a)), so that

$$\varphi x \neq 0\,.$$

Hence φ is injective. ■

Proposition 5.4.2.8 (0) *Let E be a (unital) involutive Banach algebra, (H, φ) a (unital) representation of E, and K a $\varphi(E)$-invariant closed vector subspace of H. Given $x \in E$, define*

$$\psi x : K \longrightarrow K\,, \quad \xi \longmapsto (\varphi x)\xi\,.$$

Then $K^\perp$ is $\varphi(E)$-invariant and (K, ψ) is a (unital) representation of E called ***the compression of*** (H, φ) ***to*** K.

Take $\xi \in K^\perp$ and $\eta \in K$. Then, given $x \in E$,

$$\langle(\varphi x)\xi|\eta\rangle = \langle\xi|(\varphi x)^*\eta\rangle = \langle\xi|(\varphi x^*)\eta\rangle = 0\,,$$

$$(\varphi x)\xi \in K^\perp\,.$$

Hence $K^\perp$ is $\varphi(E)$-invariant.

Take $x, y \in E$. Then

$$\psi(xy)\xi = \varphi(xy)\xi = (\varphi x)(\varphi y)\xi = (\psi x)(\psi y)\xi$$

for every $\xi \in K$ and therefore

$$\psi(xy) = (\psi x)(\psi y).$$

If $x \in E$, then

$$\langle(\psi x^*)\xi|\eta\rangle = \langle((\varphi x^*)\xi|\eta\rangle = \langle(\varphi x)^*\xi|\eta\rangle = \langle\xi|(\varphi x)\eta\rangle = \langle\xi|(\psi x)\eta\rangle$$

for all $\xi, \eta \in K$, so that

$$\psi x^* = (\psi x)^*.$$

Hence (K, ψ) is a representation of E.

It is easy to see that if E and φ are unital, then ψ is also unital. ■

Proposition 5.4.2.9 *Let E be an involutive Banach algebra and (H, φ) a representation of E. There is a family $(H_\iota)_{\iota \in I}$ of $\varphi(E)$-invariant closed vector subspaces of H such that:*

1) *$H_\iota \perp H_\lambda$ whenever $\iota, \lambda \in I$ are distinct.*

2) *The compression of (H, φ) to H_ι is cyclic for every $\iota \in I$.*

3) *The compression of (H, φ) to $\left(\bigcup_{\iota \in I} H_\iota\right)^\perp$ is a 0-representation.*

Given $\xi \in H$, put

$$H_\xi := \overline{\{(\varphi x)\xi \,|\, x \in E\}}.$$

Let $\mathfrak{A}$ be the set of subsets A of H such that

$$H_\xi \neq \{0\}$$

for every $\xi \in A$ and

$$H_\xi \perp H_\eta$$

for distinct $\xi, \eta \in A$. Let $\mathfrak{A}$ be ordered by inclusion and let A be a maximal element of $\mathfrak{A}$. The family $(H_\xi)_{\xi \in A}$ obviously has properties 1) and 2). Property 3) follows from the maximality of A and Proposition 5.4.2.8. ■

Corollary 5.4.2.10 *Let E be an involutive Banach algebra. Then any representation of E is the Hilbert sum of cyclic representations and of a 0-representation.* ■

Proposition 5.4.2.11 *Let E be a quasiunital involutive Banach algebra, F an involutive closed ideal of E, and (H,φ) a non-degenerate representation of F. Then there is a unique representation (H,ψ) of E with $\psi|F = \varphi$.*

Suppose that there is a cyclic vector ξ for (H,φ). Define

$$x' : F \longrightarrow \mathbb{K}\,, \quad x \longmapsto \langle(\varphi x)\xi|\xi\rangle\,.$$

By Proposition 5.4.1.3 a), $x' \in F'_+$.

Take $x \in E$. Then

$$\|(\varphi xy)\xi\|^2 = \langle(\varphi xy)\xi|(\varphi xy)\xi\rangle = \langle(\varphi y^* x^* xy)\xi|\xi\rangle =$$

$$= x'(y^* x^* xy) \leq \|x\|^2 x'(y^* y) = \|x\|^2 \|(\varphi y)\xi\|^2$$

for every $y \in F$ (Corollary 2.3.4.8). Hence there is a unique $\widetilde{x} \in \mathcal{L}(H)$ such that

$$\widetilde{x}(\varphi y)\xi = (\varphi xy)\xi$$

for every $y \in F$. If $x \in F$, then

$$\widetilde{x} = \varphi x\,.$$

Since

$$\widetilde{x}(\varphi y)(\varphi z)\xi = \widetilde{x}(\varphi yz)\xi = (\varphi xyz)\xi = (\varphi xy)(\varphi z)\xi$$

for all $y, z \in F$,

$$\widetilde{x}\varphi y = \varphi xy$$

for every $y \in F$.

Take $x, y \in E$ and $z \in F$. Then

$$\widetilde{xy}\varphi z = \varphi xyz = \widetilde{x}\varphi yz = \widetilde{x}\widetilde{y}\varphi z\,,$$

$$\widetilde{xy}(\varphi z)\xi = \widetilde{x}\widetilde{y}(\varphi z)\xi$$

$$\widetilde{xy} = \widetilde{x}\widetilde{y}\,.$$

Choose $x \in E$ and $y, z \in F$. Then

$$\langle \widetilde{x}(\varphi y)\xi | (\varphi z)\xi \rangle = \langle (\varphi xy)\xi | (\varphi z)\xi \rangle = \langle \xi | (\varphi xy)^*(\varphi z)\xi \rangle =$$

$$= \langle \xi | (\varphi y^* x^*)(\varphi z)\xi \rangle = \langle \xi | (\varphi y^* x^* z)\xi \rangle = \langle \xi | (\varphi y^*)(\varphi x^* z)\xi \rangle =$$

$$= \langle \xi | (\varphi y)^* \widetilde{x^*}(\varphi z)\xi \rangle = \langle (\varphi y)\xi \,|\, \widetilde{x^*}(\varphi z)\xi \rangle\,.$$

Thus

$$\widetilde{x}^* = \widetilde{x^*}$$

and the map

$$E \longrightarrow \mathcal{L}(H)\,, \quad x \longmapsto \widetilde{x}$$

is an involutive algebra homomorphism extending φ.

Let $\psi : E \to \mathcal{L}(H)$ be an involutive algebra homomorphism extending φ. Then, given any $(x, y) \in E \times F$,

$$(\psi x)(\varphi y)\xi = (\psi xy)\xi = (\varphi xy)\xi = \widetilde{x}(\varphi y)\xi\,.$$

It follows that

$$\psi x = \widetilde{x}$$

for every $x \in E$, which establishes the uniqueness of ψ.

The general case follows from the cyclic case and Corollary 5.4.2.10. ■

Proposition 5.4.2.12 *Let E be an involutive Banach algebra, $(x'_\iota)_{\iota \in I}$ an absolutely summable family in E'_+, and x' its sum (by Proposition 1.7.1.5 c), $x' \in E'_+$). Let (H, φ) be the representation of E associated to x' and $q : E \to E/N_{x'}$ the quotient map. For every $\iota \in I$, let (H_ι, φ_ι) be the representation of E associated to x'_ι and $q_\iota : E \to E/N_{x'_\iota}$ the quotient map.*

a) *The map*

$$u : E \longrightarrow \bigoplus_{\iota \in I} H_\iota\,, \quad x \longmapsto (q_\iota x)_{\iota \in I}$$

is well-defined; it factors through $H = \widetilde{E/N_{x'}}$ and this factorization v preserves the norms.

b) $x \in E \Rightarrow v \circ (\varphi x) = \left(\bigoplus_{\iota \in I} (\varphi_\iota x) \right) \circ v$.

c) $u(E)$ *is* $\left\{ \bigoplus_{\iota \in I} (\varphi_\iota x) \middle| x \in E \right\}$*-invariant and* (H, φ) *is equivalent to the compression to* $\overline{u(E)}$ *of the Hilbert sum of the family* $((H_\iota, \varphi_\iota))_{\iota \in I}$.

d) *If* E *is quasiunital and if, for every* $\iota \in I$, ξ_ι *is the cyclic vector associated to* x'_ι , *then* $v^*(\xi_\iota)_{\iota \in I}$ *is the cyclic vector associated to* x' .

a) We have

$$\sum_{\iota \in I} \|q_\iota x\|^2 = \sum_\iota x'_\iota(x^* x) = x'(x^* x) = \|qx\|^2$$

for every $x \in E$ (Theorem 5.4.1.2 a)) and the assertions follow.

b) Since

$$v(\varphi x) qy = vq(xy) = (q_\iota(xy))_{\iota \in I} = ((\varphi_\iota x) q_\iota y)_{\iota \in I} = \left(\bigoplus_{\iota \in I} (\varphi_\iota x) \right) vqy$$

for every $y \in E$,

$$v \circ (\varphi x) = \left(\bigoplus_{\iota \in I} (\varphi_\iota x) \right) \circ v .$$

c) follows from b) and a).

d) Since

$$x'(x) = \sum_{\iota \in I} x'_\iota(x) = \sum_{\iota \in I} \langle q_\iota x | \xi_\iota \rangle = \langle vqx | (\xi_\iota)_{\iota \in I} \rangle =$$

$$= \langle qx | v^*(\xi_\iota)_{\iota \in I} \rangle$$

for every $x \in E$ (Theorem 5.4.1.2 k)), $v^*(\xi_\iota)_{\iota \in I}$ is the cyclic vector associated to x' . ■

Theorem 5.4.2.13 (0) (Goodearl, 1982) *Let* E *be a symmetric involutive unital real Banach algebra such that*

$$\|x^* x\| = \|x\|^2$$

for every $x \in E$. *Then* E *is a real* C^**-algebra.*

Take $x \in \operatorname{Re} E$. Since $\{P(x) | P \in \mathbb{K}[t]\}$ is dense in $F := E(x,1)$, all elements of F are selfadjoint. By By Proposition 2.3.2.21 a $\Rightarrow$ b, F is symmetric, so that, by Proposition 2.1.5.18 b $\Rightarrow$ a,

$$\sigma_{\overset{\circ}{F}}((y,0)) = \sigma_F(y)$$

for every $y \in F$. By Gelfand's Theorem (Theorem 2.2.5.4) and Proposition 4.1.1.2 b),

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \|x\|.$$

Let (H,φ) be the representation of E associated to $\tau(E)$ and let $x \in E$. By Proposition 2.3.5.15 a $\Rightarrow$ c, $\overset{\vee}{\overbrace{x^*x}}(\tau(E))$ is the convex hull of $\sigma(x^*x)$. By the above,

$$\left\| \overset{\vee}{\overbrace{x^*x}} \right\| = r(x^*x) = \|x^*x\|.$$

By Corollary 5.4.2.2,

$$\|x\|^2 \geq \|\varphi(x)\|^2 \geq \|x^*x\| = \|x\|^2.$$

Hence φ preserves the norms and therefore E is isomorphic to a closed involutive subalgebra of $\mathcal{L}(H)$. In particular, E is a real C^*-algebra. ■

Proposition 5.4.2.14 (0) *Let E be a real C^*-algebra, A a subset of $\tau(E)$, and (H,φ) the representation of E associated to A. If $\overset{\circ}{E}$ and $\overset{\circ}{H}$ denote the complexifications of E and H, respectively, then*

$$\overset{\circ}{E} \longrightarrow \mathcal{L}(\overset{\circ}{H}), \quad (x,y) \longmapsto (\varphi x, \varphi y)$$

(Corollary 5.3.1.12) is equivalent to the representation of $\overset{\circ}{E}$ associated to $\{(x',0) \mid x' \in A\}$ (Corollary 4.3.6.2 $c_3 \Rightarrow c_1$). This representation is faithful iff (H,φ) is faithful.

The assertion follows from Proposition 5.4.1.8 c),d). ■

5.4.3 Example of Representations

Example 5.4.3.1 *Let T be a normal topological space. Take $\mu \in \mathcal{M}_b(T)_+ \backslash \{0\}$ and let S be the carrier of μ. Let E denote the involutive unital Banach algebra $\mathcal{C}(T)$ (Example 2.3.2.4). Put*

$$x' : \mathcal{C}(T) \longrightarrow \mathbb{K}\,, \quad x \longmapsto \int x d\mu\,,$$

$$F := \{x \in E \mid x|S = 0\}\,.$$

a) *$x' \in E'_+$ and $N_{x'} = F$.*

b) *The factorization v of the inclusion map $E \to L^2(\mu)$ through E/F is injective.*

c) *If $X, Y \in E/F$ and $x \in X$, $y \in Y$, then*

$$\langle X|Y\rangle = \int x\overline{y} d\mu\,.$$

d) *v may be extended uniquely to an isometry of Hilbert spaces*

$$\widetilde{E/F} \to L^2(\mu)\,.$$

e) *If we identify $\widetilde{E/F}$ with $L^2(\mu)$ using the isometry in d), then*

$$\widetilde{x}(y) = xy\,, \quad \|\widetilde{x}\| = \|x|S\|_\infty$$

for every $(x, y) \in E \times L^2(\mu)$.

f) *The following assertions are equivalent:*

f_1) *S is a one-point set.*

f_2) *the representation of E associated to x' is irreducible.*

f_3) *the Hilbert space associated to x' is one-dimensional.* ■

Example 5.4.3.2 *Let T be a Hausdorff topological space. Take $\mu, \nu \in \mathcal{M}_b(T)_+$ with*

$$\mu(T) = \nu(T) = 1\,,$$

and

$$x' : \mathcal{C}(T) \longrightarrow \mathbb{K}\,, \quad x \longmapsto \int x d\mu\,,$$

$$y' : \mathcal{C}(T) \longrightarrow \mathbb{K}\,, \quad x \longmapsto \int x d\nu\,.$$

Then the following are equivalent:

a) *The representations of* $\mathcal{C}(T)$ *associated to* x' *and* y' *are equivalent.*

b) *There are positive real Borel functions* f *and* g *on* T *such that*

$$f \in L^2(\mu)\,, \quad \nu = f^2 \cdot \mu\,,$$

$$g \in L^2(\nu)\,, \quad \mu = g^2 \cdot \nu\,.$$

Given $x \in \mathcal{C}(T)$, define

$$\varphi x : L^2(\mu) \longrightarrow L^2(\mu)\,, \quad \xi \longmapsto x\xi$$

$$\psi x : L^2(\nu) \longrightarrow L^2(\nu)\,, \quad \eta \longmapsto x\eta\,.$$

By Example 5.4.2.1, $(L^2(\mu), \varphi)$. and $(L^2(\nu), \psi)$ are the representations of $\mathcal{C}(T)$ associated to x' and y', respectively.

a $\Rightarrow$ b. Let $u : L^2(\nu) \to L^2(\mu)$ be an isometry of Hilbert spaces such that

$$\psi x = u^* \circ (\varphi x) \circ u$$

for every $x \in \mathcal{C}(T)$. Let K be a compact set of T and $\eta \in L^2(\nu)$ with

$$\eta = \eta e_K\,.$$

Then for every $x \in \mathcal{C}(T)$ with

$$K \cap \operatorname{Supp} x = \emptyset\,,$$

$$x(u\eta) = (\varphi x)u\eta = u(\psi x)\eta = u(x\eta) = 0\,.$$

Hence

$$u\eta = 0$$

$T\backslash K$. If K is a μ-null set, then K is a ν-null set as well. Hence ν is μ-absolutely continuous. By the Radon-Nikodym Theorem, there is a positive real Borel function f on T such that

$$f \in L^2(\mu), \quad \nu = f^2 \cdot \mu .$$

It follows that there is a positive real Borel function g on T such that

$$g \in L^2(\nu), \quad \mu = g^2 \cdot \nu .$$

b $\Rightarrow$ a. Define

$$u : L^2(\nu) \longrightarrow L^2(\mu), \quad \eta \longmapsto f\eta ,$$

$$v : L^2(\mu) \longrightarrow L^2(\nu), \quad \xi \longmapsto g\xi .$$

u, v are operators preserving the norms and

$$\begin{cases} \xi \in L^2(\mu) \Longrightarrow uv\xi = fg\xi = \xi , \\ \eta \in L^2(\nu) \Longrightarrow vu\eta = gf\eta = \eta . \end{cases}$$

Hence u and v are isometries and

$$u = v^{-1} .$$

Let $x \in \mathcal{C}(T)$ and $\eta \in L^2(\nu)$. We have

$$u^*(\psi x)u\eta = gxf\eta = x\eta = (\varphi x)\eta ,$$

so that

$$\varphi x = u^* \circ (\psi x) \circ u .$$

Hence the representations associated to x' and y' are equivalent. ■

Example 5.4.3.3 *Let H be a Hilbert space, A a nonempty set of pairwise orthogonal elements of $H\backslash\{0\}$, such that*

$$\sum_{\xi \in A} \|\xi\|^2 < \infty ,$$

and E a C^*-subalgebra of $\mathcal{L}(H)$. Put

$$x' : E \longrightarrow \mathbb{K}, \quad x \longmapsto \sum_{\xi \in A} \langle x\xi | \xi \rangle ,$$

$$f : E \times E \longrightarrow \mathbb{K}\,, \quad (x,y) \longmapsto x'(y^*x)\,,$$

$$F := \{x \in E \mid x(A) = \{0\}\}\,,$$

and for $\xi \in A$,

$$E\xi := \{x\xi | x \in E\}\,,$$

$$H_\xi := \overline{E\xi}\,.$$

a) x' *is a positive linear form,* f *is a positive sesquilinear form, and*

$$F = \{x \in E \mid \tilde{f}(x) = 0\}\,;$$

we endow E/F *with the scalar product associated to* f *(Proposition 5.1.2.9 d)) and denote its completion by* $\widetilde{E/F}$.

b) *The factorization of the map*

$$E \longrightarrow \bigoplus_{\xi \in A} H_\xi\,, \quad x \longmapsto (x\xi)_{\xi \in A}$$

through E/F *preserves the norms; we identify* $\widetilde{E/F}$ *with the corresponding closed vector subspace* K *of* $\bigoplus_{\xi \in A} H_\xi$ *via the extension of the above map (Corollary 5.1.1.4).*

c) *If we put*

$$X := \left(\pi_{H_\xi}\xi\right)_{\xi \in A} \in \bigoplus_{\xi \in A} H_\xi$$

and define

$$\varphi x : K \longrightarrow K\,, \quad (\eta_\xi)_{\xi \in A} \longmapsto (x\eta_\xi)_{\xi \in A}$$

for every $x \in E$, *then* (K, φ, X) *is the GNS-triple associated to* x'.

d) *If* A *contains a unique element* ξ, *then* $\varphi x = x|K$ *for every* $x \in E$ *and the above representation is faithful if* $x(K^\perp) = \{0\}$ *for all* $x \in E$.

e) *If* $\mathcal{K}(H) \subset E$, *then* $E\xi = H_\xi = H$ *for every* $\xi \in A$, $K = \bigoplus_{\xi \in A} H_\xi$, *and the representation from c) is faithful.*

a) is obvious.

b) Given $x \in E$,

$$\sum_{\xi \in A} \|x\xi\|^2 = \sum_{\xi \in A} \langle x^* x \xi | \xi \rangle = x'(x^* x) = \tilde{f}(x).$$

c) Let $u : E \to E/F$ be the quotient map. Then for all $x, y \in E$,

$$(\varphi x) u y = (\varphi x)(y\xi)_{\xi \in A} = (xy\xi)_{\xi \in A} = u(xy).$$

Hence (K, φ) is the representation of E associated to x'. Since $H_\xi^\perp$ is E-invariant for all $\xi \in A$ (Proposition 5.3.2.9),

$$\langle x, x' \rangle = \sum_{\xi \in A} \langle x\xi | \xi \rangle = \sum_{\xi \in A} \langle x \pi_{H_\xi} \xi | \pi_{H_\xi} \xi \rangle = \langle (\varphi x) X | X \rangle$$

for every $x \in E$. Thus X is the cyclic vector associated to x' (Theorem 5.4.1.2 k)).

d) Take $x \in E$. Then

$$(\varphi x)\eta = x\eta$$

for every $\eta \in K$, so that

$$\varphi x = x|K,$$

$$\|\varphi x\| = \|x|K\|.$$

The assertion now follows.

e) Take $\xi \in A$ and $\eta \in H$. Then

$$x := \frac{1}{\|\xi\|^2} \langle \cdot | \xi \rangle \eta \in E$$

and

$$x\xi = \eta.$$

Hence

$$E\xi = H, \quad H_\xi = H.$$

Choose $(\eta_\xi)_{\xi \in A} \in \bigoplus_{\xi \in A} H_\xi$ such that $\{\xi \in A \mid \eta_\xi \neq 0\}$ is finite. Then

$$x := \sum_{\xi \in A} \frac{1}{\|\xi\|^2} \langle \cdot | \xi \rangle \eta_\xi \in E$$

and

$$x\xi = \eta_\xi$$

for every $\xi \in A$. Hence $(\eta_\xi)_{\xi\in A} \in K$ and

$$K = \bigoplus_{\xi\in A} H_\xi .$$

Take $x \in E$ and $\eta \in H^\#$. Choose an $\xi_0 \in A$ and put

$$y := \frac{1}{\|\xi_0\|^2}\langle\cdot|\xi_0\rangle\eta \in E .$$

Then

$$y\xi = \begin{cases} \eta & if\ \ \xi = \xi_0 \\ 0 & if\ \ \xi \in A\backslash\{\xi_0\} , \end{cases}$$

so that

$$\|(y\xi)_{\xi\in A}\| = \|\eta\| \leq 1$$

$$\|\varphi x\| \geq \|(\varphi x)(y\xi)_{\xi\in A}\| = \|(xy\xi)_{\xi\in A}\| = \|x\eta\| .$$

Since η is arbitrary we deduce that

$$\|x\| \leq \|\varphi x\|$$

and the representation in c) is faithful. ■

Corollary 5.4.3.4 *Let H be a Hilbert space, $(\xi_\iota)_{\iota\in I}$, $(\eta_\iota)_{\iota\in I}$ nonempty families in $H\backslash\{0\}$ such that*

$$\iota, \lambda \in I , \iota \neq \lambda \Longrightarrow \xi_\iota \perp \xi_\lambda , \eta_\iota \perp \eta_\lambda ,$$

$$\sum_{\iota\in I} \|\xi_\iota\|^2 < \infty , \quad \sum_{\iota\in I} \|\eta_\iota\|^2 < \infty ,$$

E an involutive, closed subalgebra of $\mathcal{L}(H)$ containing $\mathcal{K}(H)$,

$$x' : E \longrightarrow \mathbb{K} , \quad x \longmapsto \sum_{\iota\in I} \langle x\xi_\iota|\xi_\iota\rangle ,$$

$$y' : E \longrightarrow \mathbb{K} , \quad x \longmapsto \sum_{\iota\in I} \langle x\eta_\iota|\eta_\iota\rangle ,$$

and (K, φ), (L, ψ) the representations of E associated to x' and y', respectively. Then

$$K = L = \bigoplus_{\iota\in I} H , \quad \varphi = \psi .$$

■

Example 5.4.3.5 (Calkin, 1941) *Let E be an infinite-dimensional Hilbert space, $\mathfrak{F}$ a free ultrafilter on $\mathbb{N}$, and $\mathcal{F}$ the set of bounded maps $x : \mathbb{N} \to E$ such that $x(\mathfrak{F})$ converges weakly to 0. Consider*

$$f : \mathcal{F} \times \mathcal{F} \longrightarrow \mathbb{K}, \quad (x, y) \longmapsto \lim_{x,\mathfrak{F}} \langle x(n) | y(n) \rangle$$

and

$$\mathcal{G} := \{x \in \mathcal{F} \mid f(x, x) = 0\}.$$

a) *$\mathcal{F}$ is a vector subspace of $E^{\mathbb{N}}$ and f a positive sesquilinear form. Let*

$$\mathcal{F} \longrightarrow \mathcal{F}/\mathcal{G}, \quad x \longmapsto \dot{x}$$

be the quotient map and endow $\mathcal{F}/\mathcal{G}$ with the scalar product associated to f (Proposition 5.1.2.9 d)). Let $\widetilde{\mathcal{F}/\mathcal{G}}$ denote its completion.

b) *If $u \in \mathcal{L}(E)$, then $u \circ x \in \mathcal{F}$ whenever $x \in \mathcal{F}$ and*

$$f(u \circ x, u \circ x) \leq \|u\|^2 f(x, x).$$

Let $\widetilde{u}$ denote the operator on $\widetilde{\mathcal{F}/\mathcal{G}}$ defined by

$$\widetilde{u}\dot{x} = \widehat{\dot{u \circ x}}$$

for $x \in \mathcal{F}$.

c) *For every $u \in \mathcal{L}(E)$,*

$$u \in \mathcal{K}(E) \Longleftrightarrow \widetilde{u} = 0.$$

d) *The factorization of the map*

$$\mathcal{L}(E) \longrightarrow \mathcal{L}(\widetilde{\mathcal{F}/\mathcal{G}}), \quad u \longmapsto \widetilde{u}$$

through $\mathcal{L}(E)/\mathcal{K}(E)$ is a faithful unital representation of $\mathcal{L}(E)/\mathcal{K}(E)$.

e) *If for each decreasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{F}$ with empty intersection, there is an $A \in \mathfrak{F}$ such that*

$$\mathrm{Card}\,(A \cap (A_n \backslash A_{n+1})) \leq 1$$

for every $n \in \mathbb{N}$, then $\{0\}$ and $\mathcal{F}/\mathcal{G}$ are the only $\{\widetilde{u} \mid u \in \mathcal{L}(E)\}$-invariant vector subspace of $\mathcal{F}/\mathcal{G}$.

a) is obvious.

b) $u \circ x$ is bounded and $u \circ x(\mathfrak{F})$ converges weakly to 0 (Corollary 1.4.1.6 a $\Rightarrow$ c). Thus $u \circ x \in \mathcal{F}$. Hence

$$f(u \circ x, u \circ x) = \lim_{n,\mathfrak{F}} \langle ux(n) | ux(n) \rangle =$$

$$= \lim_{n,\mathfrak{F}} \|ux(n)\|^2 \leq \|u\|^2 \lim_{n,\mathfrak{F}} \|x(n)\|^2 = \|u\|^2 f(x,x) .$$

c) If $u \in \mathcal{K}(E)$, then by Proposition 3.1.1.20 a $\Rightarrow$ c, $\widetilde{u} = 0$. Assume now $u \notin \mathcal{K}(E)$. By Proposition 3.1.1.20' d $\Rightarrow$ a, there is a sequence $(x_n)_{n \in \mathbb{N}}$ in E which converges weakly to 0 such that $(ux_n)_{n \in \mathbb{N}}$ does not converge to 0. There is an $\varepsilon > 0$ and a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that

$$\|ux_{k_n}\| > \varepsilon$$

for every $n \in \mathbb{N}$. Define

$$x : \mathbb{N} \longrightarrow E , \quad n \longmapsto x_{k_n} .$$

Then $x \in \mathcal{F}$ and

$$\|\widetilde{u}\dot{x}\|^2 = \|\widehat{u \circ x}\|^2 = f(u \circ x, u \circ x) = \lim_{n,\mathfrak{F}} \langle ux(n) | ux(n) \rangle =$$

$$= \lim_{n,\mathfrak{F}} \langle ux_{k_n} | ux_{k_n} \rangle = \lim_{n,\mathfrak{F}} \|ux_{k_n}\|^2 \geq \varepsilon^2 .$$

Hence $\widetilde{u} \neq 0$.

d) It is obvious that the map is a unital algebra homomorphism. Take $u \in \mathcal{L}(E)$ and $x, y, \in \mathcal{F}$. Then

$$\langle \widetilde{u}\dot{x} | \dot{y} \rangle = \langle \widehat{u \circ x} | \dot{y} \rangle = f(u \circ x, y) = \lim_{n,\mathfrak{F}} \langle ux(n) | y(n) \rangle =$$

$$= \lim_{n,\mathfrak{F}} \langle x(n) | u^* y(n) \rangle = f(x, u^* \circ y) = \langle \dot{x} | \widehat{u^* \circ y} \rangle = \langle \dot{x} | \widetilde{u^*}\dot{y} \rangle .$$

Hence

$$\widetilde{u}^* = \widetilde{u}^*$$

and the factorization referred to in d) is a unital representation. By c), it is faithful.

e) Take $x, y \in \mathcal{F}\backslash\mathcal{G}$. There is a decreasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{F}$ such that

$$\|x(i)\| > \frac{1}{2}\|x\|$$

for every $i \in A_1$ and

$$|\langle x(i)|x(j)\rangle| < \frac{\|x\|^2}{9n^2}, \quad |\langle y(i)|y(j)\rangle| < \frac{1}{n^2}$$

for every $n \in \mathbb{N}$ and $(i,j) \in \mathbb{N}_n \times A_n$. By hypothesis, there is an $A \in \mathfrak{F}$ such that

$$\mathrm{Card}\,(A \cap (A_n\backslash A_{n+1})) \leq 1$$

for every $n \in \mathbb{N}$. By Corollary 5.2.3.20, there is a $u \in \mathcal{L}(E)$ such that

$$ux(k) = y(k)$$

for every $k \in A$. Thus

$$\tilde{u}x = \overbrace{u \circ x} = y.$$

Since x and y are arbitrary, $\{0\}$ and $\mathcal{F}/\mathcal{G}$ are the only $\{\tilde{u}|u \in \mathcal{L}(E)\}$-invariant vector subspaces of $\mathcal{F}/\mathcal{G}$. ■

Remark. The existence of an ultrafilter $\mathfrak{F}$ with the properties formulated in e) follows from the axiom $2^{\aleph_0} = \aleph_1$ (G. Choquet, 1968).

Proposition 5.4.3.6 *An involutive Banach algebra having a faithful representation is semi-simple.*

The assertion follows from Theorem 5.3.1.13 and Proposition 2.3.2.26. ■

Example 5.4.3.7 *Let A be a dense set of the topological space T. Given $x \in \mathcal{C}(T)$, define*

$$\tilde{x} : \ell^2(A) \longrightarrow \ell^2(A), \quad y \longmapsto xy.$$

Then $\tilde{x} \in \mathcal{L}(\ell^2(A))$ for every $x \in \mathcal{C}(T)$ and the map

$$\mathcal{C}(T) \longrightarrow \mathcal{L}(\ell^2(A)), \quad x \longmapsto \tilde{x}$$

is a norm-preserving homomorphism of involutive unital algebras (Theorem 5.3.1.13). Given $x \in \mathcal{C}(T)$, $\tilde{x}$ is normal and it is selfadjoint (unitary) iff x is real valued (iff $|x| = 1$). ■

Remark. If $T := A := \mathbb{N}$, then $\mathcal{C}(T) = \ell^\infty$ is not separable but $\ell^2(A) = \ell^2$ is separable.

Corollary 5.4.3.8 *Every symmetric involutive semi-simple Gelfand algebra possesses a faithful representation.*

Let E be a symmetric involutive semi-simple Gelfand algebra. By Corollary 2.4.1.14, Proposition 2.4.2.2 b), and Proposition 2.4.2.3 a $\Rightarrow$ c, the Gelfand transform

$$u : E \longrightarrow \mathcal{C}(\sigma(E))$$

is an injective continuous homomorphism of involutive algebras and the assertion follows from Example 5.4.3.7. ■

5.5 Orthonormal Bases

Every Hilbert space has an orthonormal basis and the cardinality of this orthonormal basis characterises the Hilbert space up to isometry. This leads to a simple classification of all Hilbert spaces: up to isometry there are precisely as many Hilbert spaces as there are cardinal numbers. Operators can also be easily defined in terms of orthonormal bases. That is how, for example, the Fourier-Plancherel operator is defined. Orthonormal bases have also proved to be very important for compact self-normal operators. In this case the operators can be "diagonalized", that is to say there is an orthonormal basis with respect to which the associated infinite matrix is diagonal.

In this section E always denotes a pre-Hilbert space.

5.5.1 General Results

Definition 5.5.1.1 (0) *The family $(x_\iota)_{\iota\in I}$ in E is said to be* ***orthonormal*** *if*

$$\langle x_\iota | x_\lambda \rangle = \delta_{\iota\lambda}$$

for all $\iota, \lambda \in I$. An ***orthonormal basis of*** *E is an orthonormal family $(x_\iota)_{\iota\in I}$ in E such that*

$$\{x_\iota | \iota \in I\}^\perp = \{0\}$$

or, equivalently,

$$\{x_\iota | \iota \in I\}^{\perp\perp} = E$$

(Proposition 5.2.2.2 d)). The subset A of E is called ***orthonormal (an orthonormal basis of*** *E) if the family $(x)_{x\in A}$ is orthonormal (is an orthonormal basis of E).*

Proposition 5.5.1.2 (0) *If A is an orthonormal set in E, then A is an orthonormal basis of $A^{\perp\perp}$ (Proposition 5.2.2.2 a)).*

Observe that

$$A^\perp \cap A^{\perp\perp} = \{0\}\,.$$

■

Proposition 5.5.1.3 (0) *If $\mathfrak{A}$ is the set of orthonormal sets in E ordered by the inclusion, then an element of $\mathfrak{A}$ is maximal iff it is an orthonormal basis of E.*

Take $A \in \mathfrak{A}$. If A is not an orthonormal basis of E, then there is an $x \in A^{\perp}\backslash\{0\}$. Then $A \cup \left\{\frac{1}{\|x\|}x\right\} \in \mathfrak{A}$ and A is not maximal. Conversely, if A is not maximal, then there is a $B \in \mathfrak{A}$ with

$$A \subset B, \quad A \neq B.$$

Hence

$$\emptyset \neq B\backslash A \subset A^{\perp}$$

and A is not an orthonormal basis of E. ■

Corollary 5.5.1.4 (0) *Every orthonormal set in E is contained in an orthonormal basis of E. Hence E always has an orthonormal basis.*

Let A be an orthonormal set in E and $\mathfrak{A}$ the set of all orthonormal sets in E. It is easy to see that inclusion defines an inductive order on $\mathfrak{A}$. By Zorn's Lemma, there is a maximal element B in $\mathfrak{A}$ containing A. By Proposition 5.5.1.3, B is an orthonormal basis of E. ■

Corollary 5.5.1.5 *Let F be a subspace of E and B an orthonormal basis of F. Then there is an orthonormal basis A of E with $A \cap F = B$.*

B is an orthonormal set of E. By Corollary 5.5.1.4, there is an orthonormal basis A of E containg B. It follows $A \cap F = B$. ■

Corollary 5.5.1.6 *If F is a subspace of E, then there is an orthonormal basis A of E for which $A \cap F$ is an orthonormal basis of F.*

The assertion follows immediately from Corollaries 5.5.1.4 and 5.5.1.5. ■

Proposition 5.5.1.7 (0) *If A is a finite orthonormal set in E, then*

$$\pi_{A^{\perp\perp}} = \sum_{y \in A} \langle \cdot | y \rangle y .$$

Given any $x \in E$,

$$\left\| x - \sum_{y \in A} \langle x | y \rangle y \right\|^2 = \|x\|^2 - \sum_{y \in A} |\langle x | y \rangle|^2 \quad (\textbf{\textit{Bessel's Identity}}).$$

In particular, every finite orthonormal basis of E is an algebraic basis.

Let F be the vector subspace of E generated by A. Since F is finite-dimensional, it is complete. Now $F = A^{\perp\perp}$ (Corollary 5.2.3.9),

$$\sum_{y\in A}\langle x|y\rangle y \in A^{\perp\perp},$$

and

$$x - \sum_{y\in A}\langle x|y\rangle y \in A^{\perp} = A^{\perp\perp\perp}$$

(Proposition 5.2.2.2 d)). It follows that

$$\sum_{y\in A}\langle x|y\rangle y = \pi_{A^{\perp\perp}}x,$$

$$\|x\|^2 = \left\|x - \sum_{y\in A}\langle x|y\rangle y\right\|^2 + \left\|\sum_{y\in A}\langle x|y\rangle y\right\|^2$$

(Proposition 5.2.3.1) and by Pythagoras' Theorem

$$\left\|x - \sum_{y\in A}\langle x|y\rangle y\right\|^2 = \|x\|^2 - \sum_{y\in A}|\langle x|y\rangle|^2. \qquad \blacksquare$$

Proposition 5.5.1.8 (0) *Let A be an orthonormal set in E.*

a) *If $f, g \in \mathbb{K}^A$ and*

$$\sum_{x\in A} f(x)x = \sum_{x\in A} g(x)x,$$

then $f = g$.

b) *A is linearly independent.*

c) $x \in E \Rightarrow \sum_{y\in A}|\langle x|y\rangle|^2 \leq \|x\|^2$ ***(Bessel's Inequality)***.

d) $x, y \in E \Rightarrow \sum_{z\in A}|\langle x|z\rangle\langle z|y\rangle| \leq \|x\|\,\|y\|$.

a) Take $y \in A$. Then

$$f(y) = \sum_{x\in A} f(x)\langle x|y\rangle = \left\langle \sum_{x\in A} f(x)x \middle| y \right\rangle =$$

$$= \left\langle \sum_{x \in A} g(x)x \middle| y \right\rangle = \sum_{x \in A} g(x)\langle x|y\rangle = g(y)\,.$$

Thus $f = g$.

b) Take $f \in \mathbb{K}^A$ with $\{f \neq 0\}$ finite and

$$\sum_{x \in A} f(x)x = 0\,.$$

By a), $f = 0$, and thus A is linearly independent.

c) By Bessel's Identity,

$$\|x\|^2 - \sum_{y \in B} |\langle x|y\rangle|^2 = \|x - \sum_{y \in B} \langle x|y\rangle y\|^2 \geq 0\,,$$

$$\sum_{y \in B} |\langle x|y\rangle|^2 \leq \|x\|^2$$

for every finite subset B of A. Hence

$$\sum_{y \in A} |\langle x|y\rangle|^2 \leq \|x\|^2\,.$$

d) By c),

$$\sum_{z \in A} |\langle x|z\rangle\langle z|y\rangle| \leq$$

$$\leq \left(\sum_{z \in A} |\langle x|z\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{z \in A} |\langle y|z\rangle|^2\right)^{\frac{1}{2}} \leq \|x\|\,\|y\|\,. \qquad \blacksquare$$

Corollary 5.5.1.9 (0) *Let E, F be pre-Hilbert spaces and A an infinite orthonormal set of E. Take $u \in \mathcal{K}(E,F)$ and put*

$$\mathfrak{F} := \{A \backslash B | B \in \mathfrak{P}_f(A)\}\,.$$

Then $\mathfrak{F}$ converges weakly to 0 and $u(\mathfrak{F})$ converges to 0.

We may assume that E and F are complete (Corollary 5.1.1.7). Take $x' \in E'$. By the Fréchet–Riesz Theorem (Theorem 5.2.5.2), there is an $x \in E$ such that

$$x' = \langle \cdot | x\rangle\,.$$

By Bessel's Inequality,

$$\sum_{y\in A} |\langle y|x\rangle|^2 \leq \|x\|^2 ,$$

so that

$$0 = \lim_{y,\mathfrak{F}} \langle y|x\rangle = \lim_{y,\mathfrak{F}} x'(y) .$$

Hence $\mathfrak{F}$ converges weakly to 0. By Proposition 3.1.1.20 a $\Rightarrow$ c, $u(\mathfrak{F})$ converges to 0. ■

Corollary 5.5.1.10 (0) *The following are equivalent:*

a) *E is finite-dimensional.*

b) *Every orthonormal set in E is finite.*

c) *E has a finite orthonormal basis.*

d) *Every orthonormal basis of E is an algebraic basis of E.*

e) *E has an orthonormal basis which is an algebraic basis.*

a $\Rightarrow$ b. By Proposition 5.5.1.8 b), every orthonormal set in E is linearly independent and so finite.

b $\Rightarrow$ c follows from Proposition 5.5.1.4.

c $\Rightarrow$ a follows from Proposition 5.5.1.7.

b $\Rightarrow$ d. Let A be an orthonormal basis of E. By b), A must be finite and so, by Proposition 5.5.1.7, it is an algebraic basis for E.

d $\Rightarrow$ e follows from Corollary 5.5.1.4.

e $\Rightarrow$ a. Let A be an orthonormal basis of E which is also an algebraic basis of E. Take $f \in \ell^2(A)$ and put

$$x := \sum_{y\in A} f(y)y$$

(Corollary 5.2.2.5). Since A is an algebraic basis of E, there is a $g \in \mathbb{K}^{(A)}$ with

$$x = \sum_{y\in A} g(y)y .$$

Then $f = g$ (Proposition 5.5.1.8 a)). Hence $\{f \neq 0\}$ is finite. It follows that A is finite and that E is finite-dimensional. ■

Corollary 5.5.1.11 (0) *If E, F are Hilbert spaces, then $\mathcal{L}_f(E,F)$ is the vector subspace of $\mathcal{L}(E,F)$ generated by $\{\langle\cdot|x\rangle y | (x,y) \in E \times F\}$.*

Let $\mathcal{F}$ be the vector subspace of $\mathcal{L}(E,F)$ generated by

$$\{\langle\cdot|x\rangle y | (x,y) \in E \times F\}.$$

Since $\langle\cdot|x\rangle y \in \mathcal{L}_f(E,F)$ for every $(x,y) \in E \times F$, we have that $\mathcal{F} \subset \mathcal{L}_f(E,F)$. Take $u \in \mathcal{L}_f(E,F)$ and let A be a finite orthonormal basis of $\operatorname{Im} u$ (Corollary 5.5.1.10 a $\Rightarrow$ c). Then

$$\operatorname{Im} u = A^{\perp\perp}$$

(Corollary 5.2.3.9). Hence, by Proposition 5.5.1.7 and Proposition 5.3.2.13 d),

$$u = \pi_{A^{\perp\perp}} \circ u = \left(\sum_{y \in A} \langle\cdot|y\rangle y\right) \circ u = \sum_{y \in A} (\langle\cdot|y\rangle y) \circ u =$$

$$= \sum_{y \in A} \langle\cdot|u^* y\rangle y \in \mathcal{F},$$

$$\mathcal{L}_f(E,F) \subset \mathcal{F}, \quad \mathcal{L}_f(E,F) = \mathcal{F}. \qquad \blacksquare$$

Proposition 5.5.1.12 (0) *Let E, F be Hilbert spaces and $\mathcal{F}$ a real vector subspace of $\mathcal{L}(E,F)$ such that $\mathcal{F} \neq \{0\}$ and*

$$w \circ u \circ v \in \mathcal{F}$$

for all $u \in \mathcal{F}$, $v \in \mathcal{L}_f(E)$, and $w \in \mathcal{L}_f(F)$. Then

$$\mathcal{L}_f(E,F) \subset \mathcal{F}.$$

If in addition $\mathcal{F}$ is closed, then

$$\mathcal{K}(E,F) \subset \mathcal{F}.$$

Take $(x,y) \in E \times F$ and $u \in \mathcal{F} \backslash \{0\}$. There is a $z \in E$ with $uz \neq 0$. By Proposition 5.3.2.13 c), e),

$$\mathcal{F} \ni \frac{1}{\|uz\|^2} (\langle\cdot|uz\rangle y) \circ u \circ (\langle\cdot|x\rangle z) =$$

$$= \frac{1}{\|uz\|^2} (\langle\cdot|uz\rangle y) \circ (\langle\cdot|x\rangle uz) =$$

$$= \frac{1}{\|uz\|^2}\langle uz|uz\rangle\langle\cdot|x\rangle y = \langle\cdot|x\rangle y\,,$$

so that, by Corollary 5.5.1.11,

$$\mathcal{L}_f(E,F) \subset \mathcal{F}\,.$$

The last assertion follows from the first one and Corollary 5.2.3.4. ■

Corollary 5.5.1.13 (0) *Let E be a Hilbert space and $\mathcal{F}$ a closed ideal of $\mathcal{L}(E)$ (of $\mathcal{K}(E)$).*

a) *If $\mathcal{F} \neq \{0\}$, then $\mathcal{K}(E) \subset \mathcal{F}$.*

b) *The C^*-algebra $\mathcal{K}(E)$ is simple (Proposition 5.3.1.17).*

c) *If $\mathbb{K} = \mathbb{R}$, then $\mathcal{K}(E)$ is a purely real C^*-algebra.*

a) follows from Proposition 5.5.1.12.
b) follows from a).
c) follows from b) and Proposition 4.3.5.3 a ⇒ b. ■

Remark. b) is a generalization of Example 2.1.4.7.

Proposition 5.5.1.14 (0) *If A is an orthonormal set in E and $x \in E$, then the following are equivalent:*

a) $\sum\limits_{y\in A}\langle x|y\rangle y = x\,.$

b) $z \in E \Rightarrow \sum\limits_{y\in A}\langle x|y\rangle\langle y|z\rangle = \langle x|z\rangle\,.$

c) $\sum\limits_{y\in A}|\langle x|y\rangle|^2 = \|x\|^2\,.$

a ⇒ b is obvious.
b ⇒ c is obtained by putting $z := x$.
c ⇒ a . Take $\varepsilon > 0$. Then, for some $B \in \mathfrak{P}_f(A)$,

$$\left|\|x\|^2 - \sum_{y\in C}|\langle x|y\rangle|^2\right| < \varepsilon\,,$$

whenever $C \in \mathfrak{P}_f(A)$, $B \subset C$. By the Bessel's Identity,

$$\left\|x - \sum_{y\in C}\langle x|y\rangle y\right\|^2 = \|x\|^2 - \sum_{y\in C}|\langle x|y\rangle|^2 < \varepsilon$$

for every $C \in \mathfrak{P}_f(A)$ with $B \subset C$. Thus

$$\sum_{y \in A} \langle x|y\rangle y = x \,. \qquad \blacksquare$$

Theorem 5.5.1.15 (0) *If A is an orthonormal set in the Hilbert space E, then the following are equivalent:*

a) *A is an orthonormal basis of E.*

b) $x \in E \Rightarrow \sum_{y \in A} \langle x|y\rangle y = x$ ***(Fourier Expansion of** x**).***

c) $x, y \in E \Rightarrow \sum_{z \in A} \langle x|z\rangle\langle z|y\rangle = \langle x|y\rangle$.

d) $x \in E \Rightarrow \sum_{z \in A} |\langle x|y\rangle|^2 = \|x\|^2$ ***(Parseval's Equation**, 1799).*

If these conditions are fulfilled then:

e) $u \in \mathcal{L}(E) \Rightarrow \|u\|^2 \leq \sum_{x \in A} \|ux\|^2$.

f) $u \in \mathcal{L}(E)_+ \Rightarrow \|u\| \leq \sum_{x \in A} \langle ux|x\rangle$.

a $\Rightarrow$ b. Put

$$F = \bigcup_{B \in \mathfrak{P}_f(A)} B^{\perp\perp} \,.$$

Then F is a vector subspace of E containing A (Proposition 5.2.2.2 a),b)). It follows that

$$F^{\perp} \subset A^{\perp} = \{0\} \,.$$

Hence, by Corollary 5.2.3.10 b $\Rightarrow$ a,

$$\overline{F} = E \,.$$

Take $\varepsilon > 0$. There is a $z \in F$ with

$$\|x - z\| < \varepsilon \,.$$

Choose $B \in \mathfrak{P}_f(A)$ with $z \in B^{\perp\perp}$. Take $C \in \mathfrak{P}_f(A)$ with $B \subset C$. Then $z \in C^{\perp\perp}$, so that

$$\left\| x - \sum_{y \in C} \langle x|y\rangle y \right\| = \|x - \pi_{C^{\perp\perp}} x\| \leq \|x - z\| < \varepsilon$$

(Proposition 5.5.1.7). Since ε is arbitrary,

$$\sum_{y\in A}\langle x|y\rangle y = x\,.$$

b $\Rightarrow$ a. Take $x \in A^{\perp}$. Then

$$x = \sum_{y\in A}\langle x|y\rangle y = 0\,.$$

Hence $A^{\perp} = \{0\}$.

b $\Leftrightarrow$ c $\Leftrightarrow$ d follows from Proposition 5.5.1.14.

e) Take $y \in E^{\#}$. By d),

$$\|u^{*}y\|^2 = \sum_{x\in A}|\langle x|u^{*}y\rangle|^2 = \sum_{x\in A}|\langle ux|y\rangle|^2 \leq \sum_{x\in A}\|ux\|^2\,.$$

Hence

$$\|u\|^2 = \|u^{*}\|^2 = \sup_{y\in E^{\#}}\|u^{*}y\|^2 \leq \sum_{x\in A}\|ux\|^2\,.$$

f) Put

$$v := u^{\frac{1}{2}}\,.$$

By e) and Proposition 4.2.4.2 b),

$$\|u\| = \|v\|^2 \leq \sum_{x\in A}\|vx\|^2 = \sum_{x\in A}\langle vx|vx\rangle =$$

$$= \sum_{x\in A}\langle v^2x|x\rangle = \sum_{x\in A}\langle ux|x\rangle\,. \qquad \blacksquare$$

Corollary 5.5.1.16 (0) *Let E be a Hilbert space, A an orthonormal set in E, and take $x \in E$. Then*

$$\pi_{A^{\perp\perp}}x = \sum_{y\in A}\langle x|y\rangle y\,.$$

A is an orthonormal basis of $A^{\perp\perp}$ (Proposition 5.5.1.2) and

$$x - \pi_{A^{\perp\perp}}x \in A^{\perp\perp\perp} = A^{\perp}$$

(Proposition 5.2.3.1 a $\Rightarrow$b, Proposition 5.2.2.2 d)). By Theorem 5.5.1.15 a $\Rightarrow$ b,

$$\pi_{A^{\perp\perp}}x = \sum_{y\in A}\langle \pi_{A^{\perp\perp}}x|y\rangle y = \sum_{y\in A}\langle x|y\rangle y\,. \qquad \blacksquare$$

Corollary 5.5.1.17 (0) *If A is an orthonormal basis of the Hilbert space E and if $B \subset A$ then*

$$(A\backslash B)^{\perp\perp} = B^{\perp}$$

and

$$E = B^{\perp\perp} \oplus (A\backslash B)^{\perp\perp}.$$

We have, successively, that

$$B \perp (A\backslash B),$$

$$B \subset (A\backslash B)^{\perp},$$

$$(A\backslash B)^{\perp\perp} \subset B^{\perp}.$$

Take $x \in B^{\perp}$. Then

$$x = \sum_{y \in A} \langle x|y\rangle y = \sum_{y \in A\backslash B} \langle x|y\rangle y \in (A\backslash B)^{\perp\perp}$$

(Fourier Expansion), so that

$$B^{\perp} = (A\backslash B)^{\perp\perp}$$

and

$$E = B^{\perp} \oplus B^{\perp\perp} = (A\backslash B)^{\perp\perp} \oplus B^{\perp\perp}$$

(Proposition 5.2.3.7). ■

Proposition 5.5.1.18 (Gram–Schmidt Orthonormalization, Gram, 1883, Schmidt, 1907) *Take $q \in \mathbb{N} \cup \{0, \infty\}$ and let $(x_n)_{n \in \mathbb{N}_q}$ be a linearly independent family in E. Then there is an orthonormal family $(y_n)_{n \in \mathbb{N}_q}$ such that for every $p \in \mathbb{N}_q$ the families $(x_n)_{n \in \mathbb{N}_p}$ and $(y_n)_{n \in \mathbb{N}_p}$ generate the same vector subspace of E. If E is complete, then $(y_n)_{n \in \mathbb{N}_q}$ is an orthonormal basis of E iff the vector subspace of E generated by $(x_n)_{n \in \mathbb{N}_q}$ is dense.*

We construct the family $(y_n)_{n \in \mathbb{N}_q}$ recursively. Take $p \in \mathbb{N}_q$ and assume the family $(y_n)_{n \in \mathbb{N}_{p-1}}$ has been constructed. Put

$$y := x_p - \sum_{n=1}^{p-1} \langle x_p|y_n\rangle y_n.$$

Then

$$\langle y|y_n\rangle = \langle x_p|y_n\rangle - \langle x_p|y_n\rangle = 0$$

for every $n \in \mathbb{N}_{p-1}$. Since $(x_n)_{n\in\mathbb{N}_p}$ is linearly independent, $y \neq 0$. Define

$$y_p := \frac{1}{\|y\|} y .$$

It is obvious that $(x_n)_{n\in\mathbb{N}_p}$ and $(y_n)_{n\in\mathbb{N}_p}$ generate the same vector subspace of E. This proves the existence of $(y_n)_{n\in\mathbb{N}_q}$.

The final assertion follows from

$$\{x_n \mid n \in \mathbb{N}_q\}^{\perp\perp} = \{y_n \mid n \in \mathbb{N}_q\}^{\perp\perp}$$

and Corollary 5.2.3.9. ■

Remark. The example from the Remark to Corollary 5.2.3.9 shows that we cannot drop the requirement of completeness in Proposition 5.5.1.18.

Proposition 5.5.1.19 (0) *Let $(x_\lambda)_{\lambda\in L}$, $(y_\lambda)_{\lambda\in L}$ be orthonormal families in the pre-Hilbert spaces E and F, respectively. Then*

$$\sum_{\lambda\in L}\sum_{\iota\in I} |\langle x_\lambda|a_\iota\rangle\langle b_\iota|y_\lambda\rangle| \leq \sum_{\iota\in I} \|a_\iota\|\,\|b_\iota\|$$

for any family $((a_\iota, b_\iota))_{\iota\in I}$ in $E \times F$.

We have

$$\sum_{\lambda\in L}\sum_{\iota\in I} |\langle x_\lambda|a_\iota\rangle\langle b_\iota|y_\lambda\rangle| = \sum_{\iota\in I}\sum_{\lambda\in L} |\langle x_\lambda|a_\iota\rangle\langle b_\iota|y_\lambda\rangle| \leq$$

$$\leq \sum_{\iota\in I} \left(\sum_{\lambda\in L} |\langle x_\lambda|a_\iota\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{\lambda\in L} |\langle b_\iota|y_\lambda\rangle|^2\right)^{\frac{1}{2}} \leq \sum_{\iota\in I} \|a_\iota\|\,\|b_\iota\|,$$

by Bessel's Inequality. ■

Proposition 5.5.1.20 (5) *Let I be an interval in $\mathbb{R}$, E a Hilbert space, A an orthonormal basis of E, and f a map $I \to E$ such that $\langle f|x\rangle$ is continuously differentiable for every $x \in A$ and that*

$$(\|\langle f|x\rangle'\|_\infty)_{x\in A} \in \ell^2(A),$$

where $\langle f|x\rangle'$ *denotes the derivative of*

$$I \longrightarrow \mathbb{K}, \quad t \longmapsto \langle f(t)|x\rangle .$$

Then f *is differentiable and*

$$f' = \sum_{x\in A}\langle f|x\rangle' x .$$

Take $t_0 \in I$ and $t \in I\backslash\{t_0\}$. Given $x \in A$, there is a $t_x \in I$ between t and t_0 such that

$$\frac{\langle f(t)|x\rangle - \langle f(t_0)|x\rangle}{t - t_0} = \langle f|x\rangle'(t_x) .$$

Then (Corollary 5.2.2.5)

$$\left\| \frac{1}{t-t_0}(f(t) - f(t_0)) - \sum_{x\in A}\langle f|x\rangle'(t_0)x \right\|^2 =$$

$$= \left\| \sum_{x\in A}(\langle f|x\rangle'(t_x) - \langle f|x\rangle'(t_0))x \right\|^2 =$$

$$= \sum_{x\in A} |\langle f|x\rangle'(t_x) - \langle f|x\rangle'(t_0)|^2$$

(Fourier Expansion and Parseval's Equation).

Take $\varepsilon > 0$. There is a $B \in \mathfrak{P}_f(A)$ such that

$$\sum_{x\in A\backslash B} \|\langle f|x\rangle'\|_\infty^2 < \frac{\varepsilon^2}{5} .$$

Furthermore, there is a $\delta > 0$ such that

$$\sum_{x\in B} |\langle f|x\rangle'(t_x) - \langle f|x\rangle'(t_0)|^2 < \frac{\varepsilon^2}{5}$$

for every $t \in U_\delta^I(t_0)$. It follows that

$$\left\| \frac{1}{t-t_0}(f(t) - f(t_0)) - \sum_{x\in A}\langle f|x\rangle'(t_0)x \right\|^2 \leq$$

$$\leq \sum_{x\in B} |\langle f|x\rangle'(t_x) - \langle f|x\rangle'(t_0)|^2 + \sum_{x\in A\backslash B} 4\|\langle f|x\rangle'\|_\infty^2 <$$

$$< \frac{\varepsilon^2}{5} + \frac{4\varepsilon^2}{5} = \varepsilon^2$$

for every $t \in U_\delta^I(t_0)$, i.e.

$$\lim_{\substack{t \to t_0 \\ t \in I \setminus \{t_0\}}} \frac{1}{t - t_0}(f(t) - f(t_0)) = \sum_{x \in A} (f|x)'(t_0)x \,. \qquad \blacksquare$$

Proposition 5.5.1.21 *Let F be a finite-dimensional vector subspace of the Hilbert space E. Then $\pi_F \mathcal{L}(E)\pi_F$ is a finite-dimensional C^*-subalgebra of $\mathcal{L}(E)$ and π_F is a unit of it.*

It is obvious that $\pi_F \mathcal{L}(E)\pi_F$ is a subalgebra of $\mathcal{L}(E)$ and that π_F is a unit of it. By Proposition 5.3.2.8 a $\Rightarrow$ b, $\pi_F \mathcal{L}(E)\pi_F$ is an involutive set of $\mathcal{L}(E)$.

Let A be a finite orthonormal basis of F (Corollary 5.5.1.10 a $\Rightarrow$ c). By Proposition 5.5.1.7,

$$\pi_F = \sum_{x \in A} \langle \cdot | x \rangle x \,.$$

By Proposition 5.3.2.13 c),e),

$$\pi_F u \pi_F = \left(\sum_{x \in A} \langle \cdot | x \rangle x \right) u \left(\sum_{y \in A} \langle \cdot | y \rangle y \right) =$$

$$= \left(\sum_{x \in A} \langle \cdot | x \rangle x \right) \left(\sum_{y \in A} \langle \cdot | y \rangle uy \right) =$$

$$= \sum_{x,y \in A} \langle uy | x \rangle \langle \cdot | y \rangle x$$

for every $u \in \mathcal{L}(E)$. Hence $(\langle \cdot | y \rangle x)_{(x,y) \in A^2}$ is an algebraic basis of $\pi_F \mathcal{L}(E)\pi_F$, which is therefore finite-dimensional. $\blacksquare$

Proposition 5.5.1.22 (0) *Let $(x_\iota)_{\iota \in I}$ be an orthonormal basis of E, F a Hilbert space, $(y_\iota)_{\iota \in I}$ an orthonormal family in F and take $f \in \ell^\infty(I)$. Then there is a unique $u \in \mathcal{L}(E, F)$ such that*

$$ux_\iota = f(\iota)y_\iota$$

for every $\iota \in I$. Moreover,

$$\|x\| \inf_{\iota \in I} |f(\iota)| \leq \|ux\| \leq \|f\|_\infty \|x\|$$

for every $x \in E$ *and*

$$\|u\| = \|f\|_\infty .$$

If $(y_\iota)_{\iota \in I}$ *is an orthonormal basis of* F, E *is complete, and*

$$\inf_{\iota \in I} |f(\iota)| > 0 \qquad \left(\inf_{\iota \in I} |f(\iota)| = \sup_{\iota \in I} |f(\iota)| = 1 \right) ,$$

then u *is an isomorphism (isometry) of Banach spaces.*

The first assertion follows immediately from Corollary 5.2.3.20. If $(y_\iota)_{\iota \in I}$ is an orthonormal basis of F and

$$\inf_{\iota \in I} |f(\iota)| > 0$$

then

$$u(E) \supset \{y_\iota | \iota \in I\} ,$$

so that

$$u(E)^{\perp\perp} \supset \{y_\iota | \iota \in I\}^{\perp\perp} = F$$

and $u(E)$ is dense in F (Corollary 5.2.3.9). By Proposition 1.2.1.18 d), u is an isomorphism of Banach spaces. If in addition,

$$\inf_{\iota \in I} |f(\iota)| = \sup_{\iota \in I} |f(\iota)| = 1 ,$$

then by the first assertion, u preserves the norms, i.e., it is an isometry of Banach spaces. ■

Definition 5.5.1.23 (0) *Let* H *be a Hilbert space and* E *an involutive closed subalgebra of* $\mathcal{L}(H)$. *A representation of* E *is called an* ***identical representation*** *if it is equivalent to the representation* (H, φ), *where* φ *is the inclusion map* $E \to \mathcal{L}(H)$.

Theorem 5.5.1.24 (0) (Naimark, 1948) *If* E *is a Hilbert space, then any representation of* $\mathcal{K}(E)$ *is the Hilbert sum of identical representations and a* 0*-representation.*

Let (H, φ) be a representation of $\mathcal{K}(E)$ and A an orthonormal basis of E (Corollary 5.5.1.4). Given $x, y \in A$, put

$$p_{xy} := \varphi(\langle\cdot|y\rangle x)\,, \quad H_x := \operatorname{Im} p_{xx}\,.$$

Then

$$p_{xy}^* = \varphi((\langle\cdot|y\rangle x)^*) = \varphi(\langle\cdot|x\rangle y) = p_{yx}\,,$$

$$p_{xy}p_{ab} = \varphi((\langle\cdot|y\rangle x)(\langle\cdot|b\rangle a)) = \varphi(\langle a|y\rangle\langle\cdot|b\rangle x) = \langle a|y\rangle p_{xb}$$

for all $x, y, a, b \in A$ (Proposition 5.3.2.13 b),e)). It follows that for each $x \in A$, p_{xx} is the orthogonal projection of H on H_x (Proposition 5.3.2.8 b $\Rightarrow$ a) and that

$$H_x \perp H_y\,,$$

whenever $x, y \in A$ are distinct. Moreover,

$$p_{xy}p_{xy}^* = p_{xx}\,, \quad p_{xy}^* p_{xy} = p_{yy}\,,$$

$$\|p_{xx}\| = \|p_{xy}p_{xy}^*\| = \|p_{xy}^*\|^2 = \|p_{xy}\|^2 = \|p_{xy}^* p_{xy}\| = \|p_{yy}\|$$

for all $x, y \in A$ (Theorem 5.3.1.4). Hence either

$$p_{xy} = 0$$

for all $x, y \in A$ or

$$p_{xy} \neq 0\,, \quad H_x \neq \{0\}$$

for all $x, y \in A$.

Take $u \in \mathcal{K}(E)$ and $B, C \in \mathfrak{P}_f(A)$. Then

$$\pi_{B^{\perp\perp}} u \pi_{C^{\perp\perp}} = \left(\sum_{x\in B}\langle\cdot|x\rangle x\right) u \left(\sum_{y\in C}\langle\cdot|y\rangle y\right) = \sum_{(x,y)\in B\times C}\langle uy|x\rangle\langle\cdot|y\rangle x$$

(Proposition 5.5.1.7, Proposition 5.3.2.13 c),e)) and therefore

$$\varphi\left(\pi_{B^{\perp\perp}} u \pi_{C^{\perp\perp}}\right) = \sum_{(x,y)\in B\times C}\langle uy|x\rangle p_{xy}\,.$$

Thus

$$\varphi(u) = 0$$

whenever $p_{xy} = 0$ for all $x, y \in A$ (Corollary 5.3.2.16), i.e. (H, φ) is a 0-representation in this case. Suppose that (H, φ) is not a 0-representation of $\mathcal{K}(E)$. Then

$$p_{xy} \neq 0\,, \quad H_x \neq \{0\}$$

for all $x, y \in A$.

Take $a \in A$ and $\xi_a \in H_a$ with $\|\xi_a\| = 1$. Given $x \in A$, put

$$\xi_x = p_{xa}\xi_a$$

and let K denote the closed vector subspace of H generated by $\{\xi_x | x \in A\}$. Given $x, y, z \in A$,

$$p_{yz}\xi_x = p_{yz}p_{xa}\xi_a = \langle x|z\rangle p_{ya}\xi_a = \langle x|z\rangle \xi_y\,.$$

It follows that for $x, y \in A$,

$$p_{xy}(K) \subset K\,,$$

$$p_{xy}\xi_y = \xi_x\,,$$

$$\|\xi_x\| = 1\,,$$

$$\xi_x = p_{xx}\xi_x \in H_x\,.$$

In particular, $(\xi_x)_{x\in A}$ is an orthonormal basis of K. By the above formula,

$$\varphi\left(\pi_{B^{\perp\perp}} u \pi_{C^{\perp\perp}}\right)(K) \subset K$$

for all $u \in \mathcal{K}(E)$ and $B, C \in \mathfrak{P}_f(A)$. It follows that

$$(\varphi u)(K) \subset K$$

for every $u \in \mathcal{K}(E)$ (Corollary 5.3.2.16), i.e. K is $\varphi(\mathcal{K}(E))$-invariant. Let (K, ψ) be the compression of (H, φ) to K (Proposition 5.4.2.8).

Let v be the isometry $E \to K$ defined by

$$vx = \xi_x$$

for $x \in A$ (Proposition 5.5.1.22). Take $u \in \mathcal{K}(E)$ and $x, y \in A$. Then

$$\langle(\psi u)\xi_x|\xi_y\rangle = \langle(\psi u)p_{xa}\xi_a|p_{ya}\xi_a\rangle =$$

$$= \langle (\psi u)(\psi\langle\cdot|a\rangle x)\xi_a | (\psi\langle\cdot|a\rangle y)\xi_a\rangle =$$

$$= \langle \psi((\langle\cdot|y\rangle a)u(\langle\cdot|a\rangle x))\xi_a|\xi_a\rangle =$$

$$= \langle \psi(\langle ux|y\rangle\langle\cdot|a\rangle a)\xi_a|\xi_a\rangle = \langle ux|y\rangle\langle p_{aa}\xi_a|\xi_a\rangle =$$

$$= \langle uv^*\xi_x|v^*\xi_y\rangle = \langle vuv^*\xi_x|\xi_y\rangle$$

(Proposition 5.3.2.13 b),e)). Since x and y are arbitrary,

$$\psi u = v \circ u \circ v^*$$

(Theorem 5.5.1.15 a $\Rightarrow$ b). Hence (K, ψ) is an identical representation of $\mathcal{K}(E)$.

Let Φ be the set of sets $\mathfrak{K}$ with the following properties:

1) The elements of $\mathfrak{K}$ are $\varphi(\mathcal{K}(E))$-invariant closed vector subspaces K of H, such that the compression of (H, φ) to K is an identical representation of $\mathcal{K}(E)$.

2) If K_1, K_2 are distinct elements of $\mathfrak{K}$, then $K_1 \perp K_2$.

Let Φ be ordered by inclusion and let $\mathfrak{K}$ be a maximal element of Φ (Zorn). Put

$$L := \left(\bigcup_{K\in\mathfrak{K}} K\right)^{\perp}.$$

By Proposition 5.4.2.8, L is $\varphi(\mathcal{K}(E))$-invariant. Assume that the compression (L, ψ) of (H, φ) to L is not a 0-representation of $\mathcal{K}(E)$. By the above considerations, there is a $\varphi(\mathcal{K}(E))$-invariant closed vector subspace K of L such that the compression of (L, ψ) to K is an identical representation of $\mathcal{K}(E)$. Then $\mathfrak{K} \cup \{K\} \in \Phi$ and this contradicts the maximality of $\mathfrak{K}$. Hence (L, ψ) is the 0-representation of $\mathcal{K}(E)$. It is easy to see that (H, φ) is the Hilbert sum of (L, ψ) and of the compressions of (H, φ) to the $K \in \mathfrak{K}$. ■

Proposition 5.5.1.25 *Let E be a Hilbert space, $\mathcal{F}$ a C^*-subalgebra of $\mathcal{L}(E)$ containing $\mathcal{K}(E)$, (H, φ) a representation of $\mathcal{F}$, and K a $\varphi(\mathcal{K}(E))$-invariant closed vector subspace of H such that the compression of $(H, \varphi|\mathcal{K}(E))$ to K is an identical representation. Then K is $\varphi(\mathcal{F})$-invariant and the compression of (H, φ) to K is an identical representation.*

Let (K,ψ) be the compression of $(H,\varphi|\mathcal{K}(E))$ to K (Proposition 5.4.2.8) and $v : E \to K$ an isometry such that

$$\psi u = v \circ u \circ v^*$$

for every $u \in \mathcal{K}(E)$.

Take $u \in \mathcal{F}$ and $x, y \in E$ with $\|x\| = 1$. It follows successively that

$$\psi(\langle\cdot|x\rangle y) = v \circ (\langle\cdot|x\rangle y) \circ v^* = \langle\cdot|vx\rangle vy\,,$$

$$\psi(u(\langle\cdot|x\rangle x)) = \psi(\langle\cdot|x\rangle ux) = \langle\cdot|vx\rangle vux$$

(Proposition 5.3.2.13 c),d)),

$$\psi(\langle\cdot|x\rangle x)vx = \langle vx|vx\rangle vx = vx\,,$$

$$(\varphi u)vx = \varphi(u)\varphi(\langle\cdot|x\rangle x)vx = \varphi(u(\langle\cdot|x\rangle x))vx =$$

$$= \langle vx|vx\rangle vux = vux \in K\,.$$

Since x and u are arbitrary, K is $\varphi(\mathcal{F})$-invariant. Let (K,θ) be the compression of (H,φ) to K (Proposition 5.4.2.8). Then

$$\langle(\varphi u)vx|vy\rangle = \langle vux|vy\rangle = \langle vuv^*vx|vy\rangle$$

for all $x, y \in E$, so that

$$\theta u = v \circ u \circ v^*\,.$$

Thus (K,θ) is an identical representation. ■

Proposition 5.5.1.26 (0) *Let E, F be infinite-dimensional Hilbert spaces. Take $u \in \mathcal{L}(E,F)$ and let $\dot{u}$ be the equivalence class of u in*

$$\mathcal{L}(E,F)/\mathcal{K}(E,F)\,.$$

Let Φ be the set of orthonormal sequences in E and Ψ the set of filters on E which contain $E^\#$ and converge weakly to 0. Then

$$\|\dot{u}\| = \sup_{x\in\Phi}\left(\limsup_{n\to\infty}\|ux_n\|\right) = \sup_{\mathfrak{F}\in\Psi}\left(\limsup_{x,\mathfrak{F}}\|ux\|\right)\,.$$

Take $\alpha > \|\dot{u}\|$ and $\mathfrak{F} \in \Psi$. There is a $v \in \mathcal{K}(E,F)$ such that

$$\|u+v\| < \alpha .$$

By Proposition 3.1.1.20 a $\Rightarrow$ c, $v(\mathfrak{F})$ converges to 0. Since

$$\|ux\| \leq \|(u+v)x\| + \|vx\| < \alpha + \|vx\|$$

for every $x \in E^{\#}$,

$$\limsup_{x,\mathfrak{F}} \|ux\| \leq \alpha .$$

Since α and $\mathfrak{F}$ are arbitrary, it follows that

$$\sup_{\mathfrak{F}\in\Psi} \left(\limsup_{x,\mathfrak{F}} \|ux\| \right) \leq \|\dot{u}\| .$$

The inequality

$$\sup_{x\in\Phi} \left(\limsup_{n\to\infty} \|ux_n\| \right) \leq \sup_{\mathfrak{F}\in\Psi} \left(\limsup_{x,\mathfrak{F}} \|ux\| \right)$$

follows from Corollary 5.5.1.9 and the inequality

$$\|\dot{u}\| \leq \sup_{x\in\Phi} \left(\limsup_{n\to\infty} \|ux_n\| \right)$$

follows from Proposition 5.2.3.21. ■

Proposition 5.5.1.27 (7) *Let*

$$\mathcal{L}(E) \longrightarrow \mathcal{L}(E)/\mathcal{K}(E) , \quad u \longmapsto \dot{u}$$

be the quotient map. Take $x' \in (\mathcal{L}(E)/\mathcal{K}(E))'_+$ *and let* $\mathfrak{F}$ *be the set of closed vector subspaces* F *of* E *such that*

$$x'(\dot{\pi}_F) = 0 .$$

Then $\{\dot{\pi}_F | F \in \mathfrak{F}\}$ *is upward directed and has supremum* $\dot{\pi}_E$. *In particular,*

$$(\mathcal{L}(E)/\mathcal{K}(E))^{\pi} = \{0\}$$

and $\mathcal{L}(E)/\mathcal{K}(E)$ *is not a* W^**-algebra.*

Take $F, G \in \mathfrak{F}$ and let H be the closed vector subspace of E generated by $F \cup G$. Then

$$H \cap F^{\perp} \subset G,$$

$$\pi_H - \pi_F = \pi_{H \cap F^{\perp}} \leq \pi_G$$

(Proposition 5.2.3.13 a $\Rightarrow$ f, Corollary 5.3.3.8),

$$\pi_H \leq \pi_F + \pi_G,$$

$$0 \leq x'(\dot{\pi}_H) \leq x'(\dot{\pi}_F) + x'(\dot{\pi}_G) = 0,$$

$$H \in \mathfrak{F}.$$

Hence $\mathfrak{F}$ and $\{\dot{\pi}_F | F \in \mathfrak{F}\}$ are upward directed.

Let x be a lower bound for $\{\dot{\pi}_E - \dot{\pi}_F | F \in \mathfrak{F}\}$ in $(\mathcal{L}(E)/\mathcal{K}(E))_+$. By Proposition 4.2.5.5 d), there is a $u \in \mathcal{L}(E)_+$ with

$$\dot{u} = x.$$

Assume that $u \notin \mathcal{K}(E)$. By Proposition 5.2.3.21, there is an orthonormal sequence $(x_n)_{n \in \mathbb{N}}$ in E such that

$$\alpha := \inf_{n \in \mathbb{N}} \|u x_n\| > 0.$$

By Lemma 1.1.2.17, there is an uncountable set $\mathfrak{A}$ of infinite subsets of $\{x_n \mid n \in \mathbb{N}\}$ such that $A \cap B$ is finite for distinct $A, B \in \mathfrak{A}$. Then

$$\pi_{A^{\perp\perp}} \pi_{B^{\perp\perp}} = \pi_{(A \cap B)^{\perp\perp}} \in \mathcal{K}(E)$$

(Corollary 5.2.3.9),

$$\dot{\pi}_{A^{\perp\perp}} \dot{\pi}_{B^{\perp\perp}} = 0$$

for distinct $A, B \in \mathfrak{A}$. Thus

$$\sum_{A \in Q} \dot{\pi}_{A^{\perp\perp}} = \bigvee_{A \in Q} \dot{\pi}_{A^{\perp\perp}} \leq \dot{\pi}_E,$$

$$\sum_{A \in Q} x'\left(\dot{\pi}_{A^{\perp\perp}}\right) = x'\left(\sum_{A \in Q} \dot{\pi}_{A^{\perp\perp}}\right) \leq x'(\dot{\pi}_E)$$

for every $Q \in \mathfrak{P}_f(\mathfrak{A})$ (Corollary 4.2.7.5). Hence there is an $A \in \mathfrak{A}$ with

$$x'(\dot{\pi}_{A^{\perp\perp}}) = 0\,.$$

It follows that

$$A^{\perp\perp} \in \mathfrak{F}\,,$$

$$x = x\left(\dot{\pi}_E - \dot{\pi}_{A^{\perp\perp}}\right) = x - x\dot{\pi}_{A^{\perp\perp}}$$

(Proposition 4.2.7.1 d $\Rightarrow$ f),

$$\widehat{u\pi_{A^{\perp\perp}}} = x\dot{\pi}_{A^{\perp\perp}} = 0\,,$$

$$u\pi_{A^{\perp\perp}} \in \mathcal{K}(E)\,,$$

$$0 < \alpha \leq \inf_{n\in A} \|ux_n\| = \inf_{n\in A} \|u\pi_{A^{\perp\perp}}x_n\| = 0$$

(Propositions 5.5.1.7 and 5.2.3.21), which is a contradiction. Hence $u \in \mathcal{K}(E)$, $x = 0$ and 0 is the infimum of $\{\dot{\pi}_E - \dot{\pi}_F | F \in \mathfrak{F}\}$ in $\mathcal{L}(E)/\mathcal{K}(E)$ (Corollary 4.2.2.22). Hence $\dot{\pi}_E$ is the supremum of $\{\dot{\pi}_F | F \in \mathfrak{F}\}$ in $\mathcal{L}(E)/\mathcal{K}(E)$.

By Corollary 4.4.4.3 a $\Rightarrow$ b, $\mathcal{L}(E)/\mathcal{K}(E)$ is not a W^*-algebra. ■

Proposition 5.5.1.28 (7) *Let $(n_\iota)_{\iota\in I}$ be a finite family in $\mathbb{N}$, E the C^*-direct sum of the family $(\mathcal{L}(\mathbb{K}^{n_\iota}))_{\iota\in I}$, and $(u_\lambda)_{\lambda\in L}$ a summable family in E_+. Then*

$$\sum_{\lambda\in L} \|u_\lambda\| \leq \left(\sum_{\iota\in I} n_\iota\right) \left\|\sum_{\lambda\in L} u_\lambda\right\|\,.$$

This inequality is sharp.

Case 1 Card $I = 1$

By Theorem 5.5.1.15 f),

$$\|u_\lambda\| \leq \sum_{k=1}^{n_\iota} \langle u_\lambda e_k | e_k\rangle$$

for every $\lambda \in L$. It follows

$$\sum_{\lambda\in L}\|u_\lambda\| \le \sum_{\lambda\in L}\sum_{k=1}^{n_\iota}\langle u_\lambda e_k|e_k\rangle = \sum_{k=1}^{n_\iota}\sum_{\lambda\in L}\langle u_\lambda e_k|e_k\rangle =$$

$$= \sum_{k=1}^{n_\iota}\left\langle\left(\sum_{\lambda\in L}u_\lambda\right)e_k\,\middle|\,e_k\right\rangle \le \sum_{k=1}^{n_\iota}\left\|\sum_{\lambda\in L}u_\lambda\right\| = n_\iota\left\|\sum_{\lambda\in L}u_\lambda\right\| .$$

Case 2 I arbitrary

Put

$$(u_{\lambda\iota})_{\iota\in I} := u_\lambda$$

for every $\lambda \in L$. Then by Case 1,

$$\sum_{\lambda\in L}\|u_\lambda\| = \sum_{\lambda\in L}\sup_{\iota\in I}\|u_{\lambda\iota}\| \le \sum_{\lambda\in L}\sum_{\iota\in I}\|u_{\lambda\iota}\| =$$

$$= \sum_{\iota\in I}\sum_{\lambda\in L}\|u_{\lambda\iota}\| \le \sum_{\iota\in I} n_\iota\|\sum_{\lambda\in L}u_{\lambda\iota}\| \le \sum_{\iota\in I} n_\iota\|\sum_{\lambda\in L}u_\lambda\| =$$

$$= \left(\sum_{\iota\in I} n_\iota\right)\left\|\sum_{\lambda\in L}u_\lambda\right\| .$$

Now we show that the inequality is sharp. For $\iota \in I$ and $k \in \mathbb{N}_{n_\iota}$ define

$$p_{\iota k} := [\delta_{ij}\delta_{jk}]_{i,j\in\mathbb{N}_{n_\iota}} \in \mathcal{L}(\ddot{\imath}\zeta^{n_\iota}) ,$$

$$u_{\iota k} := (\delta_{\lambda\iota}p_{\lambda k})_{\lambda\in I} .$$

Then

$$\|u_{\iota k}\| = 1$$

for every $\iota \in I$ and $k \in \mathbb{N}_{n_\iota}$ and

$$\sum_{\iota\in I}\sum_{k\in\mathbb{N}_{n_\iota}} u_{\iota k} = 1 . \qquad \blacksquare$$

Proposition 5.5.1.29 *Let $(H_\iota)_{\iota\in I}$, $(K_\lambda)_{\lambda\in L}$ be families of Hilbert spaces, E and F the C^*-direct products (sums) of the families $(\mathcal{K}(H_\iota))_{\iota\in I}$ and $(\mathcal{K}(K_\lambda))_{\lambda\in L}$, respectively. Let $u : E \to F$ be an isomorphism of C^*-algebras. Then there is a bijective map $\varphi : L \to I$ and for each $\lambda \in L$ an isomorphism of Hilbert spaces $u_\lambda : H_{\varphi(\lambda)} \to K_\lambda$ such that*

$$ux = \left(u_\lambda \circ x_{\varphi(\lambda)} \circ u_\lambda^*\right)_{\lambda\in L}$$

for every $x \in E$.

By Corollary 5.5.1.13 b), $\mathcal{K}(H_\iota)$ and $\mathcal{K}(K_\lambda)$ are simple for any $\iota \in I$ and $\lambda \in L$. For $\iota \in I$ and $\lambda \in L$, put

$$E_\iota := \{x \in E | \iota' \in I \backslash \{\iota\} \Longrightarrow x_{\iota'} = 0\},$$

$$F_\iota := \{y \in F | \lambda' \in L \backslash \{\lambda\} \Longrightarrow y_{\lambda'} = 0\}.$$

By Proposition 4.3.5.5, there is a bijective map $\psi : I \to L$ such that

$$F_{\psi(\iota)} = u(E_\iota)$$

for every $\iota \in I$. We define $\varphi := \psi^{-1}$ and for each $\lambda \in L$

$$v_\lambda : \mathcal{K}(H_{\varphi(\lambda)}) \longrightarrow \mathcal{K}(K_\lambda), \quad x \longmapsto ux,$$

where we have identified $\mathcal{K}(H_{\varphi(\lambda)})$ canonically with $E_{\varphi(\lambda)}$ and $\mathcal{K}(K_\lambda)$ with F_λ. Then for each $\lambda \in \Lambda$, v_λ is an isomorphism of C^*-algebras. Take $\lambda \in L$. By Theorem 5.5.1.24, there is an isomorphism of Hilbert spaces $u_\lambda : H_{\varphi(\lambda)} \to K_\lambda$ such that

$$v_\lambda x = u_\lambda \circ x \circ u_\lambda^*$$

for every $x \in \mathcal{K}(H_{\varphi(\lambda)})$. We get

$$ux = (u_\lambda \circ x_{\varphi(\lambda)} \circ u_\lambda^*)_{\lambda \in L}$$

for every $x \in E$. ■

Proposition 5.5.1.30 *Let E be a complex Hilbert space. Denote by F the underlying real vector space of E endowed with the scalar product*

$$F \times F \longrightarrow \mathbb{R}, \quad (x, y) \longmapsto \langle\langle x|y\rangle\rangle := \operatorname{re}\langle x|y\rangle$$

and put

$$\widetilde{u} : E \longrightarrow E, \quad x \longmapsto \frac{1}{2}(ux - iu(ix))$$

for every $u \in \mathcal{L}(F)$.

a) *F is a real Hilbert space and the canonical norms of E and F coincide.*

b) *$\mathcal{L}(E)$ is a unital real C^*-subalgebra of $\mathcal{L}(F)$.*

c) $\widetilde{u} \in \mathcal{L}(E)$ *for every* $u \in \mathcal{L}(F)$ *and the map*

$$\mathcal{L}(F) \longrightarrow \mathcal{L}(F), \quad u \longmapsto \widetilde{u}$$

is an involutive operator of norm 1 *and a projection of* $\mathcal{L}(F)$ *onto* $\mathcal{L}(E)$.

d) *If* A *is an orthonormal set (basis) of* E, *then* $A \cup iA$ *is an orthonormal set (basis) of* F.

a),b), and d) are easy to see.

c) Take $u \in \mathcal{L}(F)$. By Lemma 1.3.1.5, $\widetilde{u} \in \mathcal{L}(E)$. For every $x \in E^{\#}$,

$$\|\widetilde{u}x\| \leq \frac{1}{2}(\|ux\| + \|u(ix)\|) \leq \frac{1}{2}(\|u\| + \|u\|) = \|u\|$$

so that

$$\|\widetilde{u}\| \leq \|u\|.$$

Moreover, if $u \in \mathcal{L}(E)$ then

$$\widetilde{u}x = \frac{1}{2}(ux - iu(ix)) = \frac{1}{2}(ux + ux) = ux$$

for every $x \in E$ so that

$$\widetilde{u} = u.$$

Hence our map is an operator of norm 1 and a projection of $\mathcal{L}(F)$ onto $\mathcal{L}(E)$.

We want to show now that

$$\widetilde{u^*} = \widetilde{u}^*$$

for every $u \in \mathcal{L}(F)$. Take $x, y \in E$. Then

$$\mathrm{re}\,\langle \widetilde{u}x|y\rangle = \frac{1}{2}\mathrm{re}\,\langle ux - iu(ix)|y\rangle =$$

$$= \frac{1}{2}\mathrm{re}\,(\langle ux|y\rangle - i\langle u(ix)|y\rangle) =$$

$$= \frac{1}{2}\mathrm{re}\,(\langle ux|y\rangle + \langle u(ix)|iy\rangle) =$$

$$= \frac{1}{2}(\langle\langle ux|y\rangle\rangle + \langle\langle u(ix)|iy\rangle\rangle) =$$

$$= \frac{1}{2}(\langle\langle x|u^*y\rangle\rangle + \langle\langle ix|u^*(iy)\rangle\rangle) =$$

$$= \frac{1}{2}\mathrm{re}\,(\langle x|u^*y\rangle + \langle ix|u^*(iy)\rangle) =$$

$$= \frac{1}{2}\mathrm{re}\,\langle x|u^*y - iu^*(iy)\rangle = \mathrm{re}\,\langle x|\widetilde{u^*}y\rangle\,.$$

It follows

$$\mathrm{im}\,\langle \tilde{u}x|y\rangle = \mathrm{re}\,(-i\langle \tilde{u}x|y\rangle) = \mathrm{re}\,\langle \tilde{u}x|iy\rangle =$$

$$= \mathrm{re}\,\langle x|\widetilde{u^*}(iy)\rangle = \mathrm{re}\,\langle x|i\widetilde{u^*}y\rangle =$$

$$= \mathrm{re}\,(-i\langle x|\widetilde{u^*}y\rangle) = \mathrm{im}\,\langle x|\widetilde{u^*}y\rangle\,,$$

$$\langle \tilde{u}x|y\rangle = \langle x|\widetilde{u^*}y\rangle\,,$$

$$\tilde{u}^* = \widetilde{u^*}\,. \qquad \blacksquare$$

5.5.2 Hilbert Dimension

Proposition 5.5.2.1 (0) *Let A be an orthonormal basis of E and B a subset of E. If*

$$B^{\perp} = \{0\},$$

then

$$\operatorname{Card} A \leq \operatorname{Card} B.$$

First suppose B is finite. It follows from $B^{\perp} = \{0\}$, that $B^{\perp\perp} = E$, i.e. E is the vector subspace of E generated by B (Corollary 5.2.3.9). Hence E is finite-dimensional and A is an algebraic basis of E (Corollary 5.5.1.10 a $\Rightarrow$ d). Thus

$$\operatorname{Card} A \leq \operatorname{Card} B.$$

Now let B be infinite. Consider

$$f : B \longrightarrow \mathfrak{P}(A), \quad y \longmapsto \{x \in A | \langle x|y\rangle \neq 0\}.$$

By Bessel's Inequality,

$$\sum_{x \in A} |\langle x|y\rangle|^2 \leq \|y\|^2$$

for every $y \in B$. Thus $f(y)$ is countable. From

$$A \setminus \bigcup_{y \in B} f(y) \subset B^{\perp} = \{0\}$$

it follows that

$$A \subset \bigcup_{y \in B} f(y).$$

Hence

$$\operatorname{Card} A \leq \operatorname{Card} (B \times \aleph_0) = \operatorname{Card} B. \qquad \blacksquare$$

Corollary 5.5.2.2 (0) *All orthonormal bases of E have the same cardinality. This cardinal number (Corollary 5.5.1.4) is called* ***the Hilbert dimension of*** *E. It is less than the algebraic dimension of E. If E is infinite-dimensional, then its Hilbert dimension is equal to its topological cardinality.*

The first assertion follows from Proposition 5.5.2.1. Let A be an algebraic basis of E and take $x \in A^{\perp}$. There is an $f \in \mathbb{K}^{(A)}$ with

$$x = \sum_{y \in A} f(y) y .$$

It follows that $x \in A^{\perp\perp}$ and $x = 0$ (Proposition 5.2.2.2 a),b),e)). Hence

$$A^{\perp} = \{0\}$$

and the second assertion follows from Proposition 5.5.2.1.

In order to prove the final assertion, let A be an orthonormal basis of E. By Corollary 5.5.1.10 c $\Rightarrow$ a, A is infinite. For every $B \in \mathfrak{P}_f(A)$, let B' be a countable dense set of $B^{\perp\perp}$ (Corollary 5.2.3.9). Put

$$C := \bigcup_{B \in \mathfrak{P}_f(A)} B' .$$

Then

$$\operatorname{Card} C \leq \aleph_0 \times \operatorname{Card} \mathfrak{P}_f(A) = \aleph_0 \times \operatorname{Card} A = \operatorname{Card} A$$

and

$$\overline{C} \supset \bigcup_{B \in \mathfrak{P}_f(A)} \overline{B}' = \bigcup_{B \in \mathfrak{P}_f(B)} B^{\perp\perp} .$$

Thus C is dense in E.

Now let D be a dense set of E. There is a map $f : A \to D$ such that

$$\|f(x) - x\| < \frac{1}{2}$$

for every $x \in A$. Then, for all distinct $x, y \in A$,

$$\sqrt{2} = \|x - y\| \leq \|x - f(x)\| + \|f(x) - f(y)\| + \|f(y) - y\| < \|f(x) - f(y)\| + 1 ,$$

$$\|f(x) - f(y)\| \geq \sqrt{2} - 1 > 0 .$$

Hence f is injective and therefore

$$\operatorname{Card} A \leq \operatorname{Card} D .$$

Thus the topological cardinality of E is equal to $\operatorname{Card} A$, i.e. to the Hilbert dimension of E. ■

Corollary 5.5.2.3 (0) *Let E, F be Hilbert spaces and $u \in \mathcal{L}(E,F)$. If u is injective (if $\operatorname{Im} u$ is dense), then the Hilbert dimension of E is smaller (greater) then the Hilbert dimension of F.*

Let A, B be orthonormal bases of E and F, respectively (Corollary 5.5.1.4).

Suppose that u is injective. Take $x \in u^*(B)^\perp$. Then

$$\langle ux|y\rangle = \langle x|u^*y\rangle = 0$$

for every $y \in B$. We deduce successively that $ux \in B^\perp$, $ux = 0$, $x = 0$, $u^*(B)^\perp = \{0\}$,

$$\operatorname{Card} A \leq \operatorname{Card} u^*(B) \leq \operatorname{Card} B$$

(Proposition 5.5.2.1).

Now let $\operatorname{Im} u$ be dense. Take $y \in u(A)^\perp$. Then

$$\langle x|u^*y\rangle = \langle ux|y\rangle = 0$$

for every $x \in A$. We deduce successively that $u^*y \in A^\perp$, $u^*y = 0$,

$$y \in \operatorname{Ker} u^* = (\operatorname{Im} u^{**})^\perp = (\operatorname{Im} u)^\perp = \{0\}$$

(Proposition 5.3.2.4, Proposition 5.3.1.5 a), Corollary 5.2.3.10 a $\Rightarrow$ b),

$$u(A)^\perp = \{0\},$$

$$\operatorname{Card} B \leq \operatorname{Card} u(A) \leq \operatorname{Card} A$$

(Proposition 5.5.2.1). ■

Example 5.5.2.4 *Let $(E_\iota)_{\iota\in I}$ be a family of pre-Hilbert spaces, E its Hilbert sum, and for each $\iota \in I$ let A_ι be an orthonormal basis of E_ι and u_ι the canonical embedding of E_ι into E. Then $\bigcup_{\iota\in I} u_\iota(A_\iota)$ is an orthonormal basis of E and*

$$\sum_{\iota\in I} \operatorname{Card} A_\iota$$

is the Hilbert dimension of E. ■

Theorem 5.5.2.5 (0) *If E is a Hilbert space and A is an orthonormal basis of E, then the map*

$$E \longrightarrow \ell^2(A), \quad x \longmapsto (\langle x|y\rangle)_{y\in A}$$

(Theorem 5.5.1.15 a $\Rightarrow$ d) is an isometry with inverse

$$\ell^2(A) \longrightarrow E, \quad f \longmapsto \sum_{x\in A} f(x)x$$

(Corollary 5.2.2.5).

Consider

$$u : E \longrightarrow \ell^2(A), \quad x \longmapsto (\langle x|y\rangle)_{y\in A},$$

$$v : \ell^2(A) \longrightarrow E, \quad f \longmapsto \sum_{x\in A} f(x)x.$$

By Parseval's equation,

$$\|ux\|^2 = \sum_{y\in A} |\langle x|y\rangle|^2 = \|x\|^2$$

for every $x \in E$. Thus u preserves the norms. Since

$$uvf = f$$

for every $f \in \ell^2(A)$, u is surjective and v is its inverse. ■

Corollary 5.5.2.6 (0) *Let E and F be Hilbert spaces. The following are equivalent:*

a) *E, F are isometric.*

b) *the Banach spaces E, F are isomorphic.*

c) *E and F have the same Hilbert dimension.*

a $\Rightarrow$ b is trivial.
b $\Rightarrow$ c follows from Corollary 5.5.2.3.
c $\Rightarrow$ a follows from Corollary 5.5.1.4 and Theorem 5.5.2.5. ■

Remark. Identifying isometric Hilbert spaces (Corollary 5.1.1.4), the map

$$\aleph \longmapsto \ell^2(\aleph)$$

is a bijection from the class of cardinal numbers on the class of Hilbert spaces (Theorem 5.5.2.5, Corollary 5.5.2.6 a $\Leftrightarrow$ c).

Corollary 5.5.2.7 (0) *Let E be a Hilbert space. The following are equivalent:*

a) *E is separable.*

b) *the Hilbert dimension of E is at most $\aleph_0$.*

c) *E is finite-dimensional or it is isometric to ℓ^2.*

a $\Rightarrow$ b follows from Corollary 5.5.2.2.

b $\Rightarrow$ c. Let A be an orthonormal basis of E. By b),

$$\operatorname{Card} A \leq \aleph_0$$

and so the assertion follows from Theorem 5.5.2.5.

c $\Rightarrow$ a follows from Example 1.1.2.5. ■

Corollary 5.5.2.8 (0) *If E, F are infinite-dimensional Hilbert spaces, then $\mathcal{K}(E,F)$ is not a complemented subspace of $\mathcal{L}(E,F)$.*

Let A and B be orthonormal bases of E and F, respectively (Corollary 5.5.1.4). By Theorem 5.5.2.5, we may identify E and F with $\ell^2(A)$ and $\ell^2(B)$, respectively. By Corollary 3.1.2.12, $\mathcal{K}(\ell^2(A), \ell^2(B))$ is not a complemented subspace of $\mathcal{L}(\ell^2(A), \ell^2(B))$. ■

Corollary 5.5.2.9 *If the Hilbert dimension of an infinite-dimensional Hilbert space its $\aleph$, then its algebraic dimension is $\aleph^{\aleph_0}$.*

This follows immediately from Theorem 5.5.2.5 and Proposition 1.1.2.21. ■

Corollary 5.5.2.10 (0) *Let A be an orthonormal basis of the Hilbert space E. Given $x \in \ell^\infty(A)$, let $\widetilde{x}$ denote the unique operator on E such that*

$$\widetilde{x}a = x(a)a$$

for every $a \in A$ (Proposition 5.5.1.22). Then the map

$$\ell^\infty(A) \longrightarrow \mathcal{L}(E), \quad x \longmapsto \widetilde{x}$$

is an norm-preserving homomorphism of involutive unital algebras.

Take $x, y \in \ell^\infty(A)$ and $\alpha, \beta \in \mathbb{K}$. Then for $a \in A$

$$\widetilde{\alpha y + \beta y}a = (\alpha x + \beta y)(a)a = (\alpha x(a) + \beta y(a))a = \alpha x(a)a + \beta y(a)a =$$

$$= \alpha \widetilde{x}a + \beta \widetilde{y}a = (\alpha \widetilde{x} + \beta \widetilde{y})a\,,$$

$$\widetilde{x}\widetilde{y}(a) = \widetilde{x}y(a)a = y(a)\widetilde{x}a = y(a)x(a)a = (xy)(a)a = \widetilde{xy}a\,,$$

$$\widetilde{e}^A_A a = a\,,$$

and

$$\langle b|\widetilde{x^*}a\rangle = \langle b|\overline{x}(a)a\rangle = x(a)\langle b|a\rangle = x(b)\langle b|a\rangle =$$

$$= \langle x(b)b|a\rangle = \langle \widetilde{x}b|a\rangle = \langle b|\widetilde{x}^* a\rangle$$

for every $b \in A$, so that

$$\widetilde{x^*}a = \widetilde{x}^* a\,.$$

Hence the above map is an involutive unital algebra homomorphism. By Proposition 5.5.1.22, it preserves norms. ■

Proposition 5.5.2.11 (0) *Let E, F be Hilbert spaces such that at least one of them is infinite-dimensional. Let a and b be the Hilbert dimensions of E and F, respectively, and put*

$$c := \inf\{a^b, b^a\}\,.$$

a) *If $a = b$, then $c = 2^a$.*

b) *If $\inf\{a, b\} \geq 2$, then the topological cardinality of $\mathcal{L}(E, F)$ is equal to c.*

c) *If both E and F are different from $\{0\}$ then $\mathcal{L}(E, F)$ is separable iff one of these spaces is finite dimensional and the other is separable.*

a) We have

$$2^a \leq a^a \leq (2^a)^a = 2^{a \times a} = 2^a\,.$$

b) The map

$$\mathcal{L}(E,F) \longrightarrow \mathcal{L}(F,E), \quad u \longmapsto u^*$$

is an isometry of real Banach spaces, so we may assume $a \leq b$. Then, by a),

$$b^a \leq b^b = 2^b \leq a^b,$$

so that

$$c = b^a.$$

By Example 1.1.2.5 and Theorem 5.5.2.5, there is a dense set B of F with

$$\operatorname{Card} B = b.$$

Let A be an orthonormal basis of E. Then

$$\mathcal{F} := \{u = \mathcal{L}(E,F) | u(A) \subset B\}$$

is a dense set of $\mathcal{L}(E,F)$ and

$$\operatorname{Card} \mathcal{F} \leq \operatorname{Card} B^A = b^a = c.$$

Now let A and B be orthonormal bases of E and F, respectively. Put

$$\mathcal{G} := \{f \in B^A | f \text{ is injective}\}.$$

For $f \in \mathcal{G}$ let u_f be the element of $\mathcal{L}(E,F)$ with

$$u_f x = f(x)$$

for every $x \in A$ (Proposition 5.5.1.22). Then

$$\|u_f - u_g\| \geq \sqrt{2}$$

for all distinct $f, g \in \mathcal{G}$. Hence, if $\mathcal{H}$ is a dense set of $\mathcal{L}(E,F)$, then

$$\operatorname{Card} \mathcal{H} \geq \operatorname{Card} \mathcal{G} = b^a = c.$$

c) If E is finite-dimensional, then $\operatorname{Card} \mathcal{L}(E,F) = \operatorname{Card} F^a$ and so $\mathcal{L}(E,F)$ is separable iff F is separable. We use again the isometry

$$\mathcal{L}(E,F) \longrightarrow \mathcal{L}(F,E), \quad u \longmapsto u^*,$$

to conclude that if F is finite-dimensional, then $\mathcal{L}(E,F)$ is separable iff E is separable.

If both E and F are infinite-dimensional, then, by b), $\mathcal{L}(E,F)$ is not separable. ■

Proposition 5.5.2.12 (0) *Let E, F be Hilbert spaces different from $\{0\}$. Let $\aleph(E)$ and $\aleph(F)$ be the Hilbert dimensions of E and F, respectively, and put*

$$\aleph := \sup\{\aleph(E), \aleph(F)\}.$$

a) *Any dense set of $\mathcal{K}(E,F)$ has cardinality at least $\aleph$.*

b) *If $\aleph$ is infinite, then $\mathcal{K}(E,F)$ has topological cardinality $\aleph$.*

c) *$\mathcal{K}(E,F)$ is separable iff E and F are separable.*

d) *The C^*-algebras $\mathcal{K}(E)$ and $\mathcal{K}(F)$ are isomorphic iff the Hilbert spaces E and F are isomorphic.*

e) *If E is infinite-dimensional, then the topological cardinality of $\mathcal{L}(E)/\mathcal{K}(E)$ is $2^{\aleph(E)}$.*

a) Let A and B be orthonormal bases of E and F, respectively. Then for all $x_1, x_2 \in A$, $x_1 \neq x_2$ and $y_1, y_2 \in B$, $y_1 \neq y_2$ we have

$$\|\langle\cdot|x_1\rangle y_1 - \langle\cdot|x_2\rangle y_1\| \geq 1,$$

$$\|\langle\cdot|x_1\rangle y_1 - \langle\cdot|x_1\rangle y_2\| \geq 1.$$

Hence there is a set of pairwise disjoint open sets of $\mathcal{K}(E,F)$ of cardinality $\aleph$. Thus any dense set of $\mathcal{K}(E,F)$ has cardinality at least $\aleph$.

b) Let A and B be dense sets of E and F, respectively, of cardinality at most $\aleph$ and let $(x,y) \in E \times F$. Then for all $(a,b) \in A \times B$,

$$\|\langle\cdot|a\rangle b - \langle\cdot|x\rangle y\| \leq$$

$$\leq \|\langle\cdot|a-x\rangle b\| + \|\langle\cdot|x\rangle(b-y)\| \leq$$

$$\leq \|a-x\|\,\|b\| + \|x\|\,\|b-y\|.$$

It follows that $\langle\cdot|x\rangle y$ belongs to the closure of

$$\{\langle\cdot|a\rangle b | (a,b) \in A \times B\}$$

in $\mathcal{K}(E,F)$. Since

$$\mathrm{Card}\,\{\langle\cdot|a\rangle b|(a,b)\in A\times B\}\leq\aleph$$

it follows from Corollary 5.5.1.11 and Corollary 5.2.3.4, that $\mathcal{K}(E,F)$ has a dense set of cardinality at most $\aleph$. By a), $\aleph$ is the topological cardinality of $\mathcal{K}(E,F)$.

c) follows from a),b), and Corollary 5.5.2.7 a $\Leftrightarrow$ b.

d) We may assume that $\aleph$ is infinite. By b), if the C^*-algebras $\mathcal{K}(E)$ and $\mathcal{K}(F)$ are isomorphic, then

$$\aleph(E)=\aleph(F)\,.$$

By Corollary 5.5.2.6 c $\Rightarrow$ a, the Hilbert spaces E and F are isomorphic.

The converse implication is trivial.

e) follows from b) and Proposition 5.5.2.11 a),b). ■

Proposition 5.5.2.13 (0) *Let E,F be Hilbert spaces, G a separable closed vector subspace of E and take*

$$u\in\mathcal{L}(E,F)\backslash\mathcal{K}(E,F)\,.$$

Then there are $v\in\mathcal{L}(E)$ and $w\in\mathcal{L}(F,E)$ such that

$$w\circ u\circ v=\pi_G=v^*v\,.$$

By Proposition 5.2.3.21, there is an orthonormal sequence $(x_n)_{n\in\mathbb{N}}$ in E such that

$$\alpha:=\limsup_{n\to\infty}\|ux_n\|>0\,.$$

By Corollary 5.5.1.9 (and Corollary 1.4.1.6 a $\Rightarrow$ c), $(ux_n)_{n\in\mathbb{N}}$ converges weakly to 0. Hence we can construct inductively an increasing sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\|ux_{k_n}\|>\frac{\alpha}{2}\,,$$

$$m\in\mathbb{N}_{n-1}\Longrightarrow|\langle ux_{k_m}|ux_{k_n}\rangle|<\frac{\alpha^2}{3^{m+n}}$$

for every $n\in\mathbb{N}$.

We may assume G to be infinite dimensional. Let $(y_n)_{n\in\mathbb{N}}$ be an orthonormal basis of G (Corollary 5.5.2.7 a $\Rightarrow$ b). By Proposition 5.5.1.22, there is a $v\in\mathcal{L}(E)$ such that

$$vy_n = x_{k_n}$$

for every $n \in \mathbb{N}$ and $v = 0$ on $G^\perp$. We have

$$\sum_{n\in\mathbb{N}} \frac{\alpha^2}{3^{2n}} = \frac{\alpha^2}{8} < \frac{\alpha^2}{4} \leq \inf_{n\in\mathbb{N}} \|ux_{k_n}\|^2 .$$

By Corollary 5.2.3.20, there is a $w \in \mathcal{L}(F, E)$ such that

$$wux_{k_n} = y_n$$

for every $n \in \mathbb{N}$. Thus

$$wuvy_n = y_n$$

for every $n \in \mathbb{N}$ and

$$w \circ u \circ v = 0$$

on $G^\perp$. Hence

$$w \circ u \circ v = \pi_G .$$

Let $(z_\iota)_{\iota\in I}$ be an orthonormal basis of $G^\perp$. Then

$$v^*vy_n = v^*x_{k_n} = \sum_{m\in\mathbb{N}} \langle v^*x_{k_n}|y_m\rangle\, y_m + \sum_{\iota\in I} \langle v^*x_{k_n}|z_\iota\rangle\, z_\iota =$$

$$= \sum_{m\in\mathbb{N}} \langle x_{k_n}|vy_m\rangle\, y_m + \sum_{\iota\in I} \langle x_{k_n}|vz_\iota\rangle\, z_\iota = \sum_{m\in\mathbb{N}} \langle x_{k_n}|x_{k_m}\rangle\, y_m = y_n$$

for every $n \in \mathbb{N}$ (Fourier expansion) and

$$v^*v = 0$$

on $G^\perp$. Thus

$$v^*v = \pi_G . \qquad \blacksquare$$

Corollary 5.5.2.14 (0) *If E is a Hilbert space and $u \in \mathcal{L}(E)\backslash\mathcal{K}(E)$, then the ideal of $\mathcal{L}(E)$ generated by u contains*

$$\{v \in \mathcal{L}(E) | \operatorname{Im} v \text{ is separable}\} .$$

Let $\mathcal{F}$ be the ideal of $\mathcal{L}(E)$ generated by u and take $v \in \mathcal{L}(E)$ such that $\operatorname{Im} v$ is separable. Put

$$F := \overline{\operatorname{Im} v}\,.$$

By Proposition 5.5.2.13, $\pi_F \in \mathcal{F}$. Thus

$$v = \pi_F v \in \mathcal{F}\,. \qquad \blacksquare$$

Remark. The ideal generated by $u \in \mathcal{L}(E)\backslash\mathcal{K}(E)$ need not be closed. But it does contain $\{v \in \mathcal{L}(E) | \operatorname{Im} v \text{ is separable}\}$ which is (by Example 2.2.2.23) a closed ideal.

Corollary 5.5.2.15 (0) *If E is separable, then the C^*-algebra $\mathcal{L}(E)/\mathcal{K}(E)$ is simple.*

Let $\mathcal{F}$ be a closed ideal of $\mathcal{L}(E)/\mathcal{K}(E)$ and let q be quotient map

$$\mathcal{L}(E) \longrightarrow \mathcal{L}(E)/\mathcal{K}(E)\,.$$

Then $\overset{-1}{q}(\mathcal{F})$ is a closed ideal of $\mathcal{L}(E)$. By Corollary 5.5.1.13 a) and Corollary 5.5.2.14,

$$\overset{-1}{q}(\mathcal{F}) \in \{\{0\}\,,\ \mathcal{K}(E)\,,\ \mathcal{L}(E)\}\,.$$

Hence

$$\mathcal{F} \in \{\{0\}\,,\ \mathcal{L}(E)/\mathcal{K}(E)\}\,,$$

so that $\mathcal{L}(E)/\mathcal{K}(E)$ is simple. $\blacksquare$

Corollary 5.5.2.16 *Let E be a Hilbert space and put*

$$\mathcal{F} := \{u \in \mathcal{L}(E) | \operatorname{Im} u \text{ is separable}\}\,.$$

Then every representation of $\mathcal{F}$ (Example 2.2.2.23) is the Hilbert sum of identical representations, of representations with kernel $\mathcal{K}(E)$, and of a zero representation.

Let (H, φ) be a representation of $\mathcal{F}$. Then $(H, \varphi|\mathcal{K}(E))$ (Proposition 3.1.1.18) is a representation of $\mathcal{K}(E)$. Thus by Theorem 5.5.1.24, it is the Hilbert sum of identical representations and of a 0–representation. By Proposition

5.5.1.25, (H, φ) is the Hilbert sum of identical representations and of representations (K, ψ) with $\psi(\mathcal{K}(E)) = 0$. $\operatorname{Ker}\psi$ is a closed ideal of $\mathcal{F}$ containing $\mathcal{K}(E)$. By Corollary 5.5.2.14,

$$\operatorname{Ker}\psi = \mathcal{K}(E) \text{ or } \operatorname{Ker}\psi = \mathcal{F}.$$ ■

Remark. If E is separable, then $\mathcal{F} = \mathcal{L}(E)$ and a reprensentation of $\mathcal{L}(E)$ with kernel $\mathcal{K}(E)$ is in fact a representation of the Calkin algebra $\mathcal{L}(E)/\mathcal{K}(E)$.

Proposition 5.5.2.17 *Let E be a Hilbert space and $\aleph$ its Hilbert dimension. Then there is a faithful unital representation (H, φ) of $\mathcal{L}(E)/\mathcal{K}(E)$ such that the Hilbert dimension of H is at most $\aleph^{\aleph_0}$.*

We may assume E to be infinite-dimensional. Then there is a dense set A of E such that

$$\operatorname{Card} A = \aleph$$

(Corollary 5.5.2.2). Let (H, φ) be the faithful unital representation of $\mathcal{L}(E)/\mathcal{K}(E)$ from Example 5.4.3.5 d). In the sequel we use the notation of this example. Define

$$\mathcal{F}_0 := \{x \in \mathcal{F} | x(\mathbb{N}) \subset A\}.$$

Then

$$\operatorname{Card} \mathcal{F}_0 \leq \operatorname{Card} A^{\aleph_0} = \aleph^{\aleph_0}.$$

Take $x \in \mathcal{F}$. There is a $y \in \mathcal{F}_0$ such that

$$\|x(n) - y(n)\| < \frac{1}{n}$$

for every $n \in \mathbb{N}$. Then $x - y \in \mathcal{G}$. Hence

$$\operatorname{Card}(\mathcal{F}/\mathcal{G}) \leq \operatorname{Card} \mathcal{F}_0 \leq \aleph^{\aleph_0}.$$

By Corollary 5.5.2.2, the Hilbert dimension of $\widetilde{\mathcal{F}/\mathcal{G}}$ is at most $\operatorname{Card}(\mathcal{F}/\mathcal{G})$. Therefore the Hilbert dimension of H is at most $\aleph^{\aleph_0}$. ■

Proposition 5.5.2.18 *Let F be a quasiunital involutive Banach algebra and A a dense set of F.*

a) *If (H, φ) is a cyclic representation of F, then the Hilbert dimension of H is at most* $\operatorname{Card} A$.

b) *If (H, φ) is the universal representation of F, then the Hilbert dimension of H is at most $2^{\operatorname{Card} A}$.*

a) By Proposition 5.4.1.3 b), (H, φ) is equivalent to the representation of E associated to an $x' \in E'_+$. By Theorem 5.4.1.2 b),h), H contains a dense set of cardinality at most $\operatorname{Card} A$, so by Corollary 5.5.2.2, the Hilbert dimension of H is at most $\operatorname{Card} A$.

b) The map

$$\tau(E) \longrightarrow \mathbb{K}^A, \quad x' \longmapsto x'|A$$

is injective, and so

$$\operatorname{Card} \tau(E) \leq \operatorname{Card} \mathbb{K}^A \leq 2^{\operatorname{Card} A}.$$

By a),

$$\operatorname{Card} H \leq (\operatorname{Card} A) 2^{\operatorname{Card} A} = 2^{\operatorname{Card} A}. \qquad \blacksquare$$

Example 5.5.2.19 *Let $(T, \mathfrak{T}, \mu)$ be a measure space and $\mathcal{H}$ the set of maps $x : T \to E$ such that*

$$T \longrightarrow \mathbb{K}, \quad t \longmapsto \langle x(t)|y\rangle$$

is measurable for every $y \in E$ and that

$$T \longrightarrow \mathbb{R}, \quad t \longmapsto \|x(t)\|^2$$

is μ-integrable. If E is separable, then:

a)

$$T \longrightarrow \mathbb{K}, \quad t \longmapsto \langle x(t)|y(t)\rangle$$

is μ-integrable for all $x, y \in \mathcal{H}$.

b) *$\mathcal{H}$ is a vector subspace of E^T and*

$$f : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{K}, \quad (x, y) \longmapsto \int \langle x(t)|y(t)\rangle d\mu(t)$$

is a positive sesquilinear form. Put

$$\mathcal{F} := \{x \in \mathcal{H} | f(x, x) = 0\},$$

$$H := \mathcal{H}/\mathcal{F},$$

and endow H with the scalar product associated to f (Proposition 5.1.2.9 d)); H is called ***the direct integral of E with respect to μ.***

c) *If E is complete, then H is also complete.*

Let A be an orthonormal basis of E (Corollary 5.5.1.4). By Corollary 5.5.2.7 a $\Rightarrow$ b, A is countable.

a) The map

$$T \longrightarrow \mathbb{K}, \quad t \longmapsto \langle x(t)|z\rangle,$$

belongs to $\mathcal{L}^2(\mu)$ for every $z \in A$ and

$$\sum_{z\in A} |\langle x(t)|z\rangle|^2 = \|x(t)\|^2$$

for every $t \in T$ (Parseval's Equation). Given $t \in T$,

$$\langle x(t)|y(t)\rangle = \sum_{z\in A} \langle x(t)|z\rangle\langle z|y(t)\rangle$$

(Theorem 5.5.1.15 a $\Rightarrow$ c),

$$|\langle x(t)|y(t)\rangle| \leq \left(\sum_{z\in A} |\langle x(t)|z\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{z\in A} |\langle y(t)|z\rangle|^2\right)^{\frac{1}{2}} \leq \|x(t)\|\,\|y(t)\|$$

and so

$$T \longrightarrow \mathbb{K}, \quad t \longmapsto \langle x(t)|y(t)\rangle$$

is μ-integrable.

b) follows from a).

c) Let $(X_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in H and for each $n \in \mathbb{N}$, choose $x_n \in X_n$. Then $(\langle x_n|z\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mu)$ for every $z \in E$. Given $z \in A$, put

$$x_z := \lim_{n\to\infty} \langle x_n|z\rangle \in L^2(\mu)$$

and

$$x : T \longrightarrow E, \quad t \longmapsto \sum_{y\in A} x_y(t)y,$$

which is well-defined up to a set of μ-measure zero. The map

$$T \longrightarrow \mathbb{K}, \quad t \longmapsto \langle x(t)|y\rangle$$

is measurable for every $y \in E$.

Given $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$, such that

$$\|X_m - X_n\| < \varepsilon$$

for every $m, n \in \mathbb{N}\backslash\mathbb{N}_{n_\varepsilon}$. Take $n \in \mathbb{N}$, $n \geq n_\varepsilon$. By Parseval's Equation,

$$\varepsilon^2 \geq \lim_{m\to\infty} \int \|x_m(t) - x_n(t)\|^2 d\mu(t) = \lim_{m\to\infty} \int \sum_{z\in A} |\langle x_m(t) - x_n(t)|z\rangle|^2 d\mu(t) =$$

$$= \lim_{m\to\infty} \sum_{z\in A} \int |\langle x_m(t) - x_n(t)|z\rangle|^2 d\mu(t) \geq$$

$$\geq \sum_{z\in A} \int |\langle x(t) - x_n(t)|z\rangle|^2 d\mu(t),$$

so that

$$x - x_n \in \mathcal{H}, \quad x \in \mathcal{H},$$

and

$$\int \|x(t) - x_n(t)\|^2 d\mu(t) = \int \sum_{z\in A} |\langle x(t) - x_n(t)|z\rangle|^2 d\mu(t) =$$

$$= \sum_{z\in A} \int |\langle x(t) - x_n(t)|z\rangle|^2 d\mu(t) \leq \varepsilon^2.$$

Since ε is arbitrary, $(X_n)_{n\in\mathbb{N}}$ converges to the equivalence class of x in H. ■

5.5.3 Standard Examples

Example 5.5.3.1 *For any set* T, $(e_t)_{t\in T}$ *is an orthonormal basis of* $\ell^2(T)$. *It is called* ***the canonical orthonormal basis of*** $\ell^2(T)$.

Given $s,t \in T$,

$$\langle e_s|e_t\rangle = \delta_{s,t},$$

i.e. $(e_t)_{t\in T}$ is an orthonormal family in $\ell^2(T)$. Take $x \in \{e_t|t \in T\}^\perp$. Then

$$x(t) = \langle x|e_t\rangle = 0$$

for every $t \in T$ and so $x = 0$. Hence $(e_t)_{t\in T}$ is an orthonormal basis of $\ell^2(T)$. ■

Example 5.5.3.2 *Let* $\mathbb{K} = \mathbb{C}$ *and* λ *the Haar measure on* $\mathbf{T}$ *with* $\mu(\mathbf{T}) = 1$. *Given* $n \in \mathbb{Z}$, *consider*

$$x_n : \mathbf{T} \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \alpha^n.$$

Then $(x_n)_{n\in\mathbb{Z}}$ *is an orthonormal basis of* $L^2(\lambda)$. *The map*

$$\ell^2(\mathbb{Z}) \longrightarrow L^2(\lambda), \quad y \longmapsto \sum_{n\in\mathbb{Z}} y_n x_n$$

is an isometry which coincides with the Gelfand transform on $\ell^1(\mathbb{Z})$, *where we here identified* $\sigma(\ell^1(\mathbb{Z}))$ *with* $\mathbf{T}$ *(Example 2.4.5.6).*

Given $n,p \in \mathbb{Z}$,

$$\langle x_n|x_p\rangle = \int \alpha^n a^{-p} d\lambda(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} e^{int}e^{-ipt}dt = \delta_{np}$$

and so $(x_n)_{n\in\mathbb{Z}}$ is an orthonormal family in $L^2(\lambda)$. By the Weierstrass-Stone Theorem (Corollary 1.3.5.16), the vector subspace of $\mathcal{C}(\mathbf{T})$ generated by $\{x_n|n \in \mathbb{Z}\}$ is dense. Since the inclusion map

$$\mathcal{C}(\mathbf{T}) \longrightarrow L^2(\lambda)$$

is continuous, it follows that

$$\mathcal{C}(\mathbf{T}) \subset \{x_n|n \in \mathbb{Z}\}^{\perp\perp}$$

(Corollary 5.2.3.9) and

$$\{x_n | n \in \mathbb{Z}\}^{\perp\perp} = L^2(\lambda)$$

(Proposition 5.2.2.2 a)). Hence $(x_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\lambda)$.

By Theorem 5.5.2.5, the map

$$u : \ell^2(\mathbb{Z}) \longrightarrow L^2(\lambda), \quad y \longmapsto \sum_{n\in\mathbb{Z}} y_n x_n$$

is an isometry. Take $y \in \ell^1(\mathbb{Z})$. Then

$$\widehat{y} = \sum_{n\in\mathbb{Z}} y_n x_n ,$$

where the sum is taken in $\mathcal{C}(\mathbf{T})$ (Example 2.4.5.6). Since $\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$ and the inclusion map $\mathcal{C}(\mathbf{T}) \to L^2(\lambda)$ is continuous,

$$\widehat{y} = uy . \qquad \blacksquare$$

Example 5.5.3.3 *Let* $\mathbb{K} = \mathbb{C}$. *Consider*

$$T := U_1^{\mathbb{C}}(0)$$

and λ *the Lebesgue measure on* T. *Let* $\mathcal{A}$ *be the closed vector subspace*

$$\{x \in L^2(\lambda) | x \text{ is analytic}\}$$

of $L^2(\lambda)$ *(Example 3.1.2.10).*

a) *Given* $x \in \ell^2$, *the radius of convergence of the power series*

$$\sum_{n=1}^{\infty} \sqrt{\frac{n}{\pi}} x_n t^{n-1}$$

is greater than 1. *Define*

$$\widetilde{x} : T \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \sum_{n=1}^{\infty} \sqrt{\frac{n}{\pi}} x_n \alpha^{n-1} .$$

b) $(\widetilde{e}_n)_{n\in\mathbb{N}}$ *is an orthonormal basis of* $\mathcal{A}$ *and the map*

$$\ell^2 \longrightarrow \mathcal{A}, \quad x \longmapsto \widetilde{x}$$

is an isometry.

a) The assertion follows from

$$\limsup_{n\to\infty}\left|\sqrt{\frac{n}{\pi}}x_n\right|^{\frac{1}{n}} = \lim_{n\to\infty}|x_n|^{\frac{1}{n}} \leq 1\,.$$

b) Given $n,p\in\mathbb{N}$

$$\langle\widetilde{e}_n|\widetilde{e}_p\rangle = \int\sqrt{\frac{n}{\pi}}\alpha^{n-1}\sqrt{\frac{p}{\pi}}(\overline{\alpha})^{p-1}d\lambda(\alpha) =$$

$$= \frac{\sqrt{np}}{\pi}\int_0^1 r^{n+p-1}\left(\int_0^{2\pi}e^{i(n-p)\theta}d\theta\right)dr = 2n\delta_{np}\int_0^1 r^{2n-1}dr = \delta_{np}\,.$$

Hence $(\widetilde{e}_n)_{n\in\mathbb{N}}$ is an orthonormal family in $\mathcal{A}$.

Take $x\in\mathcal{A}$ and let $\sum_{n=0}^{\infty}\alpha_n t^n$ be its associated power series. Choose $p\in\mathbb{N}$. Then, given any $\rho\in]0,1[$,

$$\int_{|\alpha|\leq\rho}x\overline{\widetilde{e}_p}d\lambda = \sqrt{\frac{p}{\pi}}\sum_{n=0}^{\infty}\alpha_n\int_{|\alpha|\leq\rho}\alpha^n(\overline{\alpha})^{p-1}d\lambda(\alpha) =$$

$$= \sqrt{\frac{p}{\pi}}\sum_{n=0}^{\infty}\alpha_n\int_0^{\rho}r^{n+p}\left(\int_0^{2\pi}e^{i(n-p+1)\theta}d\theta\right)dr =$$

$$= \sqrt{\frac{p}{\pi}}\alpha_{p-1}2\pi\int_0^{\rho}r^{2p-1}dr = \sqrt{\frac{\pi}{p}}\alpha_{p-1}\rho^{2p}\,.$$

Thus

$$\langle x|\widetilde{e}_p\rangle = \sqrt{\frac{\pi}{p}}\alpha_{p-1}\,.$$

We deduce that

$$\{\widetilde{e}_n|n\in\mathbb{N}\}^{\perp} = \{0\}\,,$$

and hence $(\widetilde{e}_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{A}$.

Take $x\in\ell^2$. Then $(x_n\widetilde{e}_n)_{n\in\mathbb{N}}$ is summable in $\mathcal{A}$ (Corollary 5.2.2.5). By Example 3.1.2.10, given any compact set K of T, the sequence $(x_n\widetilde{e}_n|K)_{n\in\mathbb{N}}$ is summable in $\mathcal{C}(K)$. Hence $\widetilde{x}\in\mathcal{A}$ and

$$\widetilde{x} = \sum_{n \in \mathbb{N}} x_n \widetilde{e}_n ,$$

where the sum is taken in $\mathcal{A}$. By Theorem 5.5.2.5, the map

$$\ell^2 \longrightarrow \mathcal{A}, \quad x \longmapsto \widetilde{x}$$

is an isometry. ■

Example 5.5.3.4 *Take* $\mathbb{K} = \mathbb{R}$ *and*

$$T := U_1^{\mathbb{R}^2}(0) .$$

Let λ *be Lebesgue measure on* T *and* $\mathcal{H}$ *the closed vector subspace*

$$\{x \in L^2(\lambda) | x \text{ is harmonic}\}$$

of $L^2(\lambda)$ *(Example 3.1.2.9). Then*

$$A := \left\{ \frac{1}{\sqrt{\pi}} \right\} \cup \left\{ \sqrt{\frac{2(n+1)}{\pi}} r^n \cos n\theta | n \in \mathbb{N} \right\} \cup$$

$$\cup \left\{ \sqrt{\frac{2(n+1)}{\pi}} r^n \sin n\theta | n \in \mathbb{N} \right\}$$

is an orthonormal basis of $\mathcal{H}$. *Here the functions are expressed in terms of polar coordinates* r, θ.

It is easy to see that A is orthonormal . Take $u \in \mathcal{H}$. Since u is the real part of a complex analytic function on T, u may be written in the form

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n$$

where the sum converges uniformly on the compact sets of T. Take $n \in \mathbb{N}$ and $\rho \in]0, 1[$. Then

$$\int_{U_\rho(0)} u(r, \theta) r^n \cos n\theta \, d\lambda = a_n \int_{U_\rho(0)} r^{2n} \cos^2 n\theta \, d\lambda =$$

$$= a_n \int_0^\rho \left(\int_0^{2\pi} \cos^2 n\theta \, d\theta \right) r^{2n+1} dr = \frac{a_n \pi \rho^{2n+2}}{2(n+1)} ,$$

$$\int\limits_{U_\rho(0)} u(r,\theta) r^n \sin n\theta \, d\lambda = b_n \int\limits_{U_\rho(0)} r^{2n} \sin^2 n\theta \, d\lambda =$$

$$= b_n \int\limits_0^\rho \left(\int\limits_0^{2\pi} \sin^2 n\theta \, d\theta \right) r^{2n+1} dr = \frac{b_n \pi \rho^{2n+2}}{2(n+1)} ,$$

$$\int\limits_{U_\rho(0)} u(r,\theta) d\lambda = a_0 \pi \rho^2 .$$

Hence

$$\int u(r,\theta) a^n \cos n\theta \, d\lambda = \frac{\pi a_n}{2(n+1)} ,$$

$$\int u(r,\theta) r^n \sin n\theta \, d\lambda = \frac{\pi b_n}{2(n+1)} ,$$

$$\int u(r,\theta) d\lambda = \pi a_0 .$$

Thus if $u \in A^\perp$, then $u = 0$. It follows that A is an orthonormal basis of $\mathcal{H}$ ■

Example 5.5.3.5 *Take* $a, b \in \mathbb{R}_+$, $a < b$ *and let* λ *be Lebesgue measure on* $[a,b]$. *Given* $n \in \mathbb{N} \cup \{0\}$, *define*

$$p_n : [a,b] \longrightarrow \mathbb{K} , \quad t \longmapsto \frac{d^n}{dt^n}((t-a)(b-t))^n ,$$

$$\alpha_n := \left(\int\limits_a^b p_n(t)^2 dt \right)^{\frac{1}{2}} , \quad x_n := \frac{1}{\alpha_n} p_n .$$

Then $(x_n)_{n \in \mathbb{N} \cup \{0\}}$ *is an orthonormal basis of* $L^2(\lambda)$.

Take $m, n \in \mathbb{N}$ with $m < n$. Using complete induction and integration by parts, we see that

$$\int\limits_a^b t^m p_n(t) dt = (-1)^m m! \int\limits_a^b \frac{d^{n-m}}{dt^{n-m}}((t-a)(b-t))^n dt = 0 .$$

Now given $n \in \mathbb{N} \cup \{0\}$, p_n is a polynomial of degree n. Thus it follows that $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ is an orthonormal family in $L^2(\lambda)$. Moreover, $\{x_n | n \in \mathbb{N}\cup\{0\}\}^{\perp\perp}$ contains all polynomials. Since the set of polynomials is dense in $L^2(\lambda)$,

$$\{x_n | n \in \mathbb{N} \cup \{0\}\}^{\perp\perp} = L^2(\lambda).$$

Hence, $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ is an orthonormal basis of $L^2(\lambda)$. ■

Example 5.5.3.6 (4) *Let λ be Lebesgue measure on $\mathbb{R}$. Given $n \in \mathbb{N} \cup \{0\}$, define*

$$y_n : \mathbb{R} \longrightarrow \mathbb{K}, \quad t \longmapsto e^{\frac{1}{2}t^2}\frac{d^n e^{-t^2}}{dt^n},$$

$$\alpha_n := \left(\int y_n^2 d\lambda\right)^{\frac{1}{2}}, \quad x_n := \frac{1}{\alpha_n} y_n.$$

Then $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ is an orthonormal basis of $L^2(\lambda)$.

It is easy to see that for each $n \in \mathbb{N} \cup \{0\}$, we can find a polynomial $P_n \in \mathbb{R}[t]$ of degree n, such that

$$x_n(t) = P_n(t)e^{-\frac{1}{2}t^2}$$

for every $t \in \mathbb{R}$. It follows that

$$\mathcal{F} := \{P(t)e^{-\frac{1}{2}t^2} \mid P \in \mathbb{K}[t]\} \subset \{x_n | n \in \mathbb{N} \cup \{0\}\}^{\perp\perp}.$$

Since $\mathcal{F}$ is a dense set of $L^2(\lambda)$,

$$\{x_n | n \in \mathbb{N} \cup \{0\}\}^{\perp\perp} = L^2(\lambda).$$

Take $m, n \in \mathbb{N}$ with $m < n$. Using complete induction and integration by parts, we see that

$$\int_{-\infty}^{\infty} t^m \frac{d^n e^{-t^2}}{dt^n} dt = (-1)^m m! \int_{-\infty}^{\infty} \frac{d^{n-m} e^{-t^2}}{dt^{n-m}} dt = 0.$$

This and the above considerations imply that $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ is an orthonormal basis of $L^2(\lambda)$. ■

Example 5.5.3.7 *Let λ be Lebesgue measure on $\mathbb{R}_+$. Given $n \in \mathbb{N} \cup \{0\}$, define*

$$y_n : \mathbb{R}_+ \longrightarrow \mathbb{K}, \quad t \longmapsto e^{\frac{t}{2}} \frac{d^n(t^n e^{-t})}{dt^n},$$

$$\alpha_n := \left(\int y_n^2 d\lambda \right)^{\frac{1}{2}}, \quad x_n := \frac{1}{\alpha_n} y_n .$$

Then $(x_n)_{n \in \mathbb{N} \cup \{0\}}$ is an orthonormal basis of $L^2(\lambda)$.

It is easy to see that for each $n \in \mathbb{N} \cup \{0\}$, there is a polynomial $P_n \in \mathbb{R}[t]$ of degree n such that

$$x_n(t) = P_n(t) e^{-\frac{t}{2}}$$

for ever $t \in \mathbb{R}_+$. It follows that

$$\mathcal{F} := \{P(t)e^{-\frac{t}{2}} \mid P \in \mathbb{K}[t]\} \subset \{x_n | n \in \mathbb{N} \cup \{0\}\}^{\perp\perp}.$$

Since $\mathcal{F}$ is dense in $L^2(\lambda)$,

$$\{x_n | n \in \mathbb{N} \cup \{0\}\}^{\perp\perp} = L^2(\lambda).$$

Take $m, n \in \mathbb{N}$ with $m < n$. Complete induction and integration by parts show that

$$\int_0^\infty t^m \frac{d^n(t^n e^{-t})}{dt^n} dt = (-1)^m m! \int_0^\infty \frac{d^{n-m}(t^n e^{-t})}{dt^{n-m}} dt = 0 .$$

This and the above considerations show that $(x_n)_{n \in \mathbb{N} \cup \{0\}}$ is an orthonormal basis of $L^2(\lambda)$. ■

Proposition 5.5.3.8 (0) *Let $(S, \mathfrak{S}, \mu)$, $(T, \mathfrak{T}, \nu)$ be two σ-finite measure spaces and $\mathcal{F}, \mathcal{G}$ orthonormal bases of $L^2(\mu)$ and $L^2(\nu)$, respectively. Then*

$$\{x \otimes y | (x, y) \in \mathcal{F} \times \mathcal{G}\}$$

is an orthonormal basis of $L^2(\mu \otimes \nu)$.

Take (x_1, y_1), $(x_2, y_2) \in \mathcal{F} \times \mathcal{G}$. Then

$$\langle x_1 \otimes y_1 | x_2 \otimes y_2 \rangle = \int x_1(s) y_1(t) \overline{x_2(s)}\, \overline{y_2(t)} d(\mu \otimes \nu)(s, t) =$$

$$= \left(\int x_1 \overline{x}_2 d\mu \right) \left(\int y_1 \overline{y}_2 d\nu \right) = \langle x_1 | x_2 \rangle \langle y_1 | y_2 \rangle .$$

Hence

$$\{ x \otimes y | (x, y) \in \mathcal{F} \times \mathcal{G} \}$$

is an orthonormal set in $L^2(\mu \otimes \nu)$.

Take

$$k \in \{ x \otimes y | (x, y) \in \mathcal{F} \times \mathcal{G} \}^{\perp} .$$

Then, for any $(x, y) \in \mathcal{F} \times \mathcal{G}$,

$$0 = \langle k | x \otimes y \rangle = \int k(s,t) \overline{x(s)}\, \overline{y(t)}\, d(\mu \otimes \nu)(s,t) =$$

$$= \int \left(\int k(s,t) \overline{y(t)}\, d\nu(t) \right) \overline{x(s)} d\mu(s) = \int (k\overline{y}) \overline{x} d\mu = \langle k\overline{y} | x \rangle$$

by Fubini and Proposition 3.1.6.16. Hence

$$k\overline{y} = 0$$

for every $y \in \mathcal{G}$. Thus $k = 0$ (Proposition 3.1.6.17 e)). Hence

$$\{ x \otimes y | (x, y) \in \mathcal{F} \times \mathcal{G} \}^{\perp} = \{ 0 \}$$

and so

$$\{ x \otimes y | (x, y) \in \mathcal{F} \times \mathcal{G} \}$$

is an orthonormal basis of $L^2(\mu \otimes \nu)$. ■

Corollary 5.5.3.9 (4) *Let* $(x_n)_{n \in \mathbb{N} \cup \{0\}}$ *be the orthonormal basis defined in Example 5.5.3.6. Take* $m \in \mathbb{N}$ *and let* λ *be Lebesgue measure on* $\mathbb{R}^m$. *Define*

$$K := (\mathbb{N} \cup \{0\})^m$$

and given $k \in K$, *consider*

$$y_k : \mathbb{R}^m \longrightarrow \mathbb{K}, \quad t \longmapsto \prod_{j=1}^{m} x_{k_j}(t_j) .$$

Then $(y_k)_{k \in K}$ *is an orthonormal basis of* $L^2(\lambda)$.

This follows immediately from Proposition 5.5.3.8 using complete induction. ■

Proposition 5.5.3.10 *Let $(T, \mathfrak{T}, \mu)$ be a measure space, f a positive locally μ-integrable function and $\nu := f \cdot \mu$. If $(x_\iota)_{\iota \in I}$ is an orthonormal basis of $L^2(\mu)$ and if we define*

$$g : T \longrightarrow \mathbb{K}, \quad t \longmapsto \begin{cases} \frac{1}{\sqrt{f(t)}} & \text{if } f(t) \neq 0 \\ 0 & \text{if } f(t) = 0, \end{cases}$$

then $(gx_\iota)_{\iota \in I}$ is an orthonormal basis of $L^2(\nu)$.

We have $gx_\iota \in L^2(\nu)$ for every $\iota \in I$. Moreover,

$$\int (gx_\iota)(gx_\lambda) d\nu = \int x_\iota x_\lambda g^2 d\nu = \int x_\iota x_\lambda d\mu = \delta_{\iota\lambda}$$

for all $\iota, \lambda \in I$. Hence $(gx_\iota)_{\iota \in I}$ is orthonormal. Let $x \in \{gx_\iota | \iota \in I\}^\perp$. Then $\sqrt{f}\, x \in L^2(\mu)$ and

$$\int \sqrt{f}\, x x_\iota d\mu = \int x g x_\iota d\nu = 0$$

for all $\iota \in I$. Hence $\sqrt{f} x = 0$ μ-a.e. and therefore $x = 0$ ν-a.e. ■

Corollary 5.5.3.11 *Take $\mathbb{K} = \mathbb{C}$,*

$$T := U_1^{\mathbb{C}}(0),$$

and let λ be Lebesgue measure on T. Given $n \in \mathbb{N} \cup \{0\}$ and $p \in \mathbb{Z}$, define

$$y_n : [0,1] \longrightarrow \mathbb{C}, \quad t \longmapsto \frac{1}{\sqrt{t}} \frac{d^n}{dt^n}(t(1-t))^n,$$

$$x_{np} : T \longrightarrow \mathbb{C}, \quad re^{i\theta} \longmapsto \alpha_{np} y_n(r) e^{ip\theta},$$

where α_{np} is given by

$$\int |x_{np}|^2 d\lambda = 1.$$

Then $\{x_{np} | n \in \mathbb{N} \cup \{0\}, p \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\lambda)$.

Let μ be the measure on $[0,1]$ defined by $d\mu := tdt$ and let ν be Haar measure on $\mathbf{T}$ with $\nu(\mathbf{T}) = 1$, and set

$$\varphi : [0,1] \times \mathbf{T} \longrightarrow T, \quad (r,\theta) \longmapsto re^{i\theta}.$$

By Example 5.5.3.5 and Proposition 5.5.3.10, there is a family $(\beta_n)_{n\in\mathbb{N}\cup\{0\}}$ in $\mathbb{R}_+$ such that $(\beta_n y_n)_{n\in\mathbb{N}\cup\{0\}}$ is an orthonormal basis of $L^2(\mu)$. By Example 5.5.3.2, $\left(e^{ip\theta}\right)_{p\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\nu)$. Thus by Proposition 5.5.3.8, $\left\{(\beta_n y_n) \otimes (e^{ip\theta}) | n \in \mathbb{N}\cup\{0\}, p \in \mathbb{Z}\right\}$ is an orthonormal basis of $L^2(\mu\otimes\nu)$. Since

$$\varphi(\mu\otimes\nu) = \lambda,$$

$\{x_{np} | n \in \mathbb{N}\cup\{0\}, p \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\lambda)$. ■

Proposition 5.5.3.12 *Take $\mathbb{K} = \mathbb{C}$. Let T be an open set of $\mathbb{C}$, λ the Lebesgue measure on T, $\mathcal{A}$ the closed vector subspace*

$$\{x \in L^2(\lambda) \mid x \text{ is analytic}\}$$

of $L^2(\lambda)$ (Example 3.1.2.10), and $(x_n)_{n\in\mathbb{N}}$ an orthonormal basis of $\mathcal{A}$. Then, for each $t \in T$, $(x_n(t))_{n\in\mathbb{N}} \in \ell^2$ and

$$T \times T \longrightarrow \mathbb{C}, \quad (s,t) \longmapsto \sum_{n\in\mathbb{N}} x_n(s)\overline{x_n(t)}$$

is the Bergman kernel of T (Example 5.2.5.9).

Take $r > 0$, with

$$A := U_r^{\mathbb{C}}(t) \subset T.$$

Then $e_A \in L^2(\lambda)$. By Gauss' Mean Value Theorem and Bessel's Inequality,

$$\pi r^2 = \|e_A\|^2 \geq \sum_{n\in\mathbb{N}} |\langle e_A | x_n\rangle|^2 = \sum_{n\in\mathbb{N}} \left|\int_A \overline{x}_n d\lambda\right|^2 = \sum_{n\in\mathbb{N}} (\pi r^2)^2 |x_n(t)|^2.$$

Hence $(x_n(t))_{n\in\mathbb{N}} \in \ell^2$.

Take $x \in \mathcal{A}$. Then

$$\sum_{n\in\mathbb{N}} \langle x | x_n\rangle x_n = x$$

(Fourier expansion). Since the map

$$\mathcal{A} \longrightarrow \mathbb{C}, \quad y \longmapsto y(t)$$

is continuous (Example 3.1.2.10),

$$x(t) = \sum_{n\in\mathbb{N}} \langle x|x_n\rangle x_n(t) = \left\langle x \middle| \sum_{n\in\mathbb{N}} \overline{x_n(t)} x_n \right\rangle$$

(Corollary 5.2.2.25). Hence k is the Bergman kernel of T. ■

Remark. a) Since $L^2(\lambda)$ is separable, $\mathcal{A}$ is also separable. Hence $\mathcal{A}$ has a countable orthonormal basis (Corollary 5.5.2.7 a ⇒ b). It follows from function theory, that if $\mathcal{A} \neq \{0\}$, then $\mathcal{A}$ is infinite-dimensional. Hence, taking a basis of $\mathcal{A}$ of the form $(x_n)_{n\in\mathbb{N}}$ is no real restriction of generality.

b) Let k be the Bergman kernel of $U_1^{\mathbb{C}}(0)$. By Example 5.5.3.3 and the above proposition,

$$k(s,t) = \sum_{n\in\mathbb{N}} \frac{n}{\pi} s^{n-1}\bar{t}^{n-1} = \frac{1}{\pi(1-s\bar{t})^2}$$

for all $s,t \in U_1^{\mathbb{C}}\{0\}$, which coincides with Example 5.2.5.11.

Proposition 5.5.3.13 *Let T be an uncountable set and $\aleph', \aleph''$ cardinal numbers such that:*

1) $\aleph' < \operatorname{Card} T$;

2) $\aleph'' \leq \operatorname{Card} T$;

3) *the sum of any countable family of cardinal numbers strictly smaller than $\aleph''$ is strictly smaller than $\aleph''$.*

Denote by $\mathcal{F}$ the set of $u \in \mathcal{L}(\ell^2(T))$ for which the Hilbert dimension of $\operatorname{Im} u$ is at most equal to $\aleph'$ (is strictly smaller than $\aleph''$) and let $\mathcal{G}$ be a C^*-subalgebra of $\mathcal{L}(\ell^2(T))$.

a) *$\mathcal{F}$ is a closed proper ideal of $\mathcal{L}(\ell^2(T))$.*

b) *$\mathcal{F}+\mathcal{G}$ is the C^*-subalgebra of $\mathcal{L}(\ell^2(T))$ generated by $\mathcal{F}\cup\mathcal{G}$.*

c) *If $\mathcal{F}\cap\mathcal{G} = \{0\}$ then*

$$\|v\| \leq \|u+v\|$$

for all $(u,v) \in \mathcal{F}\times\mathcal{G}$.

By Example 5.5.3.1, Card T is the Hilbert dimension of $\ell^2(T)$.
a) follows from Example 2.2.2.23 and Theorem 5.5.2.5.
b & c follows from a) and Proposition 5.3.2.29. ■

5.5.4 The Fourier–Plancherel Operator

Throughout this subsection we assume $\mathbb{K} = \mathbb{C}$.

Theorem 5.5.4.1 (4) (Plancherel, 1910) *Take $m \in \mathbb{N}$ and $(x_k)_{k\in K}$ the orthonormal basis of $L^2(\mathbb{R}^m)$ defined in Corollary 5.5.3.9. Given $k \in K$ and $p \in \{0,1,2,3\}$, define*

$$|k| := \sum_{j=1}^{m} k_j$$

and

$$\mathcal{F}_p := \{x_k \mid k \in K\,,\ |k| \equiv p \pmod 4\}^{\perp\perp}\,.$$

Then there is a unique operator u on $L^2(\mathbb{R}^m)$ such that

$$ux_k = (-i)^{|k|}x_k$$

for every $k \in K$. u is called ***the Fourier–Plancherel operator****. Furthermore:*

a) *u is unitary.*

b) $u^4 = 1$.

c) $\sigma(u) = \sigma_p(u) = \{1, -i, -1, i\}$.

d) *For each $p \in \{0,1,2,3\}$, $\mathcal{F}_p$ is the eigenspace of u corresponding to $(-i)^p$.*

e) $x \in L^2(\mathbb{R}^m) \Rightarrow u^{-1}x = u^*x = \overline{u\bar{x}}$.

By Proposition 5.5.1.22, there is a unique operator on $L^2(\mathbb{R}^m)$ with

$$ux_k = (-i)^{|k|}x_k$$

for every $k \in K$ and this operator is an isometry. By Corollary 5.3.1.22, u is unitary.

Given $x \in L^2(\mathbb{R}^m)$,

$$ux = u\left(\sum_{k\in K}\langle x|x_k\rangle x_k\right) = \sum_{k\in K}\langle x|x_k\rangle ux_k =$$

$$= \sum_{k \in K} (-i)^{|k|} \langle x | x_k \rangle x_k = \sum_{p=0}^{3} (-i)^p \pi_{\mathcal{F}_p} x$$

(Fourier expansion and Corollary 5.5.1.16). Hence, for any $p \in \{0,1,2,3\}$,

$$\sigma_p(u) = \{1, -i, -1, i\},$$

$\mathcal{F}_p$ is the eigenspace of u corresponding to $(-i)^p$, and

$$u^4 = 1.$$

It follows that

$$\sigma(u) = \{1, -i, -1, i\}$$

(Theorem 2.1.3.4 b)). By Corollary 5.3.4.4 a), $\mathcal{F}_p$ is the eigenspace of u^* corresponding to i^p. Hence, given $x \in L^2(\mathbb{R}^m)$,

$$u^{-1}x = u^* x = u^* \left(\sum_{p=0}^{3} \pi_{\mathcal{F}_p} x \right) = \sum_{p=0}^{3} i^p \pi_{\mathcal{F}_p} x = \overline{\overline{ux}},$$

since the functions of x_k $(k \in K)$ are real. ■

Corollary 5.5.4.2 *If $p \in \{0,1,2,3\}$, then the resolvent of the Fourier-Plancherel operator has a pole of order one at $(-i)^p$ and (using the notation of Theorem 5.5.4.1), its residue is $\pi_{\mathcal{F}_p}$.*

This follows immediately from Theorem 5.5.4.1 a),d), and Proposition 5.3.1.16. ■

Lemma 5.5.4.3 (4) *Given $n \in \mathbb{N} \cup \{0\}$ and $t \in \mathbb{R}$,*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} e^{\frac{1}{2}s^2} \frac{d^n e^{-s^2}}{ds^n} ds = (-i)^n e^{\frac{1}{2}t^2} \frac{d^n e^{-t^2}}{dt^n}.$$

We have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} e^{\frac{1}{2}s^2} \frac{d^n e^{-s^2}}{ds^n} ds =$$

$$= \frac{(-1)^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2} \frac{d^n e^{-ist + \frac{1}{2}s^2}}{ds^n} ds = \frac{(-i)^n}{\sqrt{2\pi}} e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-s^2} \frac{d^n e^{\frac{1}{2}(s-it)^2}}{ds^n} ds =$$

$$= \frac{(-i)^n}{\sqrt{2\pi}} e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-s^2} \frac{d^n e^{\frac{1}{2}(s-it)^2}}{dt^n} ds = \frac{(-i)^n}{\sqrt{2\pi}} e^{\frac{1}{2}t^2} \frac{d^n}{dt^n} \int_{-\infty}^{\infty} e^{-s^2} e^{\frac{1}{2}(s-it)^2} ds =$$

$$= \frac{(-i)^n}{\sqrt{2\pi}} e^{\frac{1}{2}t^2} \frac{d^n}{dt^n} \left(e^{-t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(s+it)^2} ds \right) =$$

$$= \frac{(-i)^n}{\sqrt{2\pi}} e^{\frac{1}{2}t^2} \frac{d^n}{dt^n} \left(e^{-t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} ds \right) = (-i)^n e^{\frac{1}{2}t^2} \frac{d^n e^{-t^2}}{dt^n} . \quad \blacksquare$$

Theorem 5.5.4.4 (4) *Take $m \in \mathbb{N}$. Let λ be Lebesgue measure or $\mathbb{R}^m$ and u the Fourier-Plancherel operator on $L^2(\lambda)$. Take $x \in L^1(\lambda) \cap L^2(\lambda)$. Then*

$$ux(t) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int e^{-i\langle s|t\rangle} x(s) d\lambda(s) ,$$

$$u^{-1}x(t) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int e^{i\langle s|t\rangle} x(s) d\lambda(s) ,$$

λ almost everywhere on $\mathbb{R}^m$. Hence u and the Fourier transform coincide on $L^1(\lambda) \cap L^2(\lambda)$.

Let $(x_n)_{n\in\mathbb{N}} \cup \{0\}$ be the orthonormal basis of $L^2(\mathbb{R})$ defined in Example 5.5.3.6, $(y_k)_{k\in K}$ the orthonormal basis of $L^2(\lambda)$ defined in Corollary 5.5.3.9, and $\mathcal{F}$ the vector subspace of $L^2(\lambda)$ generated by $(y_k)_{k\in K}$. Then $\mathcal{F}$ is dense in $L^2(\lambda)$ (Corollary 5.2.3.10 b $\Rightarrow$ a). Given $k \in K$, define

$$|k| := \sum_{j=1}^{m} k_j .$$

Then

$$\frac{1}{(2\pi)^{\frac{m}{2}}} \int e^{-i\langle s|t\rangle} y_k(s) d\lambda(s) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int \prod_{j=1}^{m} e^{-is_j t_j} x_{k_j}(s_j) d\lambda(s) =$$

$$= \prod_{j=1}^{m} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-is_j t_j} x_{k_j}(s_j) ds_j = \prod_{j=1}^{m} (-i)^{k_j} x_{k_j}(t) =$$

$$= (-i)^{|k|} y_k(t) = u y_k(t)$$

for every $x \in \mathcal{F}$ and $t \in \mathbb{R}^m$ (Lemma 5.5.4.3). Thus

$$ux(t) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int e^{-\imath\langle s|t\rangle} x(s) d\lambda(s)$$

for every $x \in \mathcal{F}$ and $t \in \mathbb{R}^m$.

Given $t \in \mathbb{R}^m$ and $\varepsilon > 0$, define

$$A(t,\varepsilon) := \prod_{j=1}^{m} [t_j - \varepsilon, t_j + \varepsilon],$$

$$f_\varepsilon : \mathbb{R} \longrightarrow \mathbb{C}, \quad s \longmapsto \begin{cases} \frac{2\sin \varepsilon s}{s} & if \ s \neq 0 \\ 2\varepsilon & if \ s = 0, \end{cases}$$

$$g_\varepsilon : \mathbb{R}^m \longrightarrow \mathbb{C}, \quad s \longmapsto \prod_{j=1}^{m} f_\varepsilon(s_j).$$

Then

$$f_\varepsilon \in L^2(\mathbb{R}), \quad \sup_{s\in\mathbb{R}} |f_\varepsilon(s)| \leq 2\varepsilon.$$

Thus

$$g_\varepsilon \in L^2(\lambda) \text{ and } \sup_{s\in\mathbb{R}} |g_\varepsilon(s)| \leq (2\varepsilon)^m$$

for every $\varepsilon > 0$ and

$$\int_{A(t,\varepsilon)} e^{-\imath\langle s|r\rangle} d\lambda(r) = \int_{A(t,\varepsilon)} \prod_{j=1}^{m} e^{-is_j r_j} d\lambda(r) =$$

$$= \prod_{j=1}^{m} \int_{t_j-\varepsilon}^{t_j+\varepsilon} e^{-is_j r_j} dr_j = \prod_{j=1}^{m} \frac{e^{-is_j(t_j+\varepsilon)} - e^{-is_j(t_j-\varepsilon)}}{-is_j} =$$

$$= \prod_{j=1}^{m} e^{-is_j t_j} \frac{e^{is_j\varepsilon} - e^{-is_j\varepsilon}}{is_j} = \prod_{j=1}^{m} e^{-is_j t_j} f_\varepsilon(s_j) = e^{-\imath\langle s|t\rangle} g_\varepsilon(s)$$

for all $s, t \in \mathbb{R}^m$ and $\varepsilon > 0$. Moreover,

$$\lim_{0<\varepsilon\to 0} \frac{g_\varepsilon(s)}{(2\varepsilon)^m} = \prod_{j=1}^{m} \lim_{0<\varepsilon\to 0} \frac{f_\varepsilon(s_j)}{2j} = 1$$

for every $s \in \mathbb{R}^m$.

Take $x \in \mathcal{F}$, $t \in \mathbb{R}^m$, and $\varepsilon > 0$. By Fubini's Theorem,

$$\langle ux | e_{A(t,\varepsilon)} \rangle = \int\limits_{A(t,\varepsilon)} ux d\lambda =$$

$$= \int\limits_{A(t,\varepsilon)} \frac{1}{(2\pi)^{\frac{m}{2}}} \left(\int e^{-i\langle s|r\rangle} x(s) d\lambda(s) \right) d\lambda(r) =$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} \int x(s) \left(\int_{A(t,\varepsilon)} e^{-i\langle s|r\rangle} d\lambda(r) \right) d\lambda(s) =$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} \int e^{-i\langle s|t\rangle} x(s) g_\varepsilon(s) d\lambda(s).$$

Since $g_\varepsilon \in L^2(\lambda)$,

$$\int\limits_{A(t,\varepsilon)} ux d\lambda = \frac{1}{(2\pi)^{\frac{m}{2}}} \int e^{-i\langle s|t\rangle} x(s) g_\varepsilon(s) d\lambda(s)$$

whenever $x \in L^2(\lambda)$, $t \in \mathbb{R}^m$, and $\varepsilon > 0$.

Now take $x \in L^1(\lambda) \cap L^2(\lambda)$. There is a λ-null set $A \subset \mathbb{R}^m$, such that

$$\lim_{0<\varepsilon\to 0} \frac{1}{(2\varepsilon)^m} \int\limits_{A(t,\varepsilon)} ux d\lambda = (ux)(t)$$

for every $t \in \mathbb{R}^m \backslash A$ (Lebesgue point). By Lebesgue's Convergence Theorem,

$$(ux)(t) = \int\limits_{0<\varepsilon\to 0} \frac{1}{(2\varepsilon)^m} \frac{1}{(2\pi)^{\frac{m}{2}}} \int e^{-i\langle s|t\rangle} x(s) g_\varepsilon(s) d\lambda(s) =$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} \int e^{-i\langle s|t\rangle} x(s) d\lambda(s)$$

for every $t \in \mathbb{R}^m \backslash A$. By Theorem 5.5.4.1 e), it also follows that

$$(u^{-1}x)(t) = \overline{u\bar{x}}(t) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int \overline{e^{-i\langle s|t\rangle} \bar{x}(s)} d\lambda(s) =$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} \int e^{i\langle s|t\rangle} x(s) d\lambda(s)$$

λ-almost everyhwere in $\mathbb{R}^m$. ■

5.5.5 Operators and Orthonormal Bases

Theorem 5.5.5.1 (0) *Take $u \in \mathcal{L}(E,F)$ with E, F Hilbert spaces. Then the following are equivalent:*

a) *u is compact.*

b) *If $\mathfrak{F}$ is a weakly convergent filter on E containing a bounded set of E, then $u(\mathfrak{F})$ converges.*

c) *If A is a bounded set of E, then the map*

$$A_{E'} \longrightarrow F\,, \quad x \longmapsto ux$$

is continuous.

d) *If $(x_\iota)_{\iota\in I}$ is an orthonormal family in E and $(y_\iota)_{\iota\in I}$ is a bounded family in F, then $(\langle ux_\iota|y_\iota\rangle)_{\iota\in I} \in c_0(I)$.*

e) *If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are orthonormal sequences in E and F, respectively, then*

$$\lim_{n\to\infty} \langle ux_n|y_n\rangle = 0\,.$$

f) *If $(x_\iota)_{\iota\in I}$ and $(y_\lambda)_{\lambda\in L}$ are orthonormal families in E and F, respectively, then*

$$(\langle ux_\iota|y_\lambda\rangle)_{(\iota,\lambda)\in I\times L} \in c_0(I\times L)\,.$$

g) *If $v \in \mathcal{L}(\ell^2, E)$ and $w \in \mathcal{L}(\ell^2, F)$ then*

$$\lim_{n\to\infty} \langle uve_n|we_n\rangle = 0\,.$$

If these conditions are fulfilled and if A and B are orthonormal bases of E and F, respectively, then

$$u = \sum_{(x,y)\in A\times B} \langle ux|y\rangle\langle\cdot|x\rangle y \qquad (\text{in } \mathcal{K}(E,F))\,.$$

This representation of u is unique in the following sense: if $f \in \mathbb{K}^{A\times B}$ such that $(f(x,y)\langle\cdot|x\rangle y)_{(x,y)\in A\times B}$ is summable and

$$u = \sum_{(x,y)\in A\times B} f(x,y)\langle\cdot|x\rangle y\,,$$

then

$$f(x,y) = \langle ux|y\rangle$$

whenever $(x,y) \in A \times B$. *We have*

$$\mathcal{K}(E,F) = \left\{ \sum_{(x,y)\in A\times B} f(x,y)\langle\cdot|x\rangle y \;\middle|\; \begin{array}{l} f \in \mathbb{K}^{A\times B},\ (f(x,y)\langle\cdot|x\rangle y)_{(x,y)\in A\times B} \\ \textit{is summable in } \mathcal{K}(E,F) \end{array} \right\}.$$

a $\Leftrightarrow$ b $\Leftrightarrow$ c follows from Proposition 3.1.1.20, since E is reflexive (Corollary 5.2.5.5).

a $\Rightarrow$ d. We may assume I to be infinite. Let $\mathfrak{F}$ be the filter of cofinite subsets of I, i.e.

$$\mathfrak{F} := \{I\backslash J | J \in \mathfrak{P}_f(I)\}.$$

By Corollary 5.5.1.9,

$$\lim_{\iota,\mathfrak{F}} ux_\iota = 0.$$

Thus

$$\lim_{\iota,\mathfrak{F}} \langle ux_\iota|y_\iota\rangle = 0.$$

d $\Rightarrow$ e is trivial.

e $\Rightarrow$ b. Let $\mathfrak{F}$ be a weakly convergent filter on E containing a bounded set of E. We may assume that $\mathfrak{F}$ converges weakly to 0 and that $E^\# \in \mathfrak{F}$. Assume that $u(\mathfrak{F})$ does not converge to 0. Then, for some $\beta > 0$,

$$U_\beta^F(0) \notin u(\mathfrak{F}).$$

Hence $\mathfrak{F}$ does not converges to 0, and so there is some $\alpha > 0$ with

$$U_\alpha^E(0) \notin \mathfrak{F}.$$

Let $\mathfrak{G}$ be an ultrafilter on E finer than $\mathfrak{F}$ such that

$$E^\#\backslash \left(U_\alpha^E(0) \cup \overset{-1}{u}\left(U_\beta^F(0)\right)\right) \in \mathfrak{G}.$$

We construct a sequence $(z_n)_{n\in\mathbb{N}}$ in $E^\#$ inductively, such that

$$\alpha \le \|z_n\| \le 1, \quad \|\pi_{E_n} z_n\| < \frac{\alpha}{n},$$

$$\beta \leq \|uz_n\|\,, \quad \|\pi_{F_n} uz_n\| < \frac{\beta}{n}\,,$$

for every $n \in \mathbb{N}$, where E_n and F_n denote the vector subspaces of E and F generated by $\{z_k | k \in \mathbb{N}_{n-1}\}$ and $\{uz_k | k \in \mathbb{N}_{n-1}\}$, respectively. Take $n \in \mathbb{N}$ and suppose that the sequence has been constructed up to $n-1$. Let A and B be finite orthonormal bases of E_n and F_n, respectively (Corollary 5.5.1.10 a $\Rightarrow$ c). Then

$$\pi_{E_n} x = \sum_{a \in A} \langle x|a\rangle a\,, \quad \pi_{F_n} y = \sum_{b \in B} \langle y|b\rangle b$$

for all $x \in E$, $y \in F$ (Proposition 5.5.1.7). Since $\mathfrak{G}$ and $u(\mathfrak{G})$ converge weakly to $\{0\}$ (Corollary 1.4.1.6 a $\Rightarrow$ c),

$$\lim_{x,\mathfrak{G}} \pi_{E_n} x = 0\,, \quad \lim_{x,\mathfrak{G}} \pi_{F_n} ux = 0\,.$$

Hence there is a $C \in \mathfrak{G}$, with

$$\alpha \leq \|z\| \leq 1\,, \quad \|\pi_{E_n} z\| < \frac{\alpha}{n}\,,$$

$$\beta \leq \|uz\|\,, \quad \|\pi_{F_n} uz\| < \frac{\beta}{n}\,,$$

for every $z \in C$. Choose z_n arbitrarily from C. This completes the inductive construction.

Given $n \in \mathbb{N}$,

$$z_n - \pi_{E_n} z_n \in E_n^{\perp}\,, \quad uz_n - \pi_{F_n} uz_n \in F_n^{\perp}\,,$$

$$\|z_n - \pi_{E_n} z_n\|^2 = \|z_n\|^2 - \|\pi_{E_n} z_n\|^2 > \alpha - \frac{\alpha}{n} > 0\,,$$

$$\|uz_n - \pi_{F_n} uz_n\|^2 = \|uz_n\|^2 - \|\pi_{F_n} uz_n\|^2 > \beta - \frac{\beta}{n} > 0$$

(Proposition 5.2.3.1). For $n \in \mathbb{N}$, define

$$x_n := \frac{1}{\|z_n - \pi_{E_n} z_n\|}(z_n - \pi_{E_n} z_n)\,,$$

$$y_n := \frac{1}{\|uz_n - \pi_{F_n} uz_n\|}(uz_n - \pi_{F_n} uz_n)\,.$$

Then $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are orthonormal sequences in E and F, respectively.

Choose $n \in \mathbb{N}$. From

$$\|z_n - \pi_{E_n} z_n\| \leq \|z_n\| \leq 1\,,$$

$$\|uz_n - \pi_{F_n} uz_n\| \leq \|uz_n\| \leq \|u\|$$

we get that

$$\beta^2 \leq \|uz_n\|^2 = \langle uz_n | uz_n \rangle \leq$$

$$\leq |\langle u(z_n - \pi_{E_n} z_n) | uz_n - \pi_{F_n} uz_n \rangle| + |\langle \pi_{E_n} z_n | uz_n - \pi_{F_n} uz_n \rangle| + |\langle uz_n | \pi_{F_n} uz_n \rangle| \leq$$

$$\leq \|z_n - \pi_{E_n} z_n\| \, \|uz_n - \pi_{F_n} uz_n\| \, |\langle ux_n | y_n \rangle| +$$

$$+ \|\pi_{E_n} z_n\| \, \|uz_n - \pi_{F_n} uz_n\| + \|uz_n\| \, \|\pi_{F_n} uz_n\| \leq$$

$$\leq \|u\| \, |\langle ux_n | y_n \rangle| + \frac{\alpha}{n} \|u\| + \|u\| \frac{\beta}{n}\,.$$

By e), we get the contradiction

$$\beta^2 \leq 0\,.$$

Hence $u(\mathfrak{F})$ converges to 0.

a & e $\Rightarrow$ f. Assume $f)$ does not hold. Then there is an $\varepsilon > 0$ such that for every $C \in \mathfrak{P}_f(I \times L)$ there is an $(\iota, \lambda) \in (I \times L) \backslash C$ with

$$|\langle ux_\iota | y_\lambda \rangle| > \varepsilon\,.$$

By induction, we construct injective maps $f : \mathbb{N} \to I$, $g : \mathbb{N} \to L$ such that

$$|\langle ux_{f(n)} | y_{g(n)} \rangle| > \varepsilon$$

for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and assume f and g have been constructed up to $n-1$. By Corollary 5.5.1.9, there are $A \in \mathfrak{P}_f(I)$ and $B \in \mathfrak{P}_f(L)$ such that

$$\|ux_\iota\| < \varepsilon\,, \quad |\langle ux_{f(k)} | y_\lambda \rangle < \varepsilon$$

for every $\iota \in I \backslash A$, $\lambda \in L \backslash B$, and $k \in \mathbb{N}_{n-1}$. By the hypothesis of this proof, there is an

$$(\iota, \lambda) \in (I \times L) \backslash (A \cup f(\mathbb{N}_{n-1})) \times (B \cup g(\mathbb{N}_{n-1}))$$

with

$$|\langle ux_\iota | y_\lambda \rangle| > \varepsilon .$$

Then $\iota \notin f(\mathbb{N}_{n-1})$ and $\lambda \notin g(\mathbb{N}_{n-1})$, so that we may put

$$f(n) := \iota , \quad g(n) := \lambda .$$

This finishes the inductive construction. The existence of f and g contradicts e).

f $\Rightarrow$e is trivial.

a & e $\Rightarrow$ g. By a) and Proposition 3.1.1.11, $w^* \circ u \circ v \in \mathcal{K}(\ell^2)$, and so, by e),

$$0 = \lim_{n\to\infty} \langle w^* uve_n | e_n \rangle = \lim_{n\to\infty} \langle uve_n | we_n \rangle .$$

g $\Rightarrow$ e. By Proposition 5.5.1.22, there are $v \in \mathcal{L}(\ell^2, E)$ and $w \in \mathcal{L}(\ell^2, F)$ such that

$$ve_n = x_n , \quad we_n = y_n$$

for every $n \in \mathbb{N}$.

Now we prove the final assertion. For $C \in \mathfrak{P}_f(A)$ (resp. $D \in \mathfrak{P}_f(B)$), let C' (resp. D') denote the vector subspace of E (resp. of F) generated by C (resp. D). Take $\varepsilon > 0$. By Corollary 5.3.2.17, there are $C_0 \in \mathfrak{P}_f(A)$ and $D_0 \in \mathfrak{P}_f(B)$ such that

$$\|\pi_{D'} \circ u \circ \pi_{C'} - u\| < \varepsilon$$

for every $C \in \mathfrak{P}_f(A)$ and $D \in \mathfrak{P}_f(B)$ with $C_0 \subset C$, $D_0 \subset D$. By Proposition 5.5.1.7 (and Proposition 5.3.2.13 c), e)),

$$\pi_{D'} \circ u \circ \pi_{C'} = \sum_{(x,y)\in C\times D} \langle ux|y\rangle \langle \cdot | x \rangle y$$

for every $C \in \mathfrak{P}_f(A)$ and $D \in \mathfrak{P}_f(B)$. Hence

$$u = \sum_{(x,y)\in A\times B} \langle ux|y\rangle \langle \cdot | x \rangle y \quad (\text{in } \mathcal{K}(E,F))$$

(Corollary 5.5.1.11). It follows that

$$\mathcal{K}(E,F) = \left\{ \sum_{(x,y)\in A\times B} f(x,y)\langle \cdot | x\rangle y \;\middle|\; \begin{array}{c} f \in \mathbb{K}^{A\times B},\ (f(x,y)\langle \cdot | x\rangle y)_{(x,y)\in A\times B} \\ \text{is summable in } \mathcal{K}(E,F) \end{array} \right\} .$$

The uniqueness of the representation of u is easy to see. ∎

Remark. The last assertion of the theorem allows us to work with compact operators as we do with matrices.

Theorem 5.5.5.1' *The above assertions are further equivalent to the following one:*

h) *if* $(x_n)_{n\in\mathbb{N}}$ *is a weakly convergent sequence in* E, *then* $(ux_n)_{n\in\mathbb{N}}$ *converges.*

Since E is reflexive (Corollary 5.2.5.5), this assertion follows from Proposition 3.1.1.20'. ∎

Proposition 5.5.5.2 *Let* E *be a Hilbert space and take* $u \in \mathcal{L}(E)_+$. *Then the following are equivalent:*

a) u *is compact.*

b) *If* $\mathfrak{F}$ *is a filter on* E *containing a bounded set and converging weakly to* 0, *then*

$$\lim_{x,\mathfrak{F}} \langle ux, x\rangle = 0\,.$$

a $\Rightarrow$ b. By Theorem 5.5.5.1 a $\Rightarrow$ b, $u(\mathfrak{F})$ converges to 0 and the assertion follows.

b $\Rightarrow$ a. Let $\mathfrak{F}$ be a weakly convergent filter on E containing a bounded set. We show that $u^{\frac{1}{2}}(\mathfrak{F})$ converges. Without loss of generality, we may assume that $\mathfrak{F}$ converges weakly to 0. Since

$$\|u^{\frac{1}{2}}x\|^2 = \langle u^{\frac{1}{2}}x|u^{\frac{1}{2}}x\rangle = \langle ux|x\rangle$$

for every x, we see by b), that

$$0 = \lim_{x,\mathfrak{F}} \langle ux|x\rangle^{\frac{1}{2}} = \lim_{x,\mathfrak{F}} \|u^{\frac{1}{2}}x\|\,,$$

i.e. $u^{\frac{1}{2}}x(\mathfrak{F})$. converges to 0. By Theorem 5.5.5.1 b $\Rightarrow$ a, $u^{\frac{1}{2}}$ is compact. By Proposition 5.3.3.10 b), u is compact. ∎

Proposition 5.5.5.3 *If* E, F *are infinite-dimensional Hilbert spaces, then* $\mathcal{L}(E,F)/\mathcal{K}(E,F)$ *is not separable.*

Let $(x_\iota)_{\iota\in I}$ and $(y_\lambda)_{\lambda\in L}$ be orthonormal bases of E and F, respectively (Corollary 5.5.1.4) and let

$$q : \mathcal{L}(E,F) \longrightarrow \mathcal{L}(E,F)/\mathcal{K}(E,F)$$

be the quotient map. We may assume that

$$\mathbb{N} \subset I \cap L .$$

For $A \subset \mathbb{N}$, let u_A be the operator $E \to F$ such that

$$u_A x_\iota = \begin{cases} y_\iota & if \ \iota \in A \\ 0 & if \ \iota \in I \backslash A . \end{cases}$$

(Proposition 5.5.1.22). Given $A, B \in \mathfrak{P}(\mathbb{N})$, define $A \sim B$ by

$$A \sim B :\Longleftrightarrow (A\backslash B) \cup (B\backslash A) \text{ is finite.}$$

$\sim$ is an equivalence relation on $\mathfrak{P}(\mathbb{N})$ the equivalence classes of which are countable. Hence the set $\mathfrak{P}(\mathbb{N})/\sim$ of equivalence classes with respect to $\sim$ is uncountable. Given $A \in \mathfrak{P}(\mathbb{N})$, let $\dot{A}$ denote its equivalence class in $\mathfrak{P}(\mathbb{N})/\sim$. Take $A, B \in \mathfrak{P}(\mathbb{N})$ such that

$$A \not\sim B$$

and take $v \in \mathcal{K}(E,F)$. By Theorem 5.5.5.1 a $\Rightarrow$ e,

$$\lim_{n\to\infty} \langle v x_n | y_n \rangle = 0 .$$

It follows that

$$\|u_A - u_B - v\| \geq 1 .$$

Hence

$$\|q u_A - q u_B\| \geq 1 .$$

Thus $\mathcal{L}(E,F)/\mathcal{K}(E,F)$ is not separable. ■

Theorem 5.5.5.4 (0) *Let E, F be Hilbert spaces. Take orthonormal families $(x_\iota)_{\iota\in I}$ and $(y_\iota)_{\iota\in I}$ in E and F, respectively, and $f \in \mathbb{K}^I$. Let*

$$J := \{\iota \in I \mid f(\iota) \neq 0\} .$$

Take $u \in \mathcal{L}(E,F)$. Then the following are equivalent:

a) u *vanishes on* $\{x_\iota | \iota \in I\}^\perp$ *and* $ux_\iota = f(\iota)y_\iota$ *for every* $\iota \in I$.

b) $x \in E \Rightarrow ux = \sum_{\iota \in I} f(\iota)\langle x|x_\iota\rangle y_\iota$.

If these conditions are fulfilled, then:

c) $x \in E \Rightarrow \|ux\|^2 = \sum_{\iota \in I} |f(\iota)|^2 |\langle x|x_\iota\rangle|^2$.

d) $f \in \ell^\infty(I)$ *and* $\|u\| = \|f\|_\infty$.

e) $y \in F \Rightarrow u^*y = \sum_{\iota \in I} \overline{f(\iota)}\langle y|y_\iota\rangle x_\iota$.

f) $x \in E \Rightarrow u^*ux = \sum_{\iota \in I} |f(\iota)|^2 \langle x|x_\iota\rangle x_\iota$.

g) $(x_\iota)_{\iota \in J}$ *is an orthonormal basis of* $(\operatorname{Ker} u)^\perp$ *and*

$$\operatorname{Ker} u = \{x_\iota | \iota \in J\}^\perp .$$

h) $(y_\iota)_{\iota \in J}$ *is an orthonormal basis of* $\operatorname{Im} u$ *and*

$$\overline{\operatorname{Im} u} = \{y_\iota | \iota \in J\}^{\perp\perp} .$$

i) *Given* $y \in F$, $y \in \operatorname{Im} u$ *iff*

$$y \in \{y_\iota | \iota \in J\}^{\perp\perp}, \quad \left(\frac{\langle y|y_\iota\rangle}{f(\iota)}\right)_{\iota \in J} \in \ell^2(J)$$

and in this case

$$u\left(\sum_{\iota \in J} \frac{\langle y|y_\iota\rangle}{f(\iota)} x_\iota\right) = y .$$

j) *The following are equivalent:*

j$_1$) u *is compact.*

j$_2$) $f \in c_0(I)$.

j$_2$) $u = \sum_{\iota \in I} f(\iota)\langle \cdot|x_\iota\rangle y_\iota$ *in* $\mathcal{L}(E, F)$.

k) *For every* $v \in \mathcal{L}(F, E)$ *and* $\iota \in I$,

$$\langle vux_\iota|x_\iota\rangle = \langle uvy_\iota|y_\iota\rangle .$$

a $\Rightarrow$b. Put

$$G := \{x_\iota | \iota \in I\}^\perp .$$

Then, for any $x \in E$,

$$x = \pi_G x + \pi_{G^\perp} x = \pi_G x + \sum_{\iota \in I} \langle x | x_\iota \rangle x_\iota$$

(Proposition 5.2.3.7, Corollary 5.5.1.16), so that

$$ux = \sum_{\iota \in I} f(\iota) \langle x | x_\iota \rangle y_\iota .$$

b $\Rightarrow$ a is trivial.

c) follows from b) and Proposition 5.5.1.14 a $\Rightarrow$ c.

d) follows from c).

e) By d) and the Bessel's Inequality,

$$\sum_{\iota \in I} \overline{f(\iota)} \langle y | y_\iota \rangle x_\iota$$

is well-defined (Corollary 5.2.2.5) and by b),

$$\left\langle x \middle| \sum_{\iota \in I} \overline{f(\iota)} \langle y | y_\iota \rangle x_\iota \right\rangle =$$

$$= \sum_{\iota \in I} f(\iota) \langle y_\iota | y \rangle \langle x | x_\iota \rangle =$$

$$= \left\langle \sum_{\iota \in I} f(\iota) \langle x | x_\iota \rangle y_\iota \middle| y \right\rangle = \langle ux | y \rangle$$

for every $x \in E$. Hence

$$u^* y = \sum_{\iota \in I} \overline{f(\iota)} \langle y | y_\iota \rangle x_\iota .$$

f) By a) and e),

$$u^* u x_\iota = f(\iota) u^* y_\iota = |f(\iota)|^2 x_\iota$$

for every $\iota \in I$. Thus, by a $\Rightarrow$ b,

$$u^* u x = \sum_{\iota \in I} |f(\iota)|^2 \langle x | x_\iota \rangle x_\iota$$

for every $x \in E$.

g) By c),

$$\operatorname{Ker} u = \{x_\iota | \iota \in J\}^\perp,$$

and so

$$(\operatorname{Ker} u)^\perp = \{x_\iota | \iota \in J\}^{\perp\perp}.$$

By Proposition 5.5.1.2, it follows that $(x_\iota)_{\iota \in J}$ is an orthonormal basis of $(\operatorname{Ker} u)^\perp$.

h) By Proposition 5.3.2.4,

$$\overline{\operatorname{Im} u} = \{\operatorname{Ker} u^*\}^\perp.$$

Hence, by e) and g), $(y_\iota)_{\iota \in I}$ is an orthonormal basis of $\operatorname{Im} u$ and

$$\overline{\operatorname{Im} u} = \{y_\iota | \iota \in J\}^{\perp\perp}.$$

i) First suppose that $y \in \operatorname{Im} u$ and take $x \in E$ with $y = ux$. By h),

$$y \in \{y_\iota | \iota \in J\}^{\perp\perp}$$

and so, by b) and Proposition 5.5.1.2 (using the Fourier expansion),

$$\sum_{\iota \in J} \langle y | y_\iota \rangle y_\iota = y = ux = \sum_{\iota \in J} f(\iota) \langle x | x_\iota \rangle y_\iota.$$

Hence

$$\langle x | x_\iota \rangle = \frac{\langle y | y_\iota \rangle}{f(\iota)}$$

for every $\iota \in J$ (Proposition 5.5.1.8 a)) and

$$\left(\frac{\langle y | y_\iota \rangle}{f(\iota)} \right)_{\iota \in J} \in \ell^2(J) \quad \text{(Bessel's Inequality)}.$$

Now suppose that

$$y \in \{y_\iota | \iota \in J\}^{\perp\perp}, \quad \left(\frac{\langle y | y_\iota \rangle}{f(\iota)} \right)_{\iota \in J} \in \ell^2(J).$$

Put

$$x := \sum_{\iota \in J} \frac{\langle y | y_\iota \rangle}{f(\iota)} x_\iota$$

(Corollary 5.2.2.5). By a) and Proposition 5.5.1.2 (using the Fourier expansion),

$$ux = \sum_{\iota \in J} \frac{\langle y|y_\iota\rangle}{f(\iota)} ux_\iota = \sum_{\iota \in J} \langle y|y_\iota\rangle y_\iota = y\,.$$

$j_1 \Rightarrow j_2$ follows from Theorem 5.5.5.1 a $\Rightarrow$ d.

$j_2 \Rightarrow j_3$. Take $\varepsilon > 0$. Define

$$K := \{\iota \in I \mid |f(\iota)| > \varepsilon\} \in \mathfrak{P}_f(I)\,.$$

Suppose $L \in \mathfrak{P}_f(I)$ and $K \subset L$. By a),

$$\left(u - \sum_{\iota \in L} f(\iota)\langle\cdot|x_\iota\rangle y_\iota\right) x_\lambda = \begin{cases} 0 & \text{if } \lambda \in L \\ f(\lambda)y_\lambda & \text{if } \lambda \in I\backslash L \end{cases}$$

and

$$u - \sum_{\iota \in L} f(\iota)\langle\cdot|x_\iota\rangle y_\iota = 0$$

on $\{x_\iota | \iota \in I\}^\perp$. It follows from a $\Rightarrow$ d, that

$$\|u - \sum_{\iota \in L} f(\iota)\langle\cdot|x_\iota\rangle y_\iota\| = \sup_{\iota \in I\backslash L} |f(\iota)| \leq \varepsilon\,.$$

Since L and ε are arbitrary,

$$u = \sum_{\iota \in L} f(\iota)\langle\cdot|x_\iota\rangle y_\iota$$

in $\mathcal{L}(E,F)$.

$j_3 \Rightarrow j_1$ follows from Proposition 3.1.1.4.

k) By a) and e),

$$\langle uvy_\iota|y_\iota\rangle = \langle vy_\iota|u^*y_\iota\rangle = \langle vy_\iota|\overline{f(\iota)}x_\iota\rangle =$$

$$= f(\iota)\langle vy_\iota|x_\iota\rangle = \langle vux_\iota|x_\iota\rangle\,. \qquad \blacksquare$$

Corollary 5.5.5.5 (0) *Let E, F be Hilbert spaces. Let $(x_\iota)_{\iota \in I}$, $(y_\iota)_{\iota \in I}$ be orthonormal families in E and F, respectively, and take $f \in \mathbb{K}^I$. Then the family $(f(\iota)\langle\cdot|x_\iota\rangle y_\iota)_{\iota \in I}$ is summable iff $f \in c_0(I)$.*

The necessity is trivial (Corollary 1.1.6.7), so take $f \in c_0(I)$ and $\varepsilon > 0$. Put

$$J := \{\iota \in I \mid |f(\iota)| \geq \varepsilon\} \in \mathfrak{P}_f(I)\,.$$

Take $K \in \mathfrak{P}_f(I \backslash J)$. By Theorem 5.5.5.4 b $\Rightarrow$ d,

$$\| \sum_{\iota \in K} f(\iota) \langle \cdot | x_\iota \rangle y_\iota \| = \sup_{\iota \in K} |f(\iota)| < \varepsilon .$$

Hence $(f(\iota)\langle \cdot | x_\iota \rangle y_\iota)_{\iota \in I}$ is summable (Proposition 1.1.6.6). ■

Proposition 5.5.5.6 *Let E be a Hilbert space and $\mathcal{F}$ a Gelfand C^*-subalgebra of $\mathcal{K}(E)$. Then there is an orthonormal set A of E and a norm-preserving homomorphism of involutive algebras*

$$\varphi : \mathcal{F} \longrightarrow c_0(A) , \quad u \longmapsto \varphi_u$$

such that:

1) $u \in \mathcal{F} \Rightarrow u = \sum_{x \in A} \varphi_u(x) \langle \cdot | x \rangle x .$

2) *For every $x \in A$, the map*

$$\widetilde{x} : \mathcal{F} \longrightarrow \mathbb{K} , \quad u \longmapsto \langle ux | x \rangle$$

is in $\sigma(\mathcal{F})$ and the map

$$A \longrightarrow \sigma(\mathcal{F}) , \quad x \longmapsto \widetilde{x}$$

is surjective. If, in addition, A is infinite, then the image of the filter of cofinite subsets of A converges to the Alexandroff point of $\sigma(\mathcal{F})$.

By Proposition 5.3.1.17, $\sigma(\mathcal{F})$ is discrete. Let $x' \in \sigma(\mathcal{F})$. Since the Gelfand transform on $\mathcal{F}$ is an isometry of C^*-algebras (Corollary 4.1.2.5), there is an orthogonal projection $\pi_{x'}$ of $\mathcal{F}$, such that

$$\widehat{\pi}_{x'} = e_{x'}^{\sigma(\mathcal{F})} .$$

Then $\pi_{x'}$ is an orthogonal projection in E with a finite-dimensional image. Let $A_{x'}$ be an orthonormal basis of $\operatorname{Im} x'$ (Corollary 5.5.1.4).

Given distinct

$$x', y' \in \sigma(\mathcal{F}) ,$$

$$\widehat{\pi_{x'} \pi_{y'}} = \widehat{\pi}_{x'} \widehat{\pi}_{y'} = e_{x'} e_{y'} = 0 ,$$

so that

$$\pi_{x'}\pi_{y'} = 0\,,$$

$$A_{x'} \perp A_{y'}$$

(Corollary 5.2.3.15 b $\Rightarrow$ a).

Put

$$A := \bigcup_{x' \in \sigma(\mathcal{F})} A_{x'}\,.$$

By the above considerations, A is an orthonormal set of E. For each $u \in \mathcal{F}$, let φ_u denote the map $A \to \mathbb{K}$ which takes the value $\widehat{u}(x')$ on $A_{x'}$ for every $x' \in \sigma(\mathcal{F})$. Since the Gelfand transform $\mathcal{F} \to \mathcal{C}_0(\sigma(\mathcal{F}))$ is an isometry of C^*-algebras (Corollary 4.1.2.5), $\varphi_u \in c_0(A)$ for every $u \in \mathcal{F}$ and the map

$$\mathcal{F} \longrightarrow c_0(A)\,, \quad u \longmapsto \varphi_u$$

is a norm-preserving homomorphism of involutive algebras.

1) $\widehat{u} = \sum\limits_{x' \in \sigma(\mathcal{F})} \widehat{u}(x') e_{x'} = \sum\limits_{x' \in \sigma(\mathcal{F})} \widehat{u}(x') \widehat{\pi}_{x'}\,.$

Thus, by Proposition 5.5.1.7 (and Corollary 5.5.5.5),

$$u = \sum_{x' \in \sigma(\mathcal{F})} \widehat{u}(x')\pi_{x'} = \sum_{x' \in \sigma(\mathcal{F})} \widehat{u}(x') \sum_{x \in A_{x'}} \langle \cdot | x \rangle x = \sum_{x \in A} \varphi_u(x) \langle \cdot | x \rangle x\,.$$

2) Take $x \in A$. Then

$$\langle ux | x \rangle = \varphi_u(x)$$

for every $u \in \mathcal{F}$. Hence $\widetilde{x} \in \sigma(E)$ and $x \in A_{\widetilde{x}}$. The other assertions follow. ■

Proposition 5.5.5.7 (0) *Let A be an orthonormal basis of the Hilbert space E. Take $f \in \mathbb{K}^A$ and $u \in \mathcal{L}(E)$ such that*

$$ux = f(x)x$$

for every $x \in A$.

a) *$f \in \ell^\infty(A)$, $\|u\| = \|f\|_\infty$, and for every $x \in E$,*

$$ux = \sum_{y \in A} f(y) \langle x | y \rangle y\,,$$

$$\|ux\|^2 = \sum_{y \in A} |f(y)|^2 |\langle x | y \rangle|^2\,,$$

$$u^*x = \sum_{y\in A} \overline{f(y)}\langle x|y\rangle y\,,$$

$$u^*ux = \sum_{y\in A} |f(y)|^2\langle x|y\rangle y\,,$$

$$\langle ux|x\rangle = \sum_{y\in A} f(y)|\langle x|y\rangle|^2\,.$$

b) $a \in \mathbb{K} \Rightarrow \operatorname{Ker}(\alpha 1 - u) = \overset{-1}{f}(\alpha)^{\perp\perp}$.

c) $\sigma_p(u) = f(A)$.

d) $\sigma(u) = \overline{f(A)}$.

e) *u is self-normal and for every $g \in \mathcal{C}(\sigma(u))$ and $x \in A$,*

$$(g(u))x = g(f(x))x\,.$$

f) *For every $f \in \ell^\infty(A)$, there is a unique $\widetilde{f} \in \mathcal{L}(E)$ such that*

$$\widetilde{f}x = f(x)x$$

for every $x \in A$. The map

$$\ell^\infty(A) \longrightarrow \mathcal{L}(E)\,, \quad f \longmapsto \widetilde{f}$$

is a homomorphism of involutive unital algebras preserving the norms. $\widetilde{f} \in \mathcal{K}(E)$ iff $f \in c_0(A)$.

a) follows from Theorem 5.5.5.4 a $\Rightarrow$ b & c & d & e & f.

b) By Theorem 5.5.5.4 a $\Rightarrow$ g,

$$\operatorname{Ker}(\alpha 1 - u) = \{x \in A \mid f(x) \neq \alpha\}^\perp\,.$$

Thus, by Corollary 5.5.1.17,

$$\operatorname{Ker}(\alpha 1 - u) = \overset{-1}{f}(\alpha)^{\perp\perp}\,.$$

c) follows from b).

d) Take $\alpha \in \mathbb{K}\backslash\overline{f(A)}$. By b), $\alpha 1 - u$ is injective. Take $x \in E$. Then

$$\left(\frac{\langle x|y\rangle}{\alpha - f(y)}\right)_{y\in A} \in \ell^2(A)$$

(Bessel's Inequality), and so $x \in \mathrm{Im}\,(\alpha 1 - u)$ by Theorem 5.5.5.4 i). Hence $\alpha 1 - u$ is surjective. By the Principle of Inverse Operators, $\alpha 1 - u$ is invertible, i.e. $\alpha \notin \sigma(u)$. Hence

$$\sigma(u) \subset \overline{f(A)}\,.$$

The reverse inclusion follows from c).

e) Take $x \in E$. By a),

$$u^*ux = \sum_{y \in A} |f(y)|^2 \langle x|y\rangle y = uu^*x\,,$$

and if f is real, then

$$u^*x = ux\,.$$

Hence u is self-normal. By a), again,

$$(P(u, u^*))x = P(f(x), \overline{f(x)})x$$

for every $P \in \mathbb{K}[s, t]$. But by Weierstrass-Stone Theorem, g can be approximated uniformly on $\sigma(u)$ by such P. Thus

$$(g(u))x = g(f(x))x\,.$$

f) The first two assertions were proved in Corollary 5.5.2.10. The final assertion follows from Theorem 5.5.5.4 a $\Rightarrow$ j. ■

Proposition 5.5.5.8 (0) *Let E be a Hilbert space and take $u \in \mathcal{L}(E)$. For every $\alpha \in \sigma(u)$, let π_α denote the orthogonal projection of E onto* $\mathrm{Ker}\,(\alpha 1 - u)$. *Then the following are equivalent:*

a) *u is normal and*

$$u = \sum_{\alpha \in \sigma(u)} \alpha \pi_\alpha\,.$$

b) *$\sigma(u)\backslash\{0\}$ is discrete and there are an orthonormal basis A of E and an $f \in \mathbb{K}^A$ with*

$$ux = f(x)x$$

for every $x \in A$.

If these equivalent conditions hold, then u is self-normal and

$$\sigma(u)\backslash\{0\} = \sigma_p(u)\backslash\{0\} = f(A)\backslash\{0\}\,.$$

a $\Rightarrow$ b and the last assertion. For $\alpha \in \sigma(u)$, let A_α be an orthonormal basis of $\mathrm{Ker}\,(\alpha 1 - u)$ (Corollary 5.5.1.4). Put

$$A := \bigcup_{\alpha\in\sigma(u)} A_\alpha\,.$$

A is an orthonormal set of E (Corollary 5.3.4.5). If $x \in A^\perp$, then

$$ux = \sum_{\alpha\in\sigma(u)} \alpha\pi_\alpha x = 0\,,$$

and so

$$x \in A_0^\perp \cap \mathrm{Ker}\,u\,,$$

$$x = 0\,.$$

Hence A is an orthonormal basis of E. We define $f : A \to \mathbb{K}$ by $f|A_\alpha = \alpha$ for $\alpha \in \sigma(u)$. Then, given $x \in A$,

$$x \in A_{f(x)} \subset \mathrm{Ker}\,(f(x)1 - u)\,.$$

Hence

$$ux = f(x)x\,.$$

By Proposition 5.5.5.7 c),d),e), u is self-normal and

$$f(A) = \sigma_p(u)\,, \quad \overline{f(A)} = \sigma(u)\,.$$

Since

$$(\|\alpha\pi_\alpha\|)_{\alpha\in\sigma(u)} \in c_0(\sigma(u))$$

and

$$\alpha \in \sigma_p(u) \Longrightarrow \|\pi_\alpha\| = 1\,,$$

it follows that $f(A)\backslash\{0\}$ is discrete. Hence $\sigma(u)\backslash\{0\}$ is discrete and

$$\sigma(u)\backslash\{0\} = \sigma_p(u)\backslash\{0\} = f(A)\backslash\{0\}\,.$$

b $\Rightarrow$ a. By Proposition 5.5.5.7 e), u is self-normal, so the assertion follows from Proposition 5.3.4.7 a). ■

Proposition 5.5.5.9 (0) *Let E, F be Hilbert spaces and take $u \in \mathcal{L}(E, F)$. Then the following are equivalent:*

a) *u is an isometry.*

b) *If A is an orthonormal basis of E, then $u(A)$ is an orthonormal basis of F.*

c) *There is an orthonormal basis $(x_\iota)_{\iota \in I}$ of E for which $(ux_\iota)_{\iota \in I}$ is an orthonormal basis of F.*

If $E = F$, then the above assertions are equivalent to the following one:

d) *u is unitary.*

a $\Rightarrow$ b. By Corollary 5.1.1.4,

$$\langle ux|uy\rangle = \langle x|y\rangle = \delta_{xy}$$

for all $x, y \in A$. $u(A)$ is thus an orthonormal set of F. By Theorem 5.5.5.4 a $\Rightarrow$ h,

$$u(A)^{\perp\perp} = \overline{\operatorname{Im} u} = F.$$

Hence $u(A)$ is an orthonormal basis of F.

b $\Rightarrow$ c. Let $(x_\iota)_{\iota \in I}$ be an orthonormal basis of E (Corollary 5.5.1.4). By b), $\{ux_\iota | \iota \in I\}$ is an orthonormal basis of F. Let ι, λ be distinct elements of I. Assume that

$$ux_\iota = ux_\lambda .$$

Define

$$A := \{x_\mu | \mu \in I \backslash (\iota, \lambda)\} \cup \left\{ \frac{1}{\sqrt{2}}(x_\iota - x_\lambda), \frac{1}{\sqrt{2}}(x_\iota + x_\lambda) \right\} .$$

It is easy to see that A is an orthonormal basis of E. Since $u(A)$ is not an orthonormal basis of F, this contradicts b). Hence

$$ux_\iota \neq ux_\lambda$$

and $(ux_\iota)_{\iota \in I}$ is an orthonormal basis of F.

c $\Rightarrow$ a. Take $x \in E$. By the Fourier expansion,

$$x = \sum_{\iota \in I} \langle x|x_\iota\rangle x_\iota ,$$

so that

$$ux = \sum_{\iota \in I} \langle x | x_\iota \rangle u x_\iota .$$

By Corollary 5.2.2.4,

$$\|ux\|^2 = \sum_{\iota \in I} |\langle x | x_\iota \rangle|^2 = \|x\| .$$

Since

$$\overline{\operatorname{Im} u} = \{ux_\iota | \iota \in I\}^{\perp\perp}$$

(Theorem 5.5.5.4 a $\Rightarrow$ h), it follows from Proposition 1.2.1.18 d), that u is an isometry.

a $\Leftrightarrow$ d follows from Corollary 5.3.1.22. ■

Proposition 5.5.5.10 (0) *Let E, F be Hilbert spaces. Take $u \in \mathcal{L}(E, F)$ and put $v := (u^* \circ u)^{\frac{1}{2}}$ (Corollary 5.3.3.9). Then there is a unique $w \in \mathcal{L}(E, F)$ such that*

$$\operatorname{Ker} u = \operatorname{Ker} w , \quad u = w \circ v .$$

Moreover:

a) $\|ux\| = \|vx\|$ *for every* $x \in E$.

b) $\operatorname{Ker} u = \operatorname{Ker} v , \quad \overline{\operatorname{Im} v} = (\operatorname{Ker} u)^\perp , \quad E = \operatorname{Ker} u \oplus \overline{\operatorname{Im} v}$.

c) $\|w\| \leq 1 , \overline{\operatorname{Im} u} = \operatorname{Im} w$.

d) $\|wx\| = \|x\|$ *for every* $x \in (\operatorname{Ker} w)^\perp$.

e) $w^* \circ w = \pi_{(\operatorname{Ker} u)^\perp} , w^* \circ u = v$.

f) *If $(x_\iota)_{\iota \in I}$ is an orthonormal family in E and $(\alpha_\iota)_{\iota \in I}$ is a family in $\mathbb{K} \backslash \{0\}$ such that*

$$v = \sum_{\iota \in I} \alpha_\iota \langle \cdot | x_\iota \rangle x_\iota ,$$

then $(wx_\iota)_{\iota \in I}$ is an orthonormal family in F and

$$u = \sum_{\iota \in I} \alpha_\iota \langle \cdot | x_\iota \rangle w x_\iota .$$

Step 1 a & b

Now

$$\|vx\|^2 = \langle vx|vx\rangle = \langle x|v^2x\rangle = \langle x|u^*ux\rangle = \langle ux|ux\rangle = \|ux\|^2,$$

$$\|ux\| = \|vx\|.$$

Hence

$$\operatorname{Ker} u = \operatorname{Ker} v, \quad (\operatorname{Ker} u)^\perp = (\operatorname{Ker} v)^\perp = \overline{\operatorname{Im} v},$$

$$E = \operatorname{Ker} u \oplus \overline{\operatorname{Im} v}$$

(Propositions 5.3.2.4 and 5.2.3.7).

Step 2 (Existence and Uniqueness of w) & c & d

We define

$$w_0 : \operatorname{Im} v \longrightarrow \operatorname{Im} u, \quad vx \longmapsto ux.$$

By a), w_0 is well-defined and linear. Moreover,

$$\|w_0x\| = \|x\|$$

for every $x \in \operatorname{Im} v$. Hence there is an isometry

$$\overline{\operatorname{Im} v} \longrightarrow \overline{\operatorname{Im} u}$$

which extends w_0. By b), we may define w by extending this isometry by 0 on $\operatorname{Ker} u$. Then

$$\operatorname{Ker} u = \operatorname{Ker} w, \quad u = w \circ v, \quad \|w\| \leq 1, \quad \overline{\operatorname{Im} u} = \operatorname{Im} w$$

and

$$\|wx\| = \|x\|$$

for every $x \in (\operatorname{Ker} w)^\perp$. By b), w is unique.

Step 3 e)

We put

$$F := (\operatorname{Ker} u)^{\perp}.$$

Then

$$\langle w^* wx|y\rangle = \langle wx|wy\rangle = \langle w\pi_F x|w\pi_F y\rangle = \langle \pi_F x|\pi_F y\rangle = \langle \pi_F x|y\rangle$$

for all $x, y \in E$ (d), Corollary 5.1.1.4, Proposition 5.2.3.1 a $\Rightarrow$ b). Thus by b),

$$w^* \circ w = \pi_F, \quad w^* \circ u = w^* \circ w \circ v = \pi_F v = v.$$

Step 4 f)

By b),

$$\operatorname{Ker} w = \operatorname{Ker} v = \{x_\iota | \iota \in I\}^{\perp},$$

so that

$$(\operatorname{Ker} w)^{\perp} = \{x_\iota | \iota \in I\}^{\perp\perp}.$$

By d), the map

$$(\operatorname{Ker} w)^{\perp} \longrightarrow \operatorname{Im} w, \quad x \longmapsto wx$$

is an isometry and so $(wx_\iota)_{\iota \in I}$ is an orthonormal family (Proposition 5.5.5.9 a $\Rightarrow$ b). Now

$$u = w \circ v = w \circ \sum_{\iota \in I} \alpha_\iota \langle \cdot | x_\iota \rangle x_\iota = \sum_{\iota \in I} \alpha_\iota \langle \cdot | x_\iota \rangle w x_\iota$$

(Proposition 5.3.2.13 c)). ■

5.5.6 Self-normal Compact Operators

Theorem 5.5.6.1 (0) *Let u be a self-normal operator on a Hilbert space E. Given $\alpha \in \mathbb{K}$, let π_α be the orthogonal projections of E onto $\operatorname{Ker}(\alpha 1 - u)$. Then the following are equivalent:*

a) *u is compact.*

b) *$\sigma(u)\backslash\{0\}$ is discrete and $\operatorname{Ker}(\alpha 1 - u)$ is finite-dimensional for every $\alpha \in \sigma_p(u)\backslash\{0\}$.*

c) *$\operatorname{Ker}(\alpha 1 - u)$ is finite-dimensional for every $\alpha \in \sigma_p(u)\backslash\{0\}$ and*

$$u = \sum_{\alpha \in \sigma(u)} \alpha \pi_\alpha .$$

d) *There is an orthonormal basis A of E and an $f \in c_0(A)$ such that*

$$ux = f(x)x$$

for every $x \in A$.

e) *There is an orthonormal basis A of E and an $f \in c_0(A)$ with*

$$u = \sum_{x \in A} f(x)\langle \cdot | x \rangle x \quad (\textbf{\textit{diagonalization of}}\ u) .$$

If these conditions are fulfilled, then:

f) *For each $g \in \mathcal{C}(\sigma(u))$,*

$$g(u) = \sum_{x \in A} g(f(x))\langle \cdot | x \rangle x$$

whenever E is finite-dimensional and

$$g(u) = g(0)1 + \sum_{x \in A} (g(f(x)) - g(0))\langle \cdot | x \rangle x$$

whenever E is infinite-dimensional.

g) *For each $\alpha \in \mathbb{K}\backslash\sigma(u)$,*

$$(\alpha 1 - u)^{-1} = \sum_{x \in A} \frac{1}{\alpha - f(x)} \langle \cdot | x \rangle x$$

whenever E is finite-dimensional and

$$(\alpha 1 - u)^{-1} = \frac{1}{\alpha} 1 + \frac{1}{\alpha} \sum_{x \in A} \frac{f(x)}{\alpha - f(x)} \langle \cdot | x \rangle x$$

whenever E is infinite-dimensional.

h) *If* $\alpha \in \mathbb{K}\backslash\{0\}$, *then*

$$\operatorname{Im}(\alpha 1 - u) = \overset{-1}{f}(\alpha)^{\perp}$$

and

$$\frac{1}{\alpha}x + \frac{1}{\alpha}\sum_{y \in A\backslash \overset{-1}{f}(\alpha)} \frac{f(y)\langle x|y\rangle}{\alpha - f(y)}y = \sum_{y \in A\backslash \overset{-1}{f}(\alpha)} \frac{\langle x|y\rangle}{\alpha - f(y)}y \in \overbrace{\alpha 1 - u}^{-1}(x)$$

for every $x \in \operatorname{Im}(\alpha 1 - u)$.

i) *If*

$$F := \bigcup_{\alpha \in \sigma(u)\backslash\{0\}} \operatorname{Ker}(\alpha 1 - u),$$

then

$$\operatorname{Ker} u = F^{\perp}.$$

j) $\sigma_p(u) = f(A)$, $\sigma(u) = \overline{f(A)}$.

a $\Rightarrow$ b follows from Theorem 3.1.5.1 a),b).

b $\Rightarrow$ c follows from Proposition 5.3.4.7 a).

c $\Rightarrow$ d. By Proposition 5.5.5.8 a $\Rightarrow$ b, there is an orthonormal basis A of E and an $f \in \mathbb{K}^A$ such that

$$ux = f(x)x$$

for every $x \in A$. Since, by Proposition 3.1.1.4, u is compact, we see that $f \in c_0(A)$, by Proposition 5.5.5.4 $j_1 \Rightarrow j_2$.

d $\Rightarrow$ e $\Rightarrow$ a follows from Theorem 5.5.5.4 $j_2 \Rightarrow j_3 \Rightarrow j_1$.

f) By Proposition 5.5.5.7 e),

$$(g(u))x = g(f(x))x$$

for every $x \in A$. The assertion now follows from a $\Rightarrow$ e.

g) Define

$$g : \sigma(u) \longrightarrow \mathbb{K}, \quad \beta \longmapsto \frac{1}{\alpha - \beta}.$$

Then

$$g(f(x)) = \frac{1}{\alpha - f(x)},$$

$$g(f(x)) - g(0) = \frac{1}{\alpha - f(x)} - \frac{1}{\alpha} = \frac{1}{\alpha}\frac{f(x)}{\alpha - f(x)}$$

for every $x \in A$ and the assertion now follows from f), since

$$g(u) = (\alpha 1 - u)^{-1}.$$

h) By Theorem 3.1.3.8, $\alpha 1 - u$ is a Fredholm operator, so that

$$\operatorname{Im}(\alpha 1 - u) = \overline{\operatorname{Im}(\alpha 1 - u)} = \{x \in A | f(x) \neq \alpha\}^{\perp\perp} = \overset{-1}{f}(\alpha)^{\perp}$$

by Theorem 5.5.5.4 h) and Corollary 5.5.1.17. By Theorem 5.5.5.4 i),

$$\sum_{y \in A \setminus \overset{-1}{f}(\alpha)} \frac{\langle x|y\rangle}{\alpha - f(y)} y \in \overbrace{\alpha 1 - u}^{-1}(x).$$

Now

$$\frac{1}{\alpha}x + \frac{1}{\alpha}\sum_{y \in A \setminus \overset{-1}{f}(\alpha)} \left(\langle x|y\rangle + \frac{f(y)\langle x|y\rangle}{\alpha - f(y)}\right) y = \sum_{y \in A \setminus \overset{-1}{f}(\alpha)} \frac{\langle x|y\rangle}{\alpha - f(y)} y.$$

i) follows from c).

j) follows from d) and Proposition 5.5.5.7 c),d). ■

Corollary 5.5.6.2 *Take $n \in \mathbb{N}$. Let u be a self-normal operator on $\mathbb{K}^n$, a the associated matrix of u, P the characteristic polynomial of a, i.e.*

$$P(t) = \operatorname{Det}(t1 - a),$$

and take $\alpha \in \sigma(u)$. Then the mulitiplicity of the eigenvalue α with respect to u agrees with the multiplicity of the zero α of P.

By Theorem 5.5.6.1 a $\Rightarrow$ d, there is an orthonormal basis $(x_i)_{i \in \mathbb{N}_n}$ of $\mathbb{K}^n$ and a family $(\alpha_i)_{i \in \mathbb{N}_n}$ in $\mathbb{K}$, such that

$$ux_i = \alpha_i x_i$$

for every $i \in \mathbb{N}_n$. If p is the multiplicity of α with respect to u, then

$$p = \operatorname{Card}\{i \in \mathbb{N}_n \mid \alpha_i = \alpha\}$$

(Proposition 5.5.5.7 b)). Let v be the operator on $\mathbb{K}^n$, given by

$$ve_i = x_i$$

for every $i \in \mathbb{N}_n$. v is obviously bijective. Put

$$w := v^{-1}uv\,.$$

Then

$$we_i = \alpha_i e_i$$

for every $i \in \mathbb{N}_n$ and

$$t1 - w = tv^{-1}v - v^{-1}uv = v^{-1}(t1 - u)v\,.$$

Thus

$$P(t) = \prod_{i=1}^{n}(t - \alpha_i)\,.$$

Hence the mulitplicity of the zero α of P is p. ■

Remark. The above result does not hold if u is not self-normal. Indeed, let u be the operator associated to the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then $\sigma(u) = \{1\}$. 1 is a zero of the charactertistic polynomial of the above matrix with multiplicity 2, but the multiplicity of the eigenvalue 1 with respect to u is 1.

Corollary 5.5.6.3 *Take $n \in \mathbb{N}$ and $a \in \mathbb{K}_{n,n}$. Consider*

$$f : \mathbb{K}^n \times \mathbb{K}^n \longrightarrow \mathbb{K}\,, \quad (x, y) \longmapsto \sum_{i,j=1}^{n} a_{ij}x_i\overline{y}_j\,.$$

If the operator associated to a is self-normal, then there is an orthonormal basis $(z_i)_{i\in\mathbb{N}_n}$ of $\mathbb{K}^n$ and a family $(\gamma_i)_{i\in\mathbb{N}_n}$ in $\mathbb{K}$ such that

$$f\left(\sum_{i=1}^{n}\alpha_i z_i\,,\ \sum_{i=1}^{n}\beta_i z_i\right) = \sum_{i\in 1}\gamma_i\alpha_i\overline{\beta}_i\,,$$

$$f\left(\sum_{i=1}^{n}\alpha_i z_i\,,\ \sum_{i=1}^{n}\alpha_i z_i\right) = \sum_{i\in 1}\gamma_i|\alpha_i|^2$$

for every $(\alpha_i)_{i\in\mathbb{N}_n}$, $(\beta_i)_{i\in\mathbb{N}_n} \in \mathbb{K}^n$.

This assertion follows from Theorem 5.5.6.1 a $\Rightarrow$ d and Proposition 5.5.5.7 a). ■

Corollary 5.5.6.4 (0) *Let E be a real (complex) Hilbert space and $\mathcal{F}$ the vector subspace of $\mathcal{L}(E)$ generated by*

$$\{\langle\cdot|x\rangle x \mid x \in E\}.$$

Then

$$\mathcal{F} = \operatorname{Re}\mathcal{L}_f(E) \qquad (\mathcal{F} = \mathcal{L}_f(E))$$

and

$$\overline{\mathcal{F}} = \operatorname{Re}\mathcal{K}(E) \qquad (\overline{\mathcal{F}} = \mathcal{K}_f(E)).$$

By Proposition 5.3.2.13 b),

$$\langle\cdot|x\rangle x \in \operatorname{Re}\mathcal{L}_f(E)$$

for every $x \in E$, so that

$$\mathcal{F} \subset \operatorname{Re}\mathcal{L}_f(E) \quad (\mathcal{F} \subset \mathcal{L}_f(E)),$$

$$\overline{\mathcal{F}} \subset \operatorname{Re}\mathcal{K}_f(E) \quad (\overline{\mathcal{F}} \subset \mathcal{K}(E)).$$

Take $u \in \operatorname{Re}\mathcal{L}_f(E)$ and $v \in \operatorname{Re}\mathcal{K}(E)$. By Theorem 5.5.6.1 a $\Rightarrow$ e, there are orthonormal bases A, B of E and $f \in \mathbb{K}^{(A)}$, $g \in c_0(B)$ such that

$$u = \sum_{x\in A} f(x)\langle\cdot|x\rangle x, \quad v = \sum_{y\in B} g(y)\langle\cdot|y\rangle y.$$

Thus $u \in \mathcal{F}$ and $v \in \overline{\mathcal{F}}$ and hence

$$\operatorname{Re}\mathcal{L}_f(E) \subset \mathcal{F}, \qquad \operatorname{Re}\mathcal{K}(E) \subset \overline{\mathcal{F}}.$$

Suppose that $\mathbb{K} = \mathbb{C}$ and take $u \in \mathcal{L}_f(E)$, $v \in \mathcal{K}(E)$. Then

$$\operatorname{re} u, \operatorname{im} u \in \operatorname{Re}\mathcal{L}_f(E) \subset \mathcal{F}, \quad \operatorname{re} v, \operatorname{im} v \in \operatorname{Re}\mathcal{K}(E) \subset \overline{\mathcal{F}}$$

(Proposition 5.3.2.13 b), Corollary 5.5.1.11, Proposition 5.3.2.2 a $\Rightarrow$ b), so that

$$u = \operatorname{re} u + i \operatorname{im} u \in \mathcal{F}, \quad v = \operatorname{re} v + i \operatorname{im} v \in \overline{\mathcal{F}},$$

$$\mathcal{L}_f(E) \subset \mathcal{F}, \quad \mathcal{K}(E) \subset \overline{\mathcal{F}}.$$ ■

Corollary 5.5.6.5 (0) *Let u be a self-normal compact operator on an infinite-dimensional Hilbert space and take $f \in C(\sigma(u))$. Then $f(u)$ is compact iff*

$$f(0) = 0 .$$

If

$$\sum_{x \in A} g(x)\langle \cdot | x \rangle x$$

is a diagonalization of u (Theorem 5.5.6.1 a $\Rightarrow$ e), then

$$f(u) = f(0)1 + \sum_{x \in A} (f(g(x)) - f(0))\langle \cdot | x \rangle x$$

by Theorem 5.5.6.1 f). By Theorem 5.5.6.1 e $\Rightarrow$ a, $f(u)$ is compact iff

$$f(0) = 0 . \qquad \blacksquare$$

Corollary 5.5.6.6 (0) *Let u be a self normal operator on a Hilbert space E. Then the following are equivalent:*

a) *u is compact.*

b) *Given $f \in C(\sigma(u))$ such that*

$$0 \notin \operatorname{Supp} f ,$$

$\operatorname{Im} f(u)$ is finite-dimensional.

a $\Rightarrow$ b. Let

$$u = \sum_{x \in A} g(x)\langle \cdot | x \rangle x$$

be a diagonalization of u (Theorem 5.5.6.1 a $\Rightarrow$ e). By Theorem 5.5.6.1 f),

$$f(u) = \sum_{x \in A} f(g(x))\langle \cdot | x \rangle x ,$$

so that $\operatorname{Im} f(u)$ is the vector subspace of E generated by

$$\{x \in A \mid f(g(x)) \neq 0\} .$$

Since the above set is finite, $\operatorname{Im} f(u)$ is finite-dimensional.

b $\Rightarrow$ a. Take $\varepsilon > 0$. Consider

$$j : \sigma(u) \longrightarrow \mathbb{K} , \quad \alpha \longmapsto \alpha$$

and $g \in C(\sigma(u))$ such that:

1) $0 \notin \operatorname{Supp} g$.

2) $g(\sigma(u)) \subset [0,1]$.

3) $g = 1$ on $\sigma(u) \backslash U_\varepsilon^{\mathbb{K}}(0)$.

We put

$$f := gj .$$

Then $f \in \mathcal{C}(\sigma(u))$,

$$0 \notin \operatorname{Supp} f .$$

$$\|f - j\| = \|gj - j\| = \|(1-g)j\| \leq \varepsilon .$$

By b), $f(u) \in \mathcal{L}_f(E)$ and by Theorem 4.1.3.1 b),

$$\|u - f(u)\| = \|j(u) - f(u)\| = \|(j-f)(u)\| = \|j - f\| \leq \varepsilon .$$

Since ε is arbitrary,

$$u \in \overline{\mathcal{L}_f(E)} \subset \mathcal{K}(E)$$

(Proposition 3.1.1.4). ■

Corollary 5.5.6.7 (0) *$\mathcal{L}_f(E)$ is a hereditary set of $\mathcal{L}(E)$.*

Let $u \in \mathcal{L}(E)_+$, $v \in \mathcal{L}_f(E)$ such that

$$u \leq v .$$

Then $v \in \mathcal{K}(E)$ (Proposition 3.1.1.4), so that $u \in \mathcal{K}(E)$ (Corollary 3.1.1.13, Proposition 4.3.4.5 e)). By Theorem 5.5.6.1 a $\Rightarrow$ e & j, there are orthonormal bases A, B of E and $f \in c_0(A)_+$, $g \in \mathbb{R}_+^{(B)}$ such that

$$u = \sum_{x \in A} f(x) \langle \cdot | x \rangle x , \quad v := \sum_{y \in B} g(y) \langle \cdot | y \rangle y .$$

Take $y \in g^{-1}(0)$. Then

$$\sum_{x \in A} f(x) |\langle x | y \rangle|^2 = \langle uy | y \rangle \leq \langle vy | y \rangle = 0$$

(Corollary 5.3.3.7), so that

$$x \notin \overset{-1}{f}(0) \Longrightarrow x \perp y .$$

We deduce that

$$\overset{-1}{f}(]0,\infty[) \subset \overset{-1}{g}(0)^{\perp} \subset \overset{-1}{g}(]0,\infty[)^{\perp\perp} .$$

Since $\overset{-1}{g}(]0,\infty[)$ is a finite set, $\overset{-1}{g}(]0,\infty[)^{\perp\perp}$ is finite-dimensional (Corollary 5.2.3.9) and $\overset{-1}{f}(]0,\infty[)$ is finite. Hence $u \in \mathcal{L}_f(E)$ and $\mathcal{L}_f(E)$ is a hereditary set of $\mathcal{L}(E)$. ■

Proposition 5.5.6.8 (0) *Let E be a Hilbert space and take $u \in \mathcal{K}(E)_+$. Then the following are equivalent:*

a) *There is an $x \in E$ such that*

$$u = \langle \cdot | x \rangle x .$$

b) *(resp. c)) Given $v \in \mathcal{L}(E)_+$ (resp. $v \in \mathcal{K}(E)_+$) with $v \leq u$, there is an $\alpha \in \mathbb{R}_+$ such that*

$$v = \alpha u .$$

In particular $\{\langle \cdot | x \rangle x \mid x \in E, \|x\| = 1\}$ is the set of minimal elements of $\Pr \mathcal{L}(E) \backslash \{0\}$.

a $\Rightarrow$ b. By Corollary 3.1.1.13 and Proposition 4.3.4.5 e), $v \in \mathcal{K}(E)_+$. By Theorem 5.5.6.1 a $\Rightarrow$ e & j, there is an orthonormal basis B of E and an $f \in c_0(B)_+$ with

$$v = \sum_{y \in B} f(y) \langle \cdot | y \rangle y .$$

Let A be an orthonormal basis of E containing x. Put

$$C := \{y \in B \mid f(y) \neq 0\} .$$

By Corollary 5.3.3.7,

$$0 \leq \langle vz | z \rangle \leq \langle uz | z \rangle$$

for every $z \in E$. Hence

$$0 \leq \sum_{y \in B} f(y) |\langle y | z \rangle|^2 \leq |\langle x | z \rangle|^2 = 0$$

for every $z \in A\backslash\{x\}$. Thus

$$\langle y|z\rangle = 0$$

for every $y \in C$ and $z \in A\backslash\{x\}$. It follows that

$$y = \sum_{z\in A}\langle y|z\rangle z = \langle y|x\rangle x$$

for every $y \in C$. Hence C contains at most one element and if y is this element, then

$$v = f(y)|\langle x|y\rangle|^2\langle\cdot|x\rangle x\,,$$

which proves b).

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. By Theorem 5.5.6.1 a $\Rightarrow$ e & j, there is an orthonormal basis A of E and an $f \in c_0(A)_+$ with

$$u = \sum_{x\in A} f(x)\langle\cdot|x\rangle x\,.$$

Take $y \in A$ with $f(y) \neq 0$ and put

$$v := f(y)\langle\cdot|y\rangle y\,.$$

Then

$$u - v = \sum_{x\in A\backslash\{y\}} f(x)\langle\cdot|x\rangle x \geq 0\,,$$

so that

$$v \leq u\,.$$

By c), there is an $\alpha \in \mathbb{R}_+$ such that

$$v = \alpha u\,.$$

Then

$$f(y) = \langle vy|y\rangle = \alpha\langle uy|y\rangle = \alpha f(y)$$

whence

$$\alpha = 1$$

and

$$f(x) = \langle ux|x\rangle = \langle vx|x\rangle = 0$$

for every $x \in A\backslash\{y\}$. Hence

$$u = f(y)\langle\cdot|y\rangle y = \langle\cdot|f(y)^{\frac{1}{2}}y\rangle f^{\frac{1}{2}}(y)y\,.$$

Now we prove the last assertion. By a $\Rightarrow$ b and Proposition 5.5.1.7, $\langle\cdot|x\rangle x$ is a minimal element of $\mathrm{Pr}\,E\backslash\{0\}$ for every $x \in E$ with $\|x\| = 1$. Let p be a minimal element of $\mathrm{Pr}\,\mathcal{L}(E)\backslash\{0\}$. Take $x \in \mathrm{Im}\,p$ with $\|x\| = 1$. Then

$$p \circ (\langle\cdot|x\rangle x) \circ p = \langle\cdot|px\rangle px = \langle\cdot|x\rangle x$$

(Proposition 5.3.2.13 c).d)), so that

$$\langle\cdot|x\rangle x \leq p$$

(Corollary 4.2.7.6 f $\Rightarrow$ a). Since p is minimal, it follows

$$p = \langle\cdot|x\rangle x\,. \qquad \blacksquare$$

Proposition 5.5.6.9 (0) *Let E be a complex Hilbert space and $\mathcal{F}$ a C^*-subalgebra of $\mathcal{L}(E)$ acting irreducibly on E. If*

$$\mathcal{F} \cap \mathcal{K}(E) \neq \{0\}\,,$$

then $\mathcal{K}(E) \subset \mathcal{F}$.

Step 1 { There is a finite-dimensional vector subspace F of E with $\{0\} \neq F \neq E$ and $\pi_F \in \mathcal{F}$

There is a

$$u \in (\mathrm{Re}\mathcal{F}) \cap \mathcal{K}(E)\backslash\mathbb{C}1\,.$$

By Theorem 5.5.6.1 a $\Rightarrow$ e, there is an orthonormal basis A of E and an $f \in c_0(A)$ such that f is not constant and

$$u = \sum_{x\in A} f(x)\langle\cdot|x\rangle x\,.$$

Take $\alpha \in f(A)\backslash\{0\}$ and put

$$B := \overset{-1}{f}(\alpha)\,, \quad F := \mathrm{Ker}\,(\alpha 1 - u)\,.$$

By Theorem 5.5.6.1 a $\Rightarrow$ b, F is finite-dimensional and by Theorem 5.5.6.1 a $\Rightarrow$ f & j (and Proposition 5.5.1.7),

$$\pi_F = \sum_{x \in B} \langle \cdot | x \rangle x \in \mathcal{F} .$$

Step 2 $\exists\, x \in E \backslash \{0\}$, $\langle \cdot | x \rangle x \in \mathcal{F}$

Let F be a minimal vector subspace of E with the properties described in Step 1. Given $u \in \mathcal{F}$, define

$$\widetilde{u} : F \longrightarrow F , \quad x \longmapsto \pi_F u x$$

and put

$$\widetilde{\mathcal{F}} := \{ \widetilde{u} | u \in \mathcal{F} \} .$$

Since $\pi_F \in \mathcal{F}$, it follows that $\widetilde{\mathcal{F}}$ is a subalgebra of $\mathcal{L}(F)$. By Proposition 5.3.2.25, $\widetilde{\mathcal{F}}$ acts irreducibly on F. By Step 1, F is one-dimensional. Hence

$$\langle \cdot | x \rangle x = \pi_F \in \mathcal{F}$$

for every $x \in F$, $\|x\| = 1$.

Step 3 Given $x, y \in E$, if $\langle \cdot | x \rangle x \in \mathcal{F}$, then $\langle \cdot | x \rangle y \in \mathcal{F}$

We may assume that

$$x \notin \{ \alpha y | \alpha \in \mathbb{C} \} .$$

By Proposition 5.3.2.20 b), x is cyclic for $\mathcal{F}$. Hence there is a sequence $(u_n)_{n \in \mathbb{N}}$ in $\mathcal{F}$ such that

$$\lim_{n \to \infty} u_n x = y .$$

We get

$$\langle \cdot | x \rangle y = \lim_{n \to \infty} \langle \cdot | x \rangle u_n x = \lim_{n \to \infty} u_n \circ (\langle \cdot | x \rangle x) \in \mathcal{F} .$$

Step 4 $\mathcal{K}(E) \subset \mathcal{F}$

By Step 2, 3, and Corollary 5.5.1.11,

$$\mathcal{L}_f(E) \subset \mathcal{F} .$$

Hence by Corollary 5.2.3.4,

$$\mathcal{K}(E) \subset \mathcal{F}\,.$$ ■

Remark. A similar result does not hold in the real case as the following example shows:

$$E := \mathbb{R}^2\,, \quad \mathcal{F} := \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}.$$

Corollary 5.5.6.10 *Let E be a complex Hilbert space and $\mathcal{F}$ an order σ-complete C^*-subalgebra of $\mathcal{L}(E)$ acting irreducibly on E. If $\Pr\mathcal{F}\backslash\{0\}$ has a minimal element, then*

$$\mathcal{K}(E) \subset \mathcal{F}\,.$$

Let p be a minimal element of $\Pr\mathcal{F}\backslash\{0\}$. By Corollary 4.3.2.12 a $\Rightarrow$ d, $p\mathcal{F}p$ is one-dimensional. So, by Proposition 5.3.2.20 b), there is an $x \in E$ with

$$p = \langle \cdot | x \rangle x\,.$$

Hence $p \in \mathcal{K}(E)$ and by Proposition 5.5.6.9,

$$\mathcal{K}(E) \subset \mathcal{F}\,.$$ ■

Proposition 5.5.6.11 (0) *Let E be a complex C^*-algebra and (H, φ), (K, ψ) irreducible representations of E such that*

$$\operatorname{Ker}\varphi = \operatorname{Ker}\psi\,.$$

If

$$\varphi(E) \cap \mathcal{K}(H) \neq \{0\}\,,$$

then the two representations are equivalent.

Put

$$F := \operatorname{Ker}\varphi = \operatorname{Ker}\psi\,,$$

$$G := \overset{-1}{\varphi}(\mathcal{K}(H))\,.$$

Let $\widetilde{\varphi}$ and $\widetilde{\psi}$ be the factorizations through G/F of $\varphi|G$ and $\psi|G$, respectively, and let

$$q : G \longrightarrow G/F$$

be the quotient map (F and G are closed ideals of E). By Proposition 5.5.6.9,

$$\mathcal{K}(H) \subset \varphi(E)\,,$$

so that

$$\widetilde{\varphi} : G/F \longrightarrow \mathcal{K}(H)$$

is an isometry.

Take $\xi, \eta \in K$, $\xi \neq 0$, and $\varepsilon > 0$. Take $x \in G \backslash F$. There is a $\xi_0 \in K$ such that

$$(\psi x)\xi_0 \neq 0\,.$$

By Proposition 5.3.2.20 b), there are $y, z \in E$ such that

$$\|(\psi y)(\psi x)\xi_0 - \eta\| < \frac{\varepsilon}{2}\,,$$

$$\|(\psi z)\xi - \xi_0\| < \frac{\varepsilon}{2(\|x\|\,\|y\| + 1)}\,.$$

We get that

$$\|\psi(yxz)\xi - \eta\| \leq \|(\psi y)(\psi x)(\psi z)\xi - (\psi y)(\psi x)\xi_0\| + \|(\psi y)(\psi x)\xi_0 - \eta\| <$$

$$< \|\psi y\|\,\|\psi x\|\,\|(\psi z)\xi - \xi_0\| + \frac{\varepsilon}{2} < \frac{\|y\|\,\|x\|\varepsilon}{2(\|x\|\,\|y\| + 1)} + \frac{\varepsilon}{2} < \varepsilon\,.$$

Since $yxz \in G$, it follows that ξ is cyclic for $\psi(G)$. By Proposition 5.3.2.20 a), $\widetilde{\psi}$ is an irreducible representation of G. Hence

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1} : \mathcal{K}(H) \longrightarrow \mathcal{L}(K)$$

is an irreducible representation of $\mathcal{K}(H)$. By Theorem 5.5.1.24, this representation is the identical representation. Hence there is an isometry

$$u : H \longrightarrow K\,,$$

such that

$$\left(\widetilde{\psi} \circ \widetilde{\varphi}^{-1} v\right) \circ u = u \circ v$$

for every $v \in \mathcal{K}(H)$. We get

$$(\psi x) \circ u = (\widetilde{\psi} q x) \circ u = u \circ (\widetilde{\varphi} q x) = u \circ (\varphi x)$$

for every $x \in G$.

Take $x \in E$ and $y \in G$. We have $xy \in G$, so that

$$u \circ (\varphi x)(\varphi y) = u \circ \varphi(xy) = \psi(xy) \circ u = (\psi x)(\psi y) \circ u .$$

Take $\xi \in H$ and $\eta \in H \backslash \{0\}$. Since η is cyclic for $\varphi(G)$ (Proposition 5.3.2.20 b)), there is a filter $\mathfrak{F}$ on G such that

$$\lim_{y,\mathfrak{F}} (\varphi y)\eta = \xi .$$

We get

$$u\xi = \lim_{y,\mathfrak{F}} u(\varphi y)\eta = \lim_{y,\mathfrak{F}} (\psi y) u\eta ,$$

$$u(\varphi x)\xi = \lim_{y,\mathfrak{F}} u((\varphi x)(\varphi y))\eta = \lim_{y,\mathfrak{F}} ((\psi x)(\psi y))u\eta =$$

$$= (\psi x) \lim_{y,\mathfrak{F}} (\psi y) u\eta = (\psi x) u\xi ,$$

$$u \circ (\varphi x) = (\psi x) \circ u .$$

Hence (H, φ) and (K, ψ) are equivalent. ■

Corollary 5.5.6.12 (0) *Let H be a complex Hilbert space, E a C^*-subalgebra of $\mathcal{L}(H)$ containing $\mathcal{K}(H)$, and $\varphi : E \to E$ and isometry of C^*-algebras. Then there is a unitary operator u on H such that for each $x \in E$,*

$$\varphi x = uxu^{-1} .$$

u is unique up to a factor of absolute value 1.

If ψ denotes the identity map $E \to E$, then (H, φ) and (H, ψ) are irreducible representations of E. By Proposition 5.5.6.11, the two representations are equivalent.

The uniqueness follows from Proposition 5.4.1.7. ■

Corollary 5.5.6.13 (0) *Let $(E_\iota)_{\iota\in I}$ be a family of (complex) Hilbert spaces, E the C^*-direct sum of the family $(\mathcal{K}(E_\iota))_{\iota\in I}$, and F a C^*-subalgebra of E such that:*

1) *for every $\iota \in I$, the representation*

$$\varphi_\iota : F \longrightarrow \mathcal{K}(E_\iota), \quad x \longmapsto x_\iota$$

is irreducible;

2) *if $\iota, \lambda \in I$ are distinct, then φ_ι and φ_λ are not equivalent representations.*

Then $\operatorname{Re} E \subset F$ $(E = F)$.

Take distinct $\iota, \lambda \in I$ and $a \in \mathcal{K}(E_\iota)$. By Proposition 4.2.4.18, it is sufficient to show that there is an $x \in F$ with

$$x_\iota = a, \quad x_\lambda = 0.$$

By 1), 2), and Proposition 5.5.6.11,

$$\operatorname{Ker}\varphi_\iota \neq \operatorname{Ker}\varphi_\lambda.$$

Since $\mathcal{K}(E_\lambda)$ is simple, $\operatorname{Ker}\varphi_\lambda$ is a maximal closed ideal of F. Hence $\varphi_\iota(\operatorname{Ker}\varphi_\lambda)$ is a non-zero closed ideal of $\mathcal{K}(E_\iota)$ (Theorem 4.2.6.6). Since $\mathcal{K}(E_\iota)$ is simple,

$$\varphi_\iota(\operatorname{Ker}\varphi_\lambda) = \mathcal{K}(E_\iota).$$

Hence there is an $x \in F$ with

$$x_\iota = \varphi_\iota x = a_\iota, \quad x_\lambda = \varphi_\lambda x = 0. \qquad \blacksquare$$

5.5.7 Examples of Real C^*–Algebras

Definition 5.5.7.1 (0) *An* ***involutive Hilbert space*** *is a Hilbert space H endowed with a conjugate linear involution*

$$H \longrightarrow H\,, \quad \xi \longmapsto \overline{\xi} \qquad (\textbf{\textit{the involution of}}\ H)$$

such that

$$\langle \overline{\xi} | \overline{\eta} \rangle = \overline{\langle \xi | \eta \rangle}$$

for all $\xi, \eta \in H$. For every set T, $\ell^2(T)$ will be considered an involutive Hilbert space with respect to the canonical involution

$$\ell^2(T) \longrightarrow \ell^2(T)\,, \quad \xi \longmapsto \overline{\xi}\,.$$

If G and H are involutive Hilbert spaces and $x \in \mathcal{L}(G,H)$ then (by Definition 2.3.1.1)

$$\overline{x} : G \longrightarrow H\,, \quad \xi \longmapsto \overline{x\overline{\xi}}\,.$$

Every involutive Hilbert space is an involutive Banach space with respect to its canonical norm.

Proposition 5.5.7.2 (0) *Let F, G, H be involutive Hilbert spaces.*

a) $x \in \mathcal{L}(G,H) \Rightarrow \overline{x} \in \mathcal{L}(G,H)\,,\ \|\overline{x}\| = \|x\|\,,\ \overline{\overline{x}} = x\,,\ \overline{x}^* = \overline{x^*}$.

b) *$\mathcal{L}(G,H)$ endowed with the involution*

$$\mathcal{L}(G,H) \longrightarrow \mathcal{L}(G,H)\,, \quad x \longmapsto \overline{x}$$

is an involutive Banach space.

c) $x \in \mathcal{L}(F,G)\,,\ y \in \mathcal{L}(G,H) \Rightarrow \overline{y \circ x} = \overline{y} \circ \overline{x}$.

d) *The map*

$$\mathcal{L}(H) \longrightarrow \mathcal{L}(H)\,, \quad x \longmapsto \overline{x}$$

is a conjugate involution of the C^*-algebra $\mathcal{L}(H)$.

e) *Assume $\mathbb{K} = \mathbb{C}$ and let E be an C^*-subalgebra of $\mathcal{L}(H)$ such that*

$$x \in E \Longrightarrow \overline{x} \in E\,.$$

Then the map

$$\overset{\circ}{E} \longrightarrow E \times E\,, \quad (x,y) \longmapsto (x + iy, \overline{x} + i\overline{y})$$

is an isomorphism of C^* algebras.

a) Take $(\xi, \eta) \in G \times H$. Then

$$\langle \overline{x}\xi|\eta\rangle = \langle \overline{\overline{x}\xi}|\eta\rangle = \overline{\langle x\overline{\xi}|\overline{\eta}\rangle} = \overline{\langle \overline{\xi}|x^*\overline{\eta}\rangle} =$$

$$= \langle \xi|\overline{x^*\overline{\eta}}\rangle = \langle \xi|\overline{x^*}\eta\rangle .$$

Hence

$$\overline{x}^* = \overline{x^*} .$$

The other relations are easy to see.

b) follows from a).

c) follows from Proposition 2.3.2.22 i).

d) follows from a) and c).

e) follows from d) and Theorem 4.1.1.8 e). ■

Proposition 5.5.7.3 *Let E be an involutive Banach algebra,*

$$E \longrightarrow E , \quad x \longmapsto \overline{x}$$

a conjugate involution, and $x' \in E'_+$ such that

$$x'(\overline{x}) = \overline{x'(x)}$$

for every $x \in E$. In the sequel we use the notation of Theorem 5.4.1.2.

a) *$\overline{x} \in F$ for every $x \in F$. Put*

$$\overline{X} := u\overline{x}$$

for every $X \in E/F$ and $x \in X$.

b) $X, Y \in E/F \Rightarrow \langle \overline{X}|\overline{Y}\rangle = \overline{\langle X|Y\rangle}$, $\|\overline{X}\| = \|X\|$, $\overline{\overline{X}} = X$.

c) *E/F endowed with the involution*

$$E/F \longrightarrow E/F , \quad X \longmapsto \overline{X}$$

is an involutive normed space. We extend this involution to $\widetilde{E/F}$.

d) *$\widetilde{E/F}$ is an involutive Hilbert space.*

e) $x \in E \Rightarrow \varphi\overline{x} = \overline{\varphi x}$.

a) We have

$$\tilde{f}(\bar{x}) = x'(\bar{x}^*\bar{x}) = x'(\overline{x^*x}) = \overline{x'(x^*x)} = 0\,.$$

b) For $(x, y) \in X \times Y$,

$$\langle \overline{X} | \overline{Y} \rangle = x'(\bar{y}^*\bar{x}) = x'(\overline{y^*x}) = \overline{x'(y^*x)} = \overline{\langle X|Y\rangle}\,,$$

$$\|\overline{X}\|^2 = \langle \overline{X} | \overline{X} \rangle = \overline{\langle X|X\rangle} = \|X\|^2\,.$$

c) Since the map is obviously a conjugate linear involutive the assertion follows from b).

d) follows from b) and c).

e) Take $Y \in E/F$ and $y \in Y$. Then

$$\overline{\varphi\bar{x}}Y = \overline{(\varphi x)\overline{Y}} = \overline{u(x\bar{y})} = u(\overline{x\bar{y}}) = u(\bar{x}y) = (\varphi\bar{x})Y$$

so that

$$\overline{\varphi x} = \varphi\bar{x}\,. \qquad \blacksquare$$

Proposition 5.5.7.4 (0) *Let H be an involutive complex (real) Hilbert space. Then H has an orthonormal basis A such that*

$$\bar{\xi} = \xi \qquad (\bar{\xi} = \pm\xi)$$

for every $\xi \in A$.

Let $\mathfrak{A}$ be the set of orthonormal sets A of H such that

$$\bar{\xi} = \xi \qquad (\bar{\xi} = \pm\xi)$$

for every $\xi \in A$. Order $\mathfrak{A}$ by inclusion and take a maximal element $A \in \mathfrak{A}$ (Zorn). Assume A is not an orthonormal basis of H. Then there is a $\xi \in A^\perp$ with $\|\xi\| = 1$. Since

$$\langle \bar{\xi} | \eta \rangle = \overline{\langle \xi | \bar{\eta} \rangle} = 0$$

for every $\eta \in A$, it follows $\bar{\xi} \in A^\perp$. Let $\theta \in \mathbb{R}$ such that

$$\langle \xi | \bar{\xi} \rangle = e^{i\theta} |\langle \xi | \bar{\xi} \rangle|\,.$$

Put

$$\eta := \frac{1}{\sqrt{2(1+|\langle\xi|\bar{\xi}\rangle|)}}(e^{-i\frac{\theta}{2}}\xi + e^{i\frac{\theta}{2}}\bar{\xi})$$

$$(\eta := \frac{1}{\sqrt{2(1+|\langle\xi|\bar{\xi}\rangle|)}}(\xi + e^{i\theta}\bar{\xi}))\,.$$

Then $\eta \in A^{\perp}$,

$$\overline{\eta} = \eta \qquad (\overline{\eta} = \pm\eta)\,,$$

and

$$\|\eta\|^2 = \frac{1}{2(1+|\langle\xi|\bar{\xi}\rangle|)}(1 + 2\mathrm{re}\, e^{-i\theta}\langle\xi|\bar{\xi}\rangle + 1) = 1\,.$$

Hence $A \cup \eta \in \mathfrak{A}$ and this contradicts the maximality of A. ■

Remark. It follows from this proposition that every involutive complex Hilbert space is isomorphic to $\ell^2(T)$ for some set T.

Proposition 5.5.7.5 (0) *Let H be an involutive Hilbert space and*

$$E := \{x \in \mathcal{L}(H) \mid x^* = \overline{x}\}\,.$$

a) *E is a closed involutive vector subspace of $\mathcal{L}(H)$.*

b) *If $\mathcal{F}$ is an upward directed set of $\mathcal{L}(H)$ contained in E and if x is its supremum in $\mathcal{L}(H)$, then $x \in E$.*

c) *If x is a self-normal operator on H belonging to E, then $f(x) \in E$ for every bounded Borel function f on $\sigma(x)$.*

d) *If $\mathbb{K} = \mathbb{C}$ and if x is a unitary operator on H belonging to E, then for each $n \in \mathbb{N}$ there is a unitary operator y on H belonging to E such that $y^n = x$. Moreover, there is an orthonormal basis A of H such that $x\bar{\xi} = \xi$ for every $\xi = A$.*

e) *If $\mathbb{K} = \mathbb{C}$ and if x, y are unitary operators on H belonging to E, then there is a unitary operator z on H such that*

$$zx = y\bar{z}\,.$$

a) E is obviously a vector subspace of $\mathcal{L}(H)$. By Proposition 5.5.7.2 a), it is closed.

b) Take $\xi, \eta \in H$ and let $\mathfrak{F}$ be the upper section filter of $\mathcal{F}$. By Theorem 5.3.3.14 c),

$$\langle x\xi|\eta\rangle = \lim_{y,\mathfrak{F}}\langle y\xi|\eta\rangle = \lim_{y,\mathfrak{F}}\langle \xi|y^*\eta\rangle = \lim_{y,\mathfrak{F}}\langle \xi|\overline{y}\eta\rangle = \lim_{y,\mathfrak{F}}\langle y\overline{\eta}|\overline{\xi}\rangle = \langle x\overline{\eta}|\overline{\xi}\rangle = \langle \xi|\overline{x}\eta\rangle .$$

Hence $x^* = \overline{x}$ and $x \in E$.

c) By Theorem 5.3.3.14, $\mathcal{L}(H)$ is $\mathbb{C}$-order complete, so by Corollary 4.3.2.5 b), $f(x)$ is well-defined. We denote by $\mathcal{B}$ the set of bounded Borel functions on $\sigma(x)$ and put

$$\mathcal{B}_0 := \{f \in \mathcal{B} \mid f(x)^* = \overline{f(x)}\} .$$

If $P \in \mathbb{K}[s,t]$, then by Proposition 5.5.7.2 a),c) (and Corollary 4.1.3.8) the map

$$\sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto P(\alpha, \overline{\alpha})$$

belongs to $\mathcal{B}_0$. By a) and the Weierstrass-Stone Theorem, $\mathcal{C}(\sigma(x)) \subset \mathcal{B}_0$. If $(f_n)_{n\in\mathbb{N}}$ is an increasing (decreasing) sequence in $\mathcal{B}_0$ with supremum (infimum) f in $\mathcal{B}$, then by b), $f \in \mathcal{B}_0$. Hence $\mathcal{B}_0 = \mathcal{B}$.

d) By Corollary 4.3.2.5 h) (and Theorem 5.3.3.14), there is a bounded Borel function f on $\sigma(x)$ such that $f(x)$ is unitary and $f(x)^n = x$. By c), $f(x) \in E$. In particular, there is a unitary operator y on H belonging to E such that $y^2 = x$. By Proposition 5.5.7.4, we may assume $H = \ell^2(T)$ for some set T. Then $(ye_t)_{t\in T}$ is an orthonormal basis of H (Proposition 5.5.5.9 d $\Rightarrow$ b). We have

$$x\overline{y}e_t = x\overline{y}e_t = xy^*e_t = y^2y^*e_t = ye_t$$

for every $t \in T$.

e) By d), there are unitary operators x_0, y_0 on H belonging to E such that

$$x_0^2 = x, \quad y_0^2 = y .$$

Put

$$z := y_0x_0^* .$$

Then z is a unitary operator on H and we have

$$zx = y_0x_0^*x_0^2 = y_0x_0 = yy_0^*x_0 = y\overline{y}_0\overline{x}_0^* = y\overline{z} .$$

■

Proposition 5.5.7.6 (0) *Take* $\mathbb{K} = \mathbb{C}$. *Let* T *be a set,* x *a unitary operator on* $H := \ell^2(T)$, *and*

$$u : \mathcal{L}(H) \longrightarrow \mathcal{L}(H), \quad a \longmapsto x\bar{a}x^* .$$

Then the following are equivalent.

a) $\overline{x} = x^*$ *or* $\overline{x} = -x^*$.

b) $u^2 = 1$.

If these conditions are fulfilled and if E *is a* C^**-subalgebra of* $\mathcal{L}(H)$ *such that* $u(E) \subset E$, *then*

$$F := \{a \in E \mid a = u(a)\}$$

is a real C^**-subalgebra of* E *such that the map*

$$\overset{\circ}{F} \longrightarrow E, \quad (x, y) \longmapsto x + iy$$

is an isomorphism of complex C^**-algebras.*

Given $a \in \mathcal{L}(H)$, we have (by Proposition 5.5.7.2 d))

$$u^2(a) = x\overline{x\bar{a}x^*}x^* = x\overline{x}a\overline{x^*}x^* .$$

If a) is fulfilled, then

$$x\overline{x} = \pm 1, \quad \overline{x^*}x^* = \pm 1,$$

so that $u^2 = 1$. If $u^2 = 1$, then

$$(x\overline{x})a = a(x\overline{x})$$

for every $a \in \mathcal{L}(H)$. Hence $x\overline{x} \in \mathbb{C}1$ (Corollary 5.3.2.14). Putting $a := \overline{x}^*$ in the above equality, we get

$$x = (x\overline{x})\overline{x}^* = \overline{x}^*(x\overline{x}),$$

$$\overline{x}x = x\overline{x} = \overline{\overline{x}x} .$$

Hence there is an $\alpha \in \mathbb{R}$ such that

$$\overline{x}x = \alpha 1 .$$

Hence

$$\overline{x} = \alpha x^*$$

from which it follows that $\alpha \in \{-1, +1\}$.

By b),

$$u(E) \subset E \Longrightarrow u(E) = E\,.$$

Hence the last assertion follows from Proposition 2.3.1.43 a) (and Corollary 4.1.1.21, Proposition 4.1.1.24). ■

Proposition 5.5.7.7 (0) *Assume* $\mathbb{K} = \mathbb{C}$. *Let* T *be a set and* x, y *unitary operators on* $H := \ell^2(T)$ *such that*

$$\overline{x} = -x^*\,, \quad \overline{y} = -y^*\,.$$

a) *There is a family* $(\xi_\iota)_{\iota \in I}$ *in* H *such that if we put*

$$\zeta_{(\iota,0)} := \xi_\iota\,, \quad \zeta_{(\iota,1)} := x\overline{\xi}_\iota$$

for every $\iota \in I$, *then* $(\zeta_\lambda)_{\lambda \in I \times \{0,1\}}$ *is an orthonormal basis of* H.

b) *If* T *is finite then* $\operatorname{Card} T$ *is even.*

c) *There is a unitary operator* z *on* H *such that*

$$zx = y\overline{z}\,.$$

a) Take $\xi, \eta \in H$. We have

$$\langle x\overline{\xi}|\eta\rangle = \langle \overline{\xi}|x^*\eta\rangle = -\langle \overline{\xi}|\overline{x}\eta\rangle = -\langle \overline{\xi}|\overline{x\overline{\eta}}\rangle = -\langle x\overline{\eta}|\xi\rangle\,,$$

$$\overline{x\overline{\xi}} = x\overline{x}\xi = -xx^*\xi = -\xi\,.$$

It follows that

$$\xi \perp x\overline{\xi}\,,$$

$$\eta \in \{\xi, x\overline{\xi}\}^\perp \Longrightarrow x\overline{\eta} \in \{\xi, x\overline{\xi}\}^\perp\,.$$

Let Ω be the set of orthonormal sets A of H such that $\xi \perp x\overline{\eta}$ for all $\xi, \eta \in A$. Since Ω is inductively ordered by inclusion, it contains a maximal

element A (Zorn's Lemma). Assume $A \cup \{x\bar{\xi} \mid \xi \in A\}$ is not an orthonormal basis of H. Then there is an

$$\eta \in (A \cup \{x\bar{\xi} \mid \xi \in A\})^{\perp}$$

with $\|\eta\| = 1$. By the above remark, $A \cup \{\eta\} \in \Omega$, contradicting the maximality of A. Hence $A \cup \{x\bar{\xi} \mid \xi \in A\}$ is an orthonormal basis of H. By the above remark, $(\xi)_{\xi \in A}$ is a family in H such that $(\zeta_\lambda)_{\lambda \in A \times \{0,1\}}$ is an orthonormal basis of H.

b) follows from a).

c) Let $(\xi_\iota)_{\iota \in I}$ and $(\eta_\lambda)_{\lambda \in L}$ be families in H with the property described in a) with respect to x and y, respectively. We may assume that $I = L$ (Corollary 5.5.2.2). Let z be the unitary operator on H, defined by

$$z\xi_\iota = \eta_\iota\,, \quad zx\bar{\xi}_\iota = y\bar{\eta}_\iota$$

for every $\iota \in I$ (Propositions 5.5.1.22 and 5.5.5.9 c $\Rightarrow$ d). Take $\iota \in I$. Then

$$z^* y z \bar{\xi}_\iota = z^* y \overline{z\xi_\iota} = z^* y \bar{\eta}_\iota = x\bar{\xi}_\iota\,,$$

$$z^* y z \overline{x\bar{\xi}_\iota} = z^* y \overline{zx\bar{\xi}_\iota} = z^* y \overline{y\bar{\eta}_\iota} = z^* y \bar{y} \eta_\iota =$$

$$= -z^* y y^* \eta_\iota = -z^* \eta_\iota = -\xi_\iota = x\overline{x\bar{\xi}_\iota}\,.$$

Hence

$$z^* y \bar{z} = x\,,$$

$$y\bar{z} = zx\,. \qquad \blacksquare$$

Proposition 5.5.7.8 (0) *Assume* $\mathbb{K} = \mathbb{C}$. *Let* T *be a set and* x, y *unitary operators on* $H := \ell^2(T)$ *such that*

$$\bar{x} = x^*\,, \quad \bar{y} = y^* \quad (\textit{resp.}\ \bar{x} = -x^*\,, \quad \bar{y} = -y^*)\,.$$

Let E *be a* C^**-subalgebra of* $\mathcal{L}(H)$ *such that*

$$zEz^* = E$$

for any unitary operator z *on* H *and such that*

$$a \in E \Longrightarrow \bar{a} \in E\,.$$

Put

$$E_x := \{a \in E \mid x\bar{a}x^* = a\},$$

$$E_y := \{a \in E \mid y\bar{a}y^* = a\}.$$

Then there is a unitary operator z on H such that

$$zE_xz^* = E_y$$

and the map

$$E_x \longrightarrow E_y, \quad a \longmapsto zaz^*$$

is an isomorphism of real C^*-algebras (Proposition 5.5.7.6).

By Proposition 5.5.7.5 e) (by Proposition 5.5.7.7 c)), there is a unitary operator z on H such that

$$zx = y\bar{z}.$$

Take $a \in E$. Then

$$y\overline{zaz^*}y^* = y\bar{z}\,\bar{a}\,\bar{z}^*y^* = zx\bar{a}x^*z^*,$$

so that

$$(a \in E_x) \Longleftrightarrow (a = x\bar{a}x^*) \Longleftrightarrow (zaz^* = y\overline{zaz^*}y^*) \Longleftrightarrow (zaz^* \in E_y).$$

Since the map

$$E \longrightarrow E, \quad a \longmapsto zaz^*$$

is an isomorphism of complex C^*-algebras (Proposition 4.1.1.24), it follows from the above that the map

$$E_x \longrightarrow E_y, \quad a \longmapsto zaz^*$$

is an isomorphism of real C^*-algebras. ■

Corollary 5.5.7.9 (7) *Let T be a set and $H_{\mathbb{K}} := \ell^2(T, \mathbb{K})$, (i.e. the Hilbert space H over $\mathbb{K}$). Let x be a unitary operator on $H_{\mathbb{C}}$ such that $\overline{x} = x^*$ and put*

$$E := \{a \in \mathcal{L}(H_{\mathbb{C}}) \mid x\bar{a}x^* = a\}.$$

Then the real C^*-algebras E and $E \cap \mathcal{K}(H_{\mathbb{C}})$ (Proposition 5.5.7.6) are isomorphic to $\mathcal{L}(H_{\mathbb{R}})$ and $\mathcal{K}(H_{\mathbb{R}})$, respectively.

By Proposition 5.5.7.8, E (resp. $E \cap \mathcal{K}(H_{\mathbb{C}})$) is isomorphic to

$$\{a \in \mathcal{L}(H_{\mathbb{C}}) \mid a = \overline{a}\} \quad (\text{resp. } \{a \in \mathcal{K}(H_{\mathbb{C}}) \mid a = \overline{a}\}) .$$

If we identify $H_{\mathbb{R}}$ canonically with a subset of $H_{\mathbb{C}}$, then for $a \in \mathcal{L}(H_{\mathbb{C}})$

$$a = \overline{a} \Longleftrightarrow a(H_{\mathbb{R}}) \subset H_{\mathbb{R}} .$$

Hence E (resp. $E \cap \mathcal{K}(H_{\mathbb{C}})\}$ is isomorphic to $\mathcal{L}(H_{\mathbb{R}})$ (resp. $\mathcal{K}(H_{\mathbb{R}})$). ■

Proposition 5.5.7.10 (0) *Let $\mathbb{K} = \mathbb{C}$ and let T be a set. Given $t \in T$, define*

$$H := \ell^2(T \times \{0,1\}) ,$$

$$t' := e_{(t,0)} , \quad t'' := e_{(t,1)} .$$

For $s, t \in T$ and $\varphi \in \mathbb{R}$, define

$$\begin{cases} x_{st}^{\varphi} := e^{i\frac{\pi\varphi}{2}} \langle \cdot | t' \rangle s' + e^{-i\frac{\pi\varphi}{2}} \langle \cdot | t'' \rangle s'' \\ y_{st}^{\varphi} := e^{i\frac{\pi\varphi}{2}} \langle \cdot | t' \rangle s'' - e^{-i\frac{\pi\varphi}{2}} \langle \cdot | t'' \rangle s' . \end{cases}$$

Further, let x be the unitary operator on H such that

$$xt' = t'' , \quad xt'' = -t'$$

for all $t \in T$ (Propositions 5.5.1.22 and 5.5.5.9 c $\Rightarrow$ d), E a C^*-subalgebra of $\mathcal{L}(H)$ such that

$$a \in E \Longrightarrow x\overline{a}x^* \in E$$

and put

$$F(E) := F := \{a \in E \mid x\overline{a}x^* = a\} .$$

a) *F is a real C^*-subalgebra of E such that the map*

$$\overset{\circ}{F} \longrightarrow E , \quad (a,b) \longmapsto a + ib$$

is an isomorphism of complex algebras. If E is simple then F is simple and purely real.

b) *For all* $q,r,s,t \in T$ *and* $\varphi,\psi \in \mathbb{R}$,

$$\begin{cases} x_{st}^{\varphi} = \cos\frac{\pi\varphi}{2} x_{st}^{0} + \sin\frac{\pi\varphi}{2} x_{st}^{1} \\ y_{st}^{\varphi} = \cos\frac{\pi\varphi}{2} y_{st}^{0} + \sin\frac{\pi\varphi}{2} y_{st}^{1}, \end{cases}$$

$$(x_{st}^{\varphi})^{*} = x_{ts}^{-\varphi}, \quad (y_{st}^{\varphi})^{*} = -y_{ts}^{\varphi},$$

$$\begin{cases} x_{qr}^{\varphi} x_{st}^{\psi} = \delta_{rs} x_{qt}^{\varphi+\psi}, \quad y_{qr}^{\varphi} y_{st}^{\psi} = -\delta_{rs} x_{qt}^{-\varphi+\psi} \\ x_{qr}^{\varphi} y_{st}^{\psi} = \delta_{rs} y_{qt}^{-\varphi+\psi}, \quad y_{qr}^{\varphi} x_{st}^{\psi} = \delta_{rs} y_{qt}^{\varphi+\psi}, \end{cases}$$

$$x\overline{x}_{st}^{\varphi}x^{*} = x_{st}^{\varphi}, \quad x\overline{y}_{st}^{\varphi}x^{*} = y_{st}^{\varphi}.$$

c) *For every* $t \in T$ *with* $x_{tt}^{0} \in E$, x_{tt}^{0} *is a minimal element of* $\Pr F\backslash\{0\}$.

d) *For every* $t \in T$ *with* $x_{tt}^{0} \in E$ *the map*

$$\mathbb{H} \longrightarrow x_{tt}^{0} F x_{tt}^{0}, \quad \alpha + \beta i + \gamma j + \delta k \longmapsto \alpha x_{tt}^{0} + \beta x_{tt}^{1} + \gamma y_{tt}^{1} + \delta y_{tt}^{0}$$

is an isomorphism of real C^{*}*-algebras (Example 4.1.1.31).*

e) *If there is a* $t \in T$ *with* $x_{tt}^{0} \in E$, *then* F *is a purely real* C^{*}*-algebra.*

f) *Assume there is a* $t \in T$ *with* $x_{tt}^{0} \in E$. *If* S *is a set and*

$$G := \mathcal{K}(\ell^{2}(S)) \ \textit{(resp.}\ G := \mathcal{K}(\ell^{2}(S)) \cap \mathbb{R}^{S}\ \textit{)}$$

then F *is not isomorphic to* G.

a) We have

$$\overline{x} = x = -x^{*}.$$

By Proposition 5.5.7.6, F is a real C^{*}-subalgebra of E and the map

$$\overset{\circ}{F} \longrightarrow E, \quad (a,b) \longmapsto a + ib$$

is an isomorphism of complex C^{*}-algebras. By Proposition 4.3.5.3 a $\Rightarrow$ b, if E is simple then F is simple and purely real.

b) We have

$$x_{st}^{\varphi} = \left(\cos\frac{\pi\varphi}{2} + i\sin\frac{\pi\varphi}{2}\right)\langle\cdot|t'\rangle s' + \left(\cos\frac{\pi\varphi}{2} - i\sin\frac{\pi\varphi}{2}\right)\langle\cdot|t''\rangle s'' =$$

$$= \cos\frac{\pi\varphi}{2}(\langle\cdot|t'\rangle s' + \langle\cdot|t''\rangle s'') + \sin\frac{\pi\varphi}{2}(i\langle\cdot|t'\rangle s' - i\langle\cdot|t''\rangle s'') =$$

$$= \cos\frac{\pi\varphi}{2}x^0_{st} + \sin\frac{\pi\varphi}{2}x^1_{st}$$

$$y^\varphi_{st} = \left(\cos\frac{\pi\varphi}{2} + i\sin\frac{\pi\varphi}{2}\right)\langle\cdot|t'\rangle s'' - \left(\cos\frac{\pi\varphi}{2} - i\sin\frac{\pi\varphi}{2}\right)\langle\cdot|t''\rangle s' =$$

$$= \cos\frac{\pi\varphi}{2}(\langle\cdot|t'\rangle s'' - \langle\cdot|t''\rangle s') + \sin\frac{\pi\varphi}{2}(i\langle\cdot|t'\rangle s'' + i\langle\cdot|t''\rangle s') =$$

$$= \cos\frac{\pi\varphi}{2}y^0_{st} + \sin\frac{\pi\varphi}{2}y^1_{st}$$

By Proposition 5.3.2.13 b),

$$\begin{cases} (x^\varphi_{st})^* = e^{-i\frac{\pi\varphi}{2}}\langle\cdot|s'\rangle t' + e^{i\frac{\pi\varphi}{2}}\langle\cdot|s''\rangle t'' = x^{-\varphi}_{ts}, \\ (y^\varphi_{st})^* = e^{-i\frac{\pi\varphi}{2}}\langle\cdot|s''\rangle t' - e^{i\frac{\pi\varphi}{2}}\langle\cdot|s'\rangle t'' = -y^\varphi_{ts}. \end{cases}$$

By Proposition 5.3.2.13 e),

$$x^\varphi_{qr}x^\psi_{st} = \left(e^{i\frac{\pi\varphi}{2}}\langle\cdot|r'\rangle q' + e^{-i\frac{\pi\varphi}{2}}\langle\cdot|r''\rangle q''\right)\left(e^{i\frac{\pi\psi}{2}}\langle\cdot|t'\rangle s' + e^{-i\frac{\pi\psi}{2}}\langle\cdot|t''\rangle s''\right) =$$
$$= e^{i\frac{\pi(\varphi+\psi)}{2}}\langle s'|r'\rangle\langle\cdot|t'\rangle q' + e^{-i\frac{\pi(\varphi+\psi)}{2}}\langle s''|r''\rangle\langle\cdot|t''\rangle q'' = \delta_{rs}x^{\varphi+\psi}_{qt},$$

$$y^\varphi_{qr}y^\psi_{st} = \left(e^{i\frac{\pi\varphi}{2}}\langle\cdot|r'\rangle q'' - e^{-i\frac{\pi\varphi}{2}}\langle\cdot|r''\rangle q'\right)\left(e^{i\frac{\pi\psi}{2}}\langle\cdot|t'\rangle s'' - e^{-i\frac{\pi\psi}{2}}\langle\cdot|t''\rangle s'\right) =$$
$$= -e^{-i\frac{\pi(\psi-\varphi)}{2}}\langle s'|r'\rangle\langle\cdot|t''\rangle q'' - e^{i\frac{\pi(\psi-\varphi)}{2}}\langle s''|r''\rangle\langle\cdot|t'\rangle q' = -\delta_{rs}x^{\psi-\varphi}_{qt},$$

$$x^\varphi_{qr}y^\psi_{st} = \left(e^{i\frac{\pi\varphi}{2}}\langle\cdot|r'\rangle q' + e^{-i\frac{\pi\varphi}{2}}\langle\cdot|r''\rangle q''\right)\left(e^{i\frac{\pi\psi}{2}}\langle\cdot|t'\rangle s'' - e^{-i\frac{\pi\psi}{2}}\langle\cdot|t''\rangle s'\right) =$$
$$= -e^{i\frac{\pi(\varphi-\psi)}{2}}\langle s'|r'\rangle\langle\cdot|t''\rangle q' + e^{i\frac{\pi(\psi-\varphi)}{2}}\langle s''|r''\rangle\langle\cdot|t'\rangle q'' = \delta_{rs}y^{\psi-\varphi}_{qt},$$

$$y^\varphi_{qr}x^\psi_{st} = \left(e^{i\frac{\pi\varphi}{2}}\langle\cdot|r'\rangle q'' - e^{-i\frac{\pi\varphi}{2}}\langle\cdot|r''\rangle q'\right)\left(e^{i\frac{\pi\psi}{2}}\langle\cdot|t'\rangle s' + e^{-i\frac{\pi\psi}{2}}\langle\cdot|t''\rangle s''\right) =$$
$$= e^{i\frac{\pi(\varphi+\psi)}{2}}\langle s'|r'\rangle\langle\cdot|t'\rangle q'' - e^{-i\frac{\pi(\varphi+\psi)}{2}}\langle s''|r''\rangle\langle\cdot|t''\rangle q' = \delta_{rs}y^{\varphi+\psi}_{qt}.$$

For $\xi, \eta \in H$,

$$\overline{\langle\cdot|\xi\rangle\eta} = \langle\cdot|\bar{\xi}\rangle\bar{\eta},$$

$$x \circ \overline{\langle\cdot|\xi\rangle\eta} \circ x^* = \langle\cdot|x\bar{\xi}\rangle x\bar{\eta}$$

(Proposition 5.3.2.13 c),d)). It follows that

$$x\overline{x}^{\varphi}_{st}x^* = x^{\varphi}_{st}\,, \quad x\overline{y}^{\varphi}_{st}x^* = y^{\varphi}_{st}\,.$$

c) By b), $x^0_{tt} \in \Pr F$. Take $p \in \Pr F\backslash\{\emptyset\}$ with $p \leq x^0_{tt}$. Then

$$p = px^0_{tt}p = \langle\cdot|pt'\rangle pt' + \langle\cdot|pt''\rangle pt''$$

(Corollary 4.2.7.6 a $\Rightarrow$ f, Proposition 5.3.2.13 c),d)) and

$$ps' = px^0_{tt}s' = 0\,, \quad ps'' = px^0_{tt}s'' = 0$$

for all $s \in T\backslash\{t\}$ (Corollary 4.2.7.6 a $\Rightarrow$ c). By Proposition 5.3.2.4, there are $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that

$$\begin{cases} pt' = \alpha t' + \beta t'' \\ pt'' = \gamma t' + \delta t''\,. \end{cases}$$

Thus

$$p = \langle\cdot|\alpha t' + \beta t''\rangle(\alpha t' + \beta t'') + \langle\cdot|\gamma t' + \delta t''\rangle(\gamma t' + \delta t'') =$$

$$= (|\alpha|^2 + |\gamma|^2)\langle\cdot|t'\rangle t' + (\overline{\alpha}\beta + \overline{\gamma}\delta)\langle\cdot|t'\rangle t'' +$$

$$+(\overline{\beta}\alpha + \overline{\delta}\gamma)\langle\cdot|t''\rangle t' + (|\beta|^2 + |\delta|^2)\langle\cdot|t''\rangle t''\,.$$

Since $p \in F$,

$$\begin{aligned} p = x\overline{p}x^* = {} & (|\alpha|^2 + |\gamma|^2)\langle\cdot|t''\rangle t'' - (\alpha\overline{\beta} + \gamma\overline{\delta})\langle\cdot|t''\rangle t' - \\ & -(\beta\overline{\alpha} + \delta\overline{\gamma})\langle\cdot|t'\rangle t'' + (|\beta|^2 + |\delta|^2)\langle\cdot|t'\rangle t' \end{aligned}$$

(Proposition 5.3.2.13 c),d)), so that

$$|\alpha|^2 + |\gamma|^2 = |\beta|^2 + |\delta|^2\,, \quad \overline{\alpha}\beta + \overline{\gamma}\delta = -(\beta\overline{\alpha} + \delta\overline{\gamma})$$

and therefore

$$p = (|\alpha|^2 + |\gamma|^2)x^0_{tt} + (\overline{\alpha}\beta + \overline{\gamma}\delta)\langle\cdot|t'\rangle t'' - \overline{\overline{\alpha}\beta + \overline{\gamma}\delta}\langle\cdot|t''\rangle t'\,.$$

Furthermore,

$$p = p^* = (|\alpha|^2 + |\gamma|^2)x^0_{tt} + \overline{\overline{\alpha}\beta + \overline{\gamma}\delta}\langle\cdot|t''\rangle t' - (\overline{\alpha}\beta + \overline{\gamma}\delta)\langle\cdot|t'\rangle t''$$

(Proposition 5.3.2.13 b)), so that

$$\overline{\alpha}\beta + \overline{\gamma}\delta = 0$$

and therefore

$$p = (|\alpha|^2 + |\gamma|^2)x^0_{tt},$$

$$1 = \|p\| = (|\alpha|^2 + |\gamma|^2)\|x^0_{tt}\| = |\alpha|^2 + |\gamma|^2,$$

$$p = x^0_{tt}.$$

d) Put

$$a := x^0_{tt}, \quad b := x^1_{tt}, \quad c := y^1_{tt}, \quad d := y^0_{tt}.$$

By b), $a \in \Pr F$,

$$b^2 = c^2 = d^2 = -a$$

$$ab = ba = b, \quad ac = ca = c, \quad ad = da = d,$$

$$bc = -cb = d, \quad cd = -dc = b, \quad db = -bd = c,$$

$$b^* = -b, \quad c^* = -c, \quad d^* = -d.$$

Take $y \in F$. By Proposition 5.3.2.13 c),e), there are $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, such that

$$aya = \alpha a + \beta b + \gamma c + \delta d.$$

Since $aya \in F$, by b),

$$aya = x\overline{ay\bar{a}}x^* = \overline{\alpha}a + \overline{\beta}b + \overline{\gamma}c + \overline{\delta}d.$$

Since a, b, c, d are linearly independent, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Hence aFa is the vector subspace of F generated by a, b, c, d and therefore the map

$$\mathbb{H} \longrightarrow aFa, \quad \alpha + \beta i + \gamma j + \delta k \longmapsto \alpha a + \beta b + \gamma c + \delta d$$

is an isomorphism of involutive algebras.

e) Assume the contrary and put

$$u : E \longrightarrow E, \quad y \longmapsto x\bar{y}x^*.$$

Then

$$u^2 y = x\overline{x\bar{y}x^*}x^* = x\bar{x}y\overline{x^*}x^* = xx^*yxx^* = y$$

for every $y \in E$, i.e. $u^2 = 1$. By Proposition 2.3.1.43 d), there is a linear map $v : E \to E$, such that

$$v^2 = -1\,, \quad uv = vu\,,$$

and

$$v(yz) = (vy)z = y(vz)\,, \quad vy^* = -(vy)^*$$

for all $y, z \in E$. Then for $y \in F$,

$$vy = vuy = uvy\,, \quad vy \in F\,.$$

Hence $v(F) \subset F$. For every $y \in F$,

$$v\left(x_{tt}^0 y x_{tt}^0\right) = x_{tt}^0 (vy) x_{tt}^0\,,$$

so that

$$v\left(x_{tt}^0 F x_{tt}^0\right) = x_{tt}^0 F x_{tt}^0\,.$$

By c), we may replace $x_{tt}^0 F x_{tt}^0$ by $\mathbb{H}$. Then

$$vi = -vi^* = (vi)^*\,, \quad vj = -vj^* = (vj)^*\,.$$

Hence there are $\alpha, \beta \in \mathbb{R}$ with

$$vi = \alpha 1\,, \quad vj = \beta 1\,.$$

We deduce

$$\begin{cases} -i = v^2 i = \alpha v 1 \\ -j = v^2 j = \beta v 1\,, \end{cases}$$

which is a contradiction.

f) Let $v : F \to G$ be an isomorphism of real C^*-algebras.By c), x_{tt}^0 is a minimal element of $\Pr F \backslash \{0\}$. Then $v(x_{tt}^0)$ is a minimal element of $\Pr G \backslash \{0\}$. By Proposition 5.5.6.8, there is a $\xi \in \ell^2(S)$ such that $v(x_{tt}^0) = \langle \cdot | \xi \rangle \xi$. Then $v(x_{tt}^0 F x_{tt}^0)$ is isomorphic to $\langle \cdot | \xi \rangle \xi \circ G \circ \langle \cdot | \xi \rangle \xi$ which is impossible since the former space has real dimension 4 while the latter has dimension 2 (resp. 1). ■

Proposition 5.5.7.11 (0) *We assume* $\mathbb{K} = \mathbb{C}$. *Let* T *be a set. Put*

$$H := \ell^2(T)\,, \quad H_{\mathbb{R}} := H \cap \mathbb{R}^T\,.$$

Let E be a C^*-subalgebra of $\mathcal{L}(H)$ containing $\mathcal{K}(H)$ such that

$$zEz^* = E$$

for every unitary operator z on H and

$$a \in E \Longrightarrow \overline{a} \in E\,.$$

Further put

$$G := \{a \in E \mid a = \overline{a}\}\,,$$

and let F be a real C^*-subalgebra of E for which there is an isomorphism of complex C^*-algebras $v : \overset{\circ}{F} \to E$ such that

$$v((a,0)) = a$$

for every $a \in F$.

a) *If u is a conjugate involution on E, then there is a unitary operator x on H such that*

$$ua = x\overline{a}x^*$$

for every $a \in E$ and

$$\overline{x} = x^* \quad or \quad \overline{x} = -x^*\,.$$

u is unique up to a factor of absolute value 1.

b) *There is a unitary operator x on H such that*

$$F = \{a \in E \mid x\overline{a}x^* = a\}\,,$$

$$x^* = \overline{x} \quad or \quad x^* = -\overline{x}\,,$$

and

$$v((a,b)) = a + ib$$

for every $(a,b) \in \overset{\circ}{F}$. x is unique up to a factor of absolute value 1.

c) *$a(H_{\mathbb{R}}) \subset H_{\mathbb{R}}$ for every $a \in G$. For $a \in G$, define*

$$\widetilde{a} : H_{\mathbb{R}} \longrightarrow H_{\mathbb{R}}\,, \quad \xi \longmapsto a\xi\,.$$

Put

$$\widetilde{G} := \{\widetilde{a} \mid a \in G\}\,.$$

d) *$\widetilde{G}$ is a C^*-subalgebra of $\mathcal{L}(H_{\mathbb{R}})$ containing $\mathcal{K}(H_{\mathbb{R}})$ such that*

$$z\widetilde{G}z^* = \widetilde{G}$$

for every unitary operator z on $H_{\mathbb{R}}$.

e) *If $E = \mathcal{L}(H)$ (resp. $E = \mathcal{K}(H)$) then $\widetilde{G} = \mathcal{L}(H_{\mathbb{R}})$ (resp. $\widetilde{G} = \mathcal{K}(H_{\mathbb{R}})$).*

f) *If $\overline{x} = x^*$ then there is a unitary operator z on H such that $\widetilde{zaz^*} \in \widetilde{G}$ for every $a \in F$ and the map*

$$F \longrightarrow \widetilde{G}, \quad a \longmapsto \widetilde{zaz^*}$$

is an isomorphism of real C^*-algebras.

g) *If $x^* = -\overline{x}$ then there is a unitary operator z on H such that*

$$zFz^* = F(E)$$

(Proposition 5.5.7.10) and the map

$$F \longrightarrow F(E), \quad a \longmapsto zaz^*$$

is an isomorphism of real C^*-algebras.

a) The map

$$E \longrightarrow E, \quad a \longmapsto \overline{ua}$$

is an isomorphism of complex C^*-algebras. By Corollary 5.5.6.12, there is a unitary operator x on H, unique up to a factor of absolute value 1, such that

$$ua = x\overline{a}x^*$$

for every $a \in E$. By Proposition 5.5.7.6 b $\Rightarrow$ a,

$$\overline{x} = x^* \quad \text{or} \quad \overline{x} = -x^*.$$

b) By Proposition 2.3.1.43 b), there is a conjugate involutive u on E such that

$$F = \{a \in E \mid ua = a\}.$$

By a), there is a unitary operator x on H, unique up to a factor of absolute value 1, such that

$$F = \{a \in E \mid x\bar{a}x^* = a\}$$

and

$$\overline{x} = x^* \quad \text{or} \quad \overline{x} = -x^* .$$

By Proposition 5.5.7.6, the map

$$\overset{\circ}{F} \longrightarrow E , \quad (a,b) \longmapsto a + ib$$

is an isomorphism of complex C^*-algebras. It follows

$$v((a,b)) = a + ib$$

for all $(a,b) \in \overset{\circ}{F}$.

c) For $\xi \in H_{\mathbb{R}}$

$$a\xi = \overline{a}\xi = \overline{a\xi} ,$$

so that

$$a\xi \in H_{\mathbb{R}} \quad \text{and} \quad a(H_{\mathbb{R}}) \subset H_{\mathbb{R}} .$$

d) For $b \in \mathcal{L}(H_{\mathbb{R}})$, define

$$\overset{\circ}{b} : H \longrightarrow H , \quad \xi + i\eta \longmapsto b\xi + ib\eta .$$

Take $b \in \mathcal{L}(H_{\mathbb{R}})$. Then

$$\overset{\circ}{b} \in \mathcal{L}(H) , \quad \overline{\overset{\circ}{b}} = \overset{\circ}{b} , \quad \widetilde{\overset{\circ}{b}} = b .$$

Hence if $\overset{\circ}{b} \in E$, then $b \in \widetilde{G}$. It is easy to see that $\widetilde{G}$ is a C^*-subalgebra of $\mathcal{L}(H_{\mathbb{R}})$. Since $\overset{\circ}{b} \in \mathcal{K}(H)$ whenever $b \in \mathcal{K}(H_{\mathbb{R}})$, we deduce that $\mathcal{K}(H_{\mathbb{R}}) \subset G$.

Now take $b \in \widetilde{G}$. Then $\overset{\circ}{b} \in G$. Since $\overset{\circ}{z}$ is a unitary operator on H,

$$\overbrace{zbz^*}^{\circ} = \overset{\circ}{z}\overset{\circ}{b}\overset{\circ}{z}^* \in E .$$

Since

$$\overline{\overset{\circ}{z}\overset{\circ}{b}\overset{\circ}{z}^*} = \overline{\overset{\circ}{z}}\,\overline{\overset{\circ}{b}}\,\overline{\overset{\circ}{z}^*} = \overset{\circ}{z}\overset{\circ}{b}\overset{\circ}{z}^*$$

we get that

$$\overbrace{zbz^*}^{\circ} \in G , \quad zbz^* \in \widetilde{G} , \quad z\widetilde{G}z^* = \widetilde{G} .$$

e) is easy to see.

f) By Proposition 5.5.7.8, there is a unitary operator z on H such that

$$zFz^* = G$$

and the map

$$F \longrightarrow G\,, \quad a \longmapsto zaz^*$$

is an isomorphism of real C^*-algebras. For $a \in F$,

$$\widetilde{zaz^*} \in \widetilde{G}$$

and the map

$$F \longrightarrow \widetilde{G}\,, \quad a \longmapsto \widetilde{zaz^*}$$

is an isomorphism of real C^*-algebras.

g) follows from Proposition 5.5.7.8. ■

Corollary 5.5.7.12 (0) *Assume that* $\mathbb{K} = \mathbb{C}$. *Let* S, T *be sets and let* x, y *be unitary operators on* $\ell^2(S)$ *and* $H := \ell^2(T)$, *respectively, such that*

$$x^* = \overline{x} \qquad y^* = -\overline{y}\,.$$

Further, let E *be a* C^**-subalgebra of* $\mathcal{L}(H)$ *containing* $\mathcal{K}(H)$ *such that*

$$zEz^* = E$$

for every unitary operator z *on* H *and such that*

$$a \in E \Longrightarrow \overline{a} \in E\,.$$

Put

$$F := \{a \in E \mid y\overline{a}y^* = a\}\,,$$
$$G := \{b \in \mathcal{K}(\ell^2(S)) \mid x\overline{b}x^* = b\}\,.$$

If either S *or* T *is nonempty, then the real* C^**-algebras* F *and* G *(Proposition 5.5.7.6) and* F *and* $\mathcal{K}(\ell^2(S))$ *are not isomorphic.*

By Proposition 5.5.7.11 d),e),f),g), F is isomorphic to $F(E)$ of Example 5.5.7.10 and G is isomorphic to $\mathcal{K}(\ell^2(S) \cap \mathbb{R}^S)$. The assertion follows from Proposition 5.5.7.10 f). ■

Corollary 5.5.7.13 (0) *We adopt the hypotheses and notation of Proposition 5.5.7.11. Let $\mathfrak{E}$ be the class of real C^*-algebras A such that the complex C^*-algebras $\overset{\circ}{A}$ and E are isomorphic.*

a) *If $n := \operatorname{Card} T$ is an odd natural number, then every element of $\mathfrak{E}$ is isomorphic to $\mathbb{R}_{n,n}$.*

b) *If $\operatorname{Card} T$ is not an odd natural number, then $\mathfrak{E}$ decomposes into precisely two isomorphism classes: those of the first class are isomorphic to $\widetilde{G}$ while the elements of the second are isomorphic to $F(E)$ (Proposition 5.5.7.10). If in addition $E = \mathcal{L}(H)$ (resp. $E = \mathcal{K}(H)$) then $\widetilde{G} = \mathcal{L}(H_{\mathbb{R}})$ (resp. $\widetilde{G} = \mathcal{K}(H_{\mathbb{R}})$).*

a & b. Let $\mathcal{F}$ be the set of real C^*-subalgebras B of E such that the map

$$\overset{\circ}{B} \longrightarrow E, \quad (x, y) \longmapsto x + iy$$

is an isomorphism of complex C^*-algebras. By Proposition 2.3.1.43 c), every element of $\mathfrak{E}$ is isomorphic to an element of $\mathcal{F}$.

Take $B \in \mathcal{F}$. By Proposition 5.5.7.11 b), there is a unitary operator x on H such that

$$B = \{a \in E \mid x\overline{a}x^* = a\}$$

and

$$x^* = \overline{x} \quad \text{or} \quad x^* = -\overline{x}.$$

If $x^* = \overline{x}$, then by Proposition 5.5.7.11 f), B is isomorphic to $\widetilde{G}$. If $x^* = -\overline{x}$, then by Proposition 5.5.7.7 b), $\operatorname{Card} T$ is not an odd natural number and by Proposition 5.5.7.11 g), B is isomorphic to $F(E)$. Hence if $n := \operatorname{Card} T$ is an odd natural number, then by Proposition 5.5.7.11 e), each element of $\mathfrak{E}$ is isomorphic to $\mathbb{R}_{n,n}$ By Corollary 5.5.7.12, $\widetilde{G}$ and $F(E)$ are not isomorphic. Hence if $\operatorname{Card} T$ is not an odd natural number, then $\mathfrak{E}$ decomposes in precisely two isomorphism classes: the elements of first class are isomorphic to $\widetilde{G}$ while the elements of the second are isomorphic to $F(E)$.

The last assertion of b) was proved in Proposition 5.5.7.11 e). ■

Corollary 5.5.7.14 *Let E be a complex C^*-algebra for which there is a faithful irreducible representation (H, φ) such that:*

1) $\varphi(E) \cap \mathcal{K}(H) \neq \{0\}$,

2) $x \in \operatorname{Un} \mathcal{L}(H) \Rightarrow x\varphi(E)x^* = \varphi(E)$,

3) $x \in \varphi(E) \Rightarrow \overline{x} \in \varphi(E)$.

Let further $\mathfrak{E}$ be the class of real C^*-algebras for which E is isomorphic to their complexifications. Put

$$F \sim G :\Longleftrightarrow F \text{ and } G \text{ are isomorphic}$$

for all $F, G \in \mathfrak{E}$. Then $\sim$ is an equivalence relation on $\mathfrak{E}$ and

$$\operatorname{Card} \mathfrak{E}/\sim \in \{1, 2\}.$$

Since the representation (H, φ) is faithful, we may identify E with $\varphi(E)$. By 1) and Proposition 5.5.6.9, $\mathcal{K}(H) \subset \varphi(E)$ and the assertion follows from Corollary 5.5.7.13 a),b). ■

Corollary 5.5.7.15 (0) *We assume $\mathbb{K} = \mathbb{C}$. Let T be a nonempty set,*

$$H := \ell^2(T), \quad H_{\mathbb{R}} := H \cap \mathbb{R}^T,$$

$n \in \mathbb{N}$, and let $\mathfrak{E}$ be the class of real C^*-algebras E such that $\overset{\circ}{E}$ is isomorphic to $\mathcal{K}(H)^n$. We set

$$E \sim F :\Longleftrightarrow E \text{ and } F \text{ are isomorphic}$$

for all $E, F \in \mathfrak{E}$,

$$E(p,q) := \mathcal{K}(H)^p \times \mathcal{K}(H_{\mathbb{R}})^q \times F(\mathcal{K}(H))^{n-2p-q} \quad (2p + q \leq n),$$

$$F(p,q) := \{E \in \mathfrak{E} \mid E \sim E(p,q)\},$$

for all $p, q \in \mathbb{N} \cup \{0\}$, and

$$A := \{(p,q) \in (\mathbb{N} \cup \{0\})^2 \mid F(p,q) \neq \emptyset\}.$$

If $\operatorname{Card} T$ is an odd natural number (is not an odd natural number), then

$$A = \{(p,q) \in (\mathbb{N} \cup \{0\})^2 \mid 2p + q = n\}$$

$$(A = \{(p,q) \in (\mathbb{N} \cup \{0\})^2 \mid 2p + q \leq n\})$$

and the map

$$A \longrightarrow \mathfrak{E}/\sim, \quad (p,q) \longmapsto F(p,q)$$

is bijective. In particular

$$\operatorname{Card}(\mathfrak{E}/\sim) = \frac{1}{2}\left(n+1+\cos^2\frac{n\pi}{2}\right)$$

$$\left(\operatorname{Card}(\mathfrak{E}/\sim) = \frac{1}{4}\left((n+2)^2 - \sin^2\frac{n\pi}{2}\right)\right).$$

By Corollary 5.5.7.13 a), if $\operatorname{Card} T$ is an odd natural number, $\mathcal{K}(H_{\mathbb{R}})$ is the only real C^*-algebra that has complexification isomorphic to $\mathcal{K}(H)$. By Corollary 5.5.7.13 b), if $\operatorname{Card} T$ is not an odd natural number then there are exactly two non-isomorphic real C^*-algebras, namely $\mathcal{K}(H_{\mathbb{R}})$ and $F(\mathcal{K}(H))$, the complexification of which are isomorphic with $\mathcal{K}(H)$.

Now we use Proposition 4.3.5.8. If $\operatorname{Card} T$ is an odd natural number then

$$A = \{(p,q) \in (\mathbb{N} \cup \{0\})^2 \mid 2p+q = n\}$$

and

$$\operatorname{Card}(\mathcal{E}/\sim) = \frac{1}{2}(n+1+\cos^2\frac{n\pi}{2}).$$

If $\operatorname{Card} T$ is not an odd natural number then

$$A = \{(p,q) \in (\mathbb{N} \cup \{0\})^2 \mid 2p+q \leq n\}$$

and

$$\begin{aligned}
\operatorname{Card}(\mathcal{E}/\sim) &= 1 + \frac{1}{2}\sum_{k=0}^{n-1}(n-k+1+\cos^2\frac{(n-k)\pi}{2}) = \\
&= 1 + \frac{1}{2}\left(n^2 - \frac{n(n-1)}{2} + n + \frac{1}{2}(n - \sin^2\frac{n\pi}{2}\right) = \\
&= \frac{1}{4}\left(4 + 2n^2 - n^2 + n + 2n + n - \sin^2\frac{n\pi}{2}\right) = \\
&= \frac{1}{4}\left((n+2)^2 - \sin^2\frac{n\pi}{2}\right).
\end{aligned}$$

■

Example 5.5.7.16 *We use the notation and conventions of Proposition 5.5.7.10. Let Σ denote the set of all*

$$(f,g,\varphi,\psi) \in \mathbb{R}_+^{T\times T} \times \mathbb{R}_+^{T\times T} \times \mathbb{R}^{T\times T} \times \mathbb{R}^{T\times T}$$

such that the family

$$\left(f(s,t)x_{st}^{\varphi(s,t)} + g(s,t)y_{st}^{\psi(s,t)}\right)_{s,t\in T}$$

is summable and for each $(f,g,\varphi,\psi) \in \Sigma$ *let* $z(f,g,\varphi,\psi)$ *denote the sum of the above family.*

a) *If* $(f,g,\varphi,\psi) \in \Sigma$, $q,r,s,t \in T$, *and* $\theta \in \mathbb{R}$, *then*

$$2\left(f^2(s,t)\sin^2(\frac{\pi}{2}(\theta+\varphi(s,t)) + g^2(s,t)\sin^2\frac{\pi}{2}(-\theta+\psi(s,t))\right)x_{qr}^1 =$$

$$= f(s,t)\sin\frac{\pi}{2}(\theta+\varphi(s,t))\left(x_{qs}^{\theta}z(f,g,\varphi,\psi)x_{tr}^0 + y_{qs}^{\theta}z(f,g,\varphi,\psi)y_{tr}^0\right) -$$

$$-g(s,t)\sin\frac{\pi}{2}(-\theta+\psi(s,t))\left(y_{qs}^{\theta}z(f,g,\varphi,\psi)x_{tr}^0 - x_{qs}^{\theta}z(f,g,\varphi,\psi)y_{tr}^0\right).$$

Hence $z(f,g,\varphi,\psi) = 0$ *implies that* $f = g = 0$.

b) *Take* $f,g,h,k \in \mathbb{R}^{T\times T}$ *such that the family*

$$\left(f(s,t)x_{st}^0 + g(s,t)x_{st}^1 + h(s,t)y_{st}^0 + k(s,t)y_{st}^1\right)_{s,t\in T}$$

is summable.

b_1) *If the sum is* 0 *then:*

$$f = g = h = k = 0.$$

b_2) *The sum is selfadjoint iff*

$$f(s,t) = f(t,s), \quad g(s,t) = -g(t,s),$$

$$h(s,t) = -h(t,s), \quad k(s,t) = -k(t,s)$$

for all $s,t \in T$.

c) *If* $E = \mathcal{K}(H)$, *then:*

c_1) F *is the closed real vector subspace of* E *generate by*

$$\{x_{st}^{\varphi} \mid s,t \in T, \varphi \in \mathbb{R}\} \cup \{y_{st}^{\varphi} \mid s,t \in T, \varphi \in \mathbb{R}\}$$

and

$$F = \{z(f,g,\varphi,\psi) \mid (f,g,\varphi,\psi) \in \Sigma\}.$$

c_2) *F is simple and purely real.*

c_3) *If $(f,g,\varphi,\psi)\in\Sigma$ then $z(f,g,\varphi,\psi)$ is selfadjoint iff*

$$f(s,t)=f(t,s)\,,\quad f(s,t)\neq 0\Longrightarrow \varphi(s,t)+\varphi(t,s)\equiv 0 \pmod 4\,,$$

$$g(s,t)=g(t,s)\,,\quad g(s,t)\neq 0\Longrightarrow \varphi(s,t)-\varphi(t,s)\equiv 2 \pmod 4$$

for all $s,t\in T$.

d) *If T is finite, then*

$$\operatorname{Dim} F=(2\operatorname{Card} T)^2\,,\quad \operatorname{Dim}\operatorname{Re} F=2(\operatorname{Card} T)^2-\operatorname{Card} T$$

and $\sum_{t\in T} x_{tt}^0$ is a unit of $\mathcal{L}(H)$.

a) Put $z:=z(f,g,\varphi,\psi)$. By Proposition 5.5.7.10 b),

$$\begin{cases} x_{qs}^{\theta} z x_{tr}^0 = f(s,t)x_{qr}^{\theta+\varphi(s,t)} + g(s,t)y_{qr}^{-\theta+\psi(s,t)} \\ y_{qs}^{\theta} z y_{tr}^0 = -f(s,t)x_{qr}^{-\theta-\varphi(s,t)} - g(s,t)y_{qr}^{\theta-\psi(s,t)} \\ x_{qs}^{\theta} z y_{tr}^0 = f(s,t)y_{qr}^{-\theta-\varphi(s,t)} - g(s,t)x_{qr}^{\theta-\psi(s,t)} \\ y_{qs}^{\theta} z x_{tr}^0 = f(s,t)y_{qr}^{\theta+\varphi(s,t)} - g(s,t)x_{qr}^{-\theta+\psi(s,t)} \end{cases}$$

$$\begin{cases} x_{qs}^{\theta} z x_{tr}^0 + y_{qs}^{\theta} z y_{tr}^0 = \\ = 2f(s,t)\sin\tfrac{\pi}{2}(\theta+\varphi(s,t))x_{qr}^1 + 2g(s,t)\sin\tfrac{\pi}{2}(-\theta+\psi(s,t))y_{qr}^1 \\ y_{qs}^{\theta} z x_{tr}^0 - x_{qs}^{\theta} z y_{tr}^0 = \\ = -2g(s,t)\sin\tfrac{\pi}{2}(-\theta+\psi(s,t))x_{qr}^1 + 2f(s,t)\sin\tfrac{\pi}{2}(\theta+\varphi(s,t))y_{qr}^1 \end{cases}$$

$$\begin{aligned} &2(f^2(s,t)\sin^2\tfrac{\pi}{2}(\theta+\varphi(s,t)) + g^2(s,t)\sin^2\tfrac{\pi}{2}(-\theta+\psi(s,t)))x_{qr}^1 = \\ &= f(s,t)\sin\tfrac{\pi}{2}(\theta+\varphi(s,t))(x_{qs}^{\theta} z x_{tr}^0 + y_{qs}^{\theta} z y_{tr}^0) - \\ &\quad - g(s,t)\sin\tfrac{\pi}{2}(-\theta+\psi(s,t))(y_{qs}^{\theta} z x_{tr}^0 - x_{qs}^{\theta} z y_{tr}^0)\,. \end{aligned}$$

Assume that $z=0$. Taking θ such that

$$\sin\frac{\pi}{2}(\theta+\varphi(s,t))\neq 0\,,\quad \sin\frac{\pi}{2}(-\theta+\psi(s,t))\neq 0\,,$$

we see that

$$f(s,t) = g(s,t) = 0\,.$$

Since s and t are arbitrary, $f = g = 0$.

b_1) For $s,t \in \mathbb{R}$, define

$$\begin{cases} F(s,t) := (f^2(s,t) + g^2(s,t))^{\frac{1}{2}} \in \mathbb{R}_+\,, \\ G(s,t) := (h^2(s,t) + k^2(s,t))^{\frac{1}{2}} \in \mathbb{R}_+\,. \end{cases}$$

Hence there are $\varphi, \psi \in \mathbb{R}^{T\times T}$ such that

$$\begin{cases} f = F\cos\frac{\pi}{2}\varphi \\ g = F\sin\frac{\pi}{2}\varphi\,, \end{cases} \qquad \begin{cases} h = G\cos\frac{\pi}{2}\psi \\ k = G\sin\frac{\pi}{2}\psi\,. \end{cases}$$

By Proposition 5.5.7.10 b),

$$F(s,t)x_{st}^{\varphi(s,t)} = F(s,t)\left(\cos\frac{\pi\varphi(s,t)}{2}x_{st}^0 + \sin\frac{\pi\varphi(s,t)}{2}x_{st}^1\right) =$$

$$= f(s,t)x_{st}^0 + g(s,t)x_{st}^1\,,$$

$$G(s,t)y_{st}^{\psi(s,t)} = G(s,t)\left(\cos\frac{\pi\psi(s,t)}{2}y_{st}^0 + \sin\frac{\pi\psi(s,t)}{2}y_{st}^1\right) =$$

$$= h(s,t)y_{st}^0 + k(s,t))y_{st}^1\,.$$

Hence $(F,G,\varphi,\psi) \in \Sigma$ and $z(F,G,\varphi,\psi) = 0$. By a),

$$F = G = 0\,.$$

Hence

$$f = g = h = k = 0\,.$$

b_2) By Proposition 5.5.7.10 b),

$$\begin{aligned} &\sum_{s,t\in T}(f(s,t)x_{st}^0 + g(s,t)x_{st}^1 + h(s,t)y_{st}^0 + k(s,t)y_{st}^1)^* = \\ &= \sum_{s,t\in T}(f(s,t)x_{ts}^0 - g(s,t)x_{ts}^1 - h(s,t)y_{ts}^0 - k(s,t)y_{ts}^1) = \\ &= \sum_{s,t\in T}(f(t,s)x_{st}^0 - g(t,s)x_{st}^1 - h(t,s)y_{st}^0 - k(s,t)y_{st}^1)\,. \end{aligned}$$

By b_1), the sum is selfadjoint iff

$$f(s,t) = f(t,s)\,, \quad g(s,t) = -g(t,s)\,,$$

$$h(s,t) = -h(t,s)\,, \quad k(s,t) = -k(t,s)$$

for all $s,t \in T$.

c_1) Let G be the closed real vector subspace of E generated by

$$\{x_{st}^{\varphi} \mid s,t \in T\,,\ \varphi \in \mathbb{R}\} \cup \{y_{st}^{\varphi} \mid s,t \in T\,,\ \varphi \in \mathbb{R}\}\,.$$

Then

$$\{z(f,g,\varphi,\psi) \mid (f,g,\varphi,\psi) \in \Sigma\} \subset G \subset F$$

(Proposition 5.5.7.10 b)). Take $z \in F$. By the last assertion of Theorem 5.5.5.1,

$$z = \sum_{s,t\in T} \begin{pmatrix} \langle zt'|s'\rangle\langle\cdot|t'\rangle s' + \langle zt'|s''\rangle\langle\cdot|t'\rangle s'' + \\ +\langle zt''|s'\rangle\langle\cdot|t''\rangle s' + \langle zt''|s''\rangle\langle\cdot|t''\rangle s'' \end{pmatrix}.$$

Hence by Proposition 5.3.2.13 c),d),

$$z = x\bar{z}x^* =$$

$$= \sum_{s,t\in T} \begin{pmatrix} \overline{\langle zt'|s'\rangle}\langle\cdot|xt'\rangle xs' + \overline{\langle zt'|s''\rangle}\langle\cdot|xt'\rangle xs'' + \\ +\overline{\langle zt''|s'\rangle}\langle\cdot|xt''\rangle xs' + \overline{\langle zt''|s''\rangle}\langle\cdot|xt''\rangle xs'' \end{pmatrix} =$$

$$= \sum_{s,t\in T} \begin{pmatrix} \overline{\langle zt'|s'\rangle}\langle\cdot|t''\rangle s'' - \overline{\langle zt'|s''\rangle}\langle\cdot|t''\rangle s' - \\ -\overline{\langle zt''|s'\rangle}\langle\cdot|t'\rangle s'' + \overline{\langle zt''|s''\rangle}\langle\cdot|t'\rangle s' \end{pmatrix}.$$

Thus

$$\begin{cases} \langle zt'|s'\rangle = \overline{\langle zt''|s''\rangle} & , \quad \langle zt''|s''\rangle = \overline{\langle zt'|s'\rangle} \\ \langle zt'|s''\rangle = -\overline{\langle zt''|s'\rangle} & , \quad \langle zt''|s'\rangle = -\overline{\langle zt'|s''\rangle} \end{cases}$$

for all $s,t \in T$, so that

$$z = \sum_{s,t\in T} \begin{pmatrix} \langle zt'|s'\rangle\langle\cdot|t'\rangle s' + \overline{\langle zt'|s'\rangle}\langle\cdot|t''\rangle s'' + \\ +\langle zt'|s''\rangle\langle\cdot|t'\rangle s'' - \overline{\langle zt', s''\rangle}\langle\cdot|t''\rangle s' \end{pmatrix}.$$

Given $s,t \in T$, define $(f,g,\varphi,\psi) \in \mathbb{R}_+^{T\times T} \times \mathbb{R}_+^{T\times T} \times \mathbb{R}^{T\times T} \times \mathbb{R}^{T\times T}$ by

$$\begin{cases} f(s,t)e^{i\frac{\pi}{2}\varphi(s,t)} := \langle zt'|s'\rangle \\ g(s,t)e^{i\frac{\pi}{2}\psi(s,t)} := \langle zt'|s''\rangle . \end{cases}$$

Then $(f,g,\varphi,\psi) \in \Sigma$ and

$$z = z(f,g,\varphi,\psi) .$$

Hence

$$F = \{z(f,g,\varphi,\psi) | (f,g,\varphi,\psi) \in \sum\} .$$

c_2) By Corollary 5.5.1.13 b), E is simple so that the assertion follows from Proposition 5.5.7.10 a).

c_3) By Proposition 5.5.7.10 b),

$$z(f,g,\varphi,\psi)^* = \sum_{s,t\in T} \left(f(s,t)x_{ts}^{-\varphi(s,t)} - g(s,t)y_{ts}^{\psi(s,t)} \right) =$$

$$= \sum_{s,t\in T} \left(f(t,s)x_{st}^{-\varphi(t,s)} - g(t,s)y_{st}^{\psi(t,s)} \right) =$$

$$= \sum_{s,t\in T} \left(f(t,s)(\cos\frac{\pi}{2}\varphi(t,s)x_{st}^0 - \sin\frac{\pi}{2}\varphi(t,s)x_{st}^1 \right) -$$

$$- g(t,s)\left(\cos\frac{\pi}{2}\psi(t,s)y_{st}^0 + \sin\frac{\pi}{2}\psi(t,s)y_{st}^1\right) .$$

By b_1) and Proposition 5.5.7.10 b), $z(f,g,\varphi,\psi)$ is selfadjoint iff for all $s,t \in T$,

$$\begin{cases} f(s,t)\cos\frac{\pi}{2}\varphi(s,t) = f(t,s)\cos\frac{\pi}{2}\varphi(t,s) \\ f(s,t)\sin\frac{\pi}{2}\varphi(s,t) = -f(t,s)\sin\frac{\pi}{2}\varphi(t,s) \\ g(s,t)\cos\frac{\pi}{2}\psi(s,t) = -g(t,s)\cos\frac{\pi}{2}\psi(t,s) \\ g(s,t)\sin\frac{\pi}{2}\psi(s,t) = -g(t,s)\sin\frac{\pi}{2}\psi(t,s) , \end{cases}$$

$$f(s,t)^2 = (f(s,t)\cos\tfrac{\pi}{2}\varphi(s,t))^2 + (f(s,t)\sin\tfrac{\pi}{2}\varphi(s,t))^2 =$$
$$= (f(t,s)\cos\tfrac{\pi}{2}\varphi(t,s))^2 + (f(t,s)\sin\tfrac{\pi}{2}\varphi(t,s))^2 = f(t,s)^2 ,$$
$$g(s,t)^2 = (g(s,t)\cos\tfrac{\pi}{2}\psi(s,t))^2 + (g(s,t)\sin\tfrac{\pi}{2}\psi(s,t))^2 =$$
$$= g(t,s)\cos\tfrac{\pi}{2}\psi(t,s))^2 + (g(t,s)\sin\tfrac{\pi}{2}\psi(t,s))^2 = g(t,s)^2 ,$$

i.e.

$$\begin{cases} f(s,t) = f(t,s)\,, \quad f(s,t) \neq 0 \Longrightarrow \varphi(s,t) + \varphi(t,s) \equiv 0 \pmod 4\,, \\ g(s,t) = g(t,s)\,, \quad g(s,t) \neq 0 \Longrightarrow \psi(s,t) - \psi(t,s) \equiv 2 \pmod 4\,. \end{cases}$$

d) By Proposition 5.5.7.10 a),

$$\operatorname{Dim} F = 4(\operatorname{Card} T)^2\,.$$

Put $n := \operatorname{Card} T$. By b_2),

$$\operatorname{Dim} \operatorname{Re} F = \frac{n(n+1)}{2} + 3\frac{n(n-1)}{2} = \frac{n^2 + n + 3n^2 - 3n}{2} = 2n^2 - n\,.$$

By c_1) and Proposition 5.5.7.10 a),b), $\sum_{t \in T} x_{tt}^0$ is a unit of $\mathcal{L}(H)$. ■

5.6 Hilbert right C^*–Modules

Hilbert C^*-modules are generalizations of Hilbert spaces in the sense that the scalar product takes its values in a C^*-algebra rather than in the ground field.

In this section, E is always a C^*-algebra.

5.6.1 Some General Results

Definition 5.6.1.1 (0) *A* ***weak semi–inner–product right*** E***–module*** *is a vector space* H *endowed with a bilinear map*

$$H \times E \longrightarrow H\,, \quad (\xi, x) \longmapsto \xi x \quad \textbf{\textit{(right multiplication)}}$$

such that

$$(\xi x)y = \xi(xy)$$

for all $\xi \in H$ *and* $x, y \in E$ *and with a sesquilinear map*

$$H \times H \longrightarrow E\,, \quad (\xi, \eta) \longmapsto \langle \xi | \eta \rangle \quad \textbf{\textit{(inner–product)}}$$

such that

$$\langle \xi x | \eta \rangle = \langle \xi | \eta \rangle x\,,$$

$$\langle \xi | \eta \rangle = \langle \eta | \xi \rangle^*\,,$$

$$\langle \xi | \xi \rangle \in E_+$$

$$E \text{ unital} \Longrightarrow \xi 1 = \xi$$

for all $\xi, \eta \in H$ *and* $x \in E$. H *is called a* ***semi–inner–product (inner–product) right*** E***–module*** *if in addition*

$$(\langle \xi | \xi \rangle + \langle \eta | \eta \rangle\,,\ \langle \eta | \xi \rangle - \langle \xi | \eta \rangle) \in \overset{\circ}{E}_+$$

$$(\text{and } \langle \xi | \xi \rangle = 0 \Longrightarrow \xi = 0)$$

for all $\xi, \eta \in H$, *where* $\overset{\circ}{E}$ *denotes the complexification of* E.

The above notions were introduced by Kaplansky (1953) for E unital, complex, and commutative and by Paschke (1973) and Rieffel (1974), for the general case.

If E is not unital and $\widetilde{E}$ is the unital C^*-algebra associated to E then, by setting $\xi 1 = \xi$ for every $\xi \in H$, H becomes a weak semi-inner-product right $\widetilde{E}$-module. If E is unital then under a very weak hypothesis, which will be always fulfilled in this book the axiom

$$\xi \in H \Longrightarrow \xi 1 = \xi$$

is automatically fulfilled (Proposition 5.6.1.2 f)).

Let E be a complex C^*-algebra. If E_0 denotes the underlying real C^*-algebra of E (Theorem 4.1.1.8 a)) then restricting the scalars of H to $\mathbb{R}$ we get a weak semi-inner-product right E_0-module.

The axiom

$$\xi, \eta \in H \Longrightarrow \langle \xi | \eta \rangle = \langle \eta | \xi \rangle^*$$

is exactly the assertion that the above sesquilinear map is Hermitian.

Proposition 5.6.1.2 (0) *Let H be a weak semi-inner-product right E-module. Then for all $x, y \in E$ and $\xi, \eta, \zeta \in H$:*

a) $\langle \zeta | \xi x + \eta y \rangle = x^* \langle \zeta | \xi \rangle + y^* \langle \zeta | \eta \rangle$.

b) $\langle \xi x + \eta y | \xi x + \eta y \rangle = x^* \langle \xi | \xi \rangle x + 2\mathrm{re}\,(y^* \langle \xi | \eta \rangle x) + y^* \langle \eta | \eta \rangle y$,
$\langle \xi + \eta | \xi + \eta \rangle \leq \langle \xi + \eta | \xi + \eta \rangle + \langle \xi - \eta | \xi - \eta \rangle = 2(\langle \xi | \xi \rangle + \langle \eta | \eta \rangle)$.

c) $\langle \xi | \eta \rangle \langle \eta | \xi \rangle \leq \| \langle \xi | \xi \rangle \| \langle \eta | \eta \rangle$.

d) $\| \langle \xi | \eta \rangle \|^2 \leq \| \langle \xi | \xi \rangle \| \, \| \langle \eta | \eta \rangle \|$.

e) $\| \langle \xi x | \xi x \rangle \| = \| \langle \xi | \xi \rangle^{\frac{1}{2}} x \|^2 \leq \|x\|^2 \| \langle \xi | \xi \rangle \|$.

f) *The map*

$$H \longrightarrow \mathbb{R}_+, \quad \xi \longmapsto \| \langle \xi | \xi \rangle \|^{\frac{1}{2}}$$

is a semi-norm. If it is a norm then for every $\xi \in H$ and every approximate unit $\mathfrak{F}$ of E,

$$\lim_{x, \mathfrak{F}} \xi x = \xi .$$

g) *If* $\mathbb{K} = \mathbb{C}$ *then* H *is a semi-inner-product right* E*-module.*

h) *For every* $\xi \in H$, $\{x \in E | \xi x = \xi x^* = 0\}$ *is a hereditary* C^**-subalgebra of* E.

i) $\{x \in E | \xi \in H \Rightarrow \xi x = 0\}$ *is a closed ideal of* E.

j) *The vector subspace of* E *generated by* $\{\langle \xi | \eta \rangle | \xi, \eta \in H\}$ *is an ideal of* E.

a) $\langle \zeta | \xi x + \eta y \rangle = \langle \xi x + \eta y | \zeta \rangle^* = (\langle \xi | \zeta \rangle x + \langle \eta | \zeta \rangle y)^* = x^* \langle \zeta | \xi \rangle + y^* \langle \zeta | \eta \rangle$.

b) By a),

$$\langle \xi x + \eta y | \xi x + \eta y \rangle = \langle \xi | \xi x + \eta y \rangle x + \langle \eta | \xi x + \eta y \rangle y =$$

$$= x^* \langle \xi | \xi \rangle x + y^* \langle \xi | \eta \rangle x + x^* \langle \eta | \xi \rangle y + y^* \langle \eta | \eta \rangle y =$$

$$= x^* \langle \xi | \xi \rangle x + 2\mathrm{re}(y^* \langle \xi | \eta \rangle x) + y^* \langle \eta | \eta \rangle y .$$

It follows

$$\langle \xi + \eta | \xi + \eta \rangle \leq \langle \xi + \eta | \xi + \eta \rangle + \langle \xi - \eta | \xi - \eta \rangle = 2(\langle \xi | \xi \rangle + \langle \eta | \eta \rangle) .$$

c) By b) and Corollary 4.2.2.3,

$$0 \leq \langle \xi x - \eta | \xi x - \eta \rangle = x^* \langle \xi | \xi \rangle x - 2\mathrm{re}(\langle \xi | \eta \rangle x) + \langle \eta | \eta \rangle \leq$$

$$\leq \| \langle \xi | \xi \rangle \| x^* x - 2\mathrm{re}(\langle \xi | \eta \rangle x) + \langle \eta | \eta \rangle .$$

If $\| \langle \xi | \xi \rangle \| = 0$, then putting

$$x := \alpha \langle \eta | \xi \rangle$$

for an $\alpha \in \mathbb{R}_+$ we get

$$2\alpha \langle \xi | \eta \rangle \langle \eta | \xi \rangle \leq \langle \eta | \eta \rangle$$

so that

$$\langle \xi | \eta \rangle \langle \eta | \xi \rangle = 0 .$$

Hence we may assume $\| \langle \xi | \xi \rangle \| = 1$. Putting $x := \langle \eta | \xi \rangle$, we see that

$$0 \leq \langle \xi | \eta \rangle \langle \eta | \xi \rangle - 2\langle \xi | \eta \rangle \langle \eta | \xi \rangle + \langle \eta | \eta \rangle = -\langle \xi | \eta \rangle \langle \eta | \xi \rangle + \langle \eta | \eta \rangle ,$$

$$\langle \xi|\eta\rangle\langle\eta|\xi\rangle \leq \langle\eta|\eta\rangle\,.$$

d) follows from c) and Corollary 4.2.1.18.

e) follows from b).

f) By b) and d),

$$\|\langle\xi+\eta|\xi+\eta\rangle\| = \|\langle\xi|\xi\rangle + 2\mathrm{re}\langle\xi|\eta\rangle + \langle\eta|\eta\rangle\| \leq$$

$$\leq \|\xi|\xi\rangle\| + 2\|\langle\xi|\eta\rangle\| + \|\langle\eta|\eta\rangle\| \leq$$

$$\leq \|\langle\xi|\xi\rangle\| + 2\|\langle\xi|\xi\rangle\|^{\frac{1}{2}}\|\langle\eta|\eta\rangle\|^{\frac{1}{2}} + \|\langle\eta|\eta\rangle\| = (\|\langle\xi|\xi\rangle\|^{\frac{1}{2}} + \|\langle\eta|\eta\rangle\|^{\frac{1}{2}})^2\,.$$

By b),

$$\langle\xi x - \xi|\xi x - \xi\rangle =$$

$$= x^*\langle\xi|\xi\rangle x - \langle\xi|\xi\rangle x - x^*\langle\xi|\xi\rangle + \langle\xi|\xi\rangle$$

for every $x \in E$ so that

$$\lim_{x,\mathfrak{F}}\langle\xi x - \xi|\xi x - \xi\rangle = 0$$

(Proposition 2.3.4.9),

$$\lim_{x,\mathfrak{F}}\xi x = \xi\,.$$

g) Take $\xi, \eta \in H$. By b) and Proposition 4.2.2.15 b),

$$(\langle\xi|\xi\rangle + \langle\eta|\eta\rangle\,,\ \langle\eta|\xi\rangle - \langle\xi|\eta\rangle) =$$

$$= (\langle\xi|\xi\rangle + \langle\eta|\eta\rangle + i(\langle\eta|\xi\rangle - \langle\xi|\eta\rangle), 0) =$$

$$= (\langle\xi + i\eta|\xi + i\eta\rangle, 0) \in \overset{\circ}{E}_+\,.$$

h) Put

$$G := \{x \in E|\xi x = \xi x^* = 0\}\,.$$

It is obvious that G is an involutive subalgebra of E. By e), it is closed, i.e. it is a C^*-subalgebra of E. Take $x \in E$ and $y, z \in G$. Then

$$\xi(yxz) = (\xi y)(xz) = 0\,,$$

$$\xi(yxz)^* = \xi(z^*x^*y^*) = (\xi z^*)(x^*y^*) = 0$$

so that $yxz \in G$. By Proposition 4.3.4.6 b $\Rightarrow$ a, G is a hereditary C^*-subalgebra of E.

i) follows from e).

j) follows from a). ■

Proposition 5.6.1.3 *Every weak semi-inner-product right* $\mathbb{H}$*-module (Example 4.1.1.31) is a semi-inner-product right* $\mathbb{H}$*-module.*

Let H be a weak semi-inner-product right $\mathbb{H}$-module and let $\xi, \eta \in H$. Put

$$\langle \eta|\xi\rangle =: \alpha 1 + \beta i + \gamma j + \delta k\,.$$

Then

$$\langle \xi|\eta\rangle = \alpha 1 - \beta i - \gamma j - \delta k\,,$$

$$\langle \eta|\xi\rangle - \langle \xi|\eta\rangle = 2(\beta i + \gamma j + \delta k)\,,$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \|\langle \xi|\eta\rangle\|^2 \leq \langle \xi|\xi\rangle\langle \eta|\eta\rangle$$

(Proposition 5.6.1.2 d)).

We identify $\mathbb{H}$ with a unital real involutive subalgebra of $\mathbb{C}_{2,2}$ and $\overset{\circ}{\mathbb{H}}$ with $\mathbb{C}_{2,2}$ as in Example 2.3.1.46. Then

$$(\langle \xi|\xi\rangle + \langle \eta|\eta\rangle\,,\, \langle \eta|\xi\rangle - \langle \xi|\eta\rangle) =$$

$$= \begin{bmatrix} \langle \xi|\xi\rangle + \langle \eta|\eta\rangle - 2\beta & -2\delta + 2\gamma i \\ -2\delta - 2\gamma i & \langle \xi|\xi\rangle + \langle \eta|\eta\rangle + 2\beta \end{bmatrix}.$$

By the above,

$$\langle \xi|\xi\rangle + \langle \eta|\eta\rangle \pm 2\beta \geq \langle \xi|\xi\rangle + \langle \eta|\eta\rangle - 2\sqrt{\langle \xi|\xi\rangle\langle \eta|\eta\rangle} =$$

$$= \left(\sqrt{\langle \xi|\xi\rangle} - \sqrt{\langle \eta|\eta\rangle}\right)^2 \geq 0\,,$$

$$(\langle\xi|\xi\rangle + \langle\eta|\eta\rangle - 2\beta)(\langle\xi|\xi\rangle + \langle\eta|\eta\rangle + 2\beta) - |-2\delta + 2\gamma i|^2 =$$

$$= (\langle\xi|\xi\rangle + \langle\eta|\eta\rangle)^2 - 4\beta^2 - 4\delta^2 - 4\gamma^2 \geq$$

$$\geq \langle\xi|\xi\rangle^2 + 2\langle\xi|\xi\rangle\langle\eta|\eta\rangle + \langle\eta|\eta\rangle^2 - 4\langle\xi|\xi\rangle\langle\eta|\eta\rangle =$$

$$= (\langle\xi|\xi\rangle - \langle\eta|\eta\rangle)^2 \geq 0\,.$$

By Example 4.2.3.5,

$$(\langle\xi|\xi\rangle + \langle\eta|\eta\rangle\,,\ \langle\eta|\xi\rangle - \langle\xi|\eta\rangle) \in (\mathbb{C}_{2,2})_+ = \mathring{\mathbb{H}}_+\,. \qquad \blacksquare$$

Definition 5.6.1.4 (0) *A **Hilbert right** E**-module** is an inner-product right E-module for which the norm*

$$\xi \longmapsto \|\langle\xi|\xi\rangle\|^{\frac{1}{2}}$$

*(Proposition 5.6.1.2 f)) is complete. If E is not specified, then we simply speak of a **Hilbert right** C^***-module.***

*A **(unital) Hilbert** E**-module** is a Hilbert right-E-module H endowed with a bilinear map*

$$E \times H \longrightarrow H\,, \quad (x,\xi) \longmapsto x\xi \qquad \textbf{\textit{(left multiplication)}}$$

such that

$$(xy)\xi = x(y\xi)\,, \quad (x\xi)y = x(\xi y)\,,$$

$$\langle x\xi|\eta\rangle = \langle\xi|x^*\eta\rangle\,, \quad \langle x\xi|x\xi\rangle \leq \|x\|^2\langle\xi|\xi\rangle$$

$$(1\xi = \xi)$$

for all $x, y \in E$ and $\xi, \eta \in H$.

By Corollary 4.2.1.18,

$$\|x\xi\|^2 = \|\langle x\xi|x\xi\rangle\| \leq \|x\|^2\|\langle\xi|\xi\rangle\| = \|x\|^2\|\xi\|^2\,,$$

so that by Proposition 5.6.1.2 e), a Hilbert E-module is an E-module with respect to its right and left multiplications.

Let E be a complex C^*-algebra, F its underlying real C^*-algebra (Theorem 4.1.1.8 a)), and H a Hilbert right F-module. If E is unital, then H endowed with the scalar multiplication

$$\mathbb{C} \times H \longrightarrow H\,, \quad (\alpha, \xi) \longmapsto \xi(\alpha 1)$$

becomes a complex vector space and so a Hilbert right E-module. But if E is not unital the above extension of the structure of H is not always possible. Take

$$E := c_0\,, \quad H := F \times \mathbb{R}\,.$$

Then H endowed with the right multiplication

$$H \times F \longrightarrow H\,, \quad ((x, \alpha), y) \longmapsto (xy + \alpha y, 0)$$

and with the inner product

$$H \times H \longrightarrow F\,, \quad ((x, \alpha), (y, \beta)) \longmapsto (\frac{1}{n}(x_n + \alpha)(\overline{y_n} + \beta))_{n \in \mathbb{N}}$$

is a Hilbert right F-module. Assume it is possible to extend the structure of H to a Hilbert right E-module and put

$$(x, \alpha) := i(0, 1)\,.$$

Then

$$(0, -1) = i(i(0, 1)) = i(x, \alpha) = (ix, 0) + (\alpha x, \alpha^2)$$

and we get the contradiction

$$\alpha^2 \equiv -1\,.$$

Example 5.6.1.5 (0) *Let F be a right ideal (closed ideal) of E. Then F endowed with the maps*

$$F \times E \longrightarrow F\,, \quad (\xi, x) \longmapsto \xi x\,,$$
$$F \times F \longrightarrow E\,, \quad (\xi, \eta) \longmapsto \eta^* \xi$$

(and with the maps

$$E \times F \longrightarrow F\,, \quad (x, \xi) \longmapsto x\xi\,,$$
$$F \longrightarrow F\,, \quad \xi \longmapsto \xi^*\,,$$

(Corollary 4.2.6.2)) is an inner-product right E-module (and an involutive unital E-module).

The first and the third map are bilinear, the second one is sesquilinear, and the last one is an involution such that F is an involutive E-module whenever F is a closed ideal of E. Take $\xi, \eta \in F$ and $x \in E$. Then

$$\langle \xi x | \eta \rangle = \eta^*(\xi x) = (\eta^* \xi) x = \langle \xi | \eta \rangle x \,,$$

$$\langle \xi | \eta \rangle = \eta^* \xi = (\xi^* \eta)^* = \langle \eta | \xi \rangle^* \,,$$

$$\langle \xi | \xi \rangle = \xi^* \xi \in E_+ \,,$$

$$\langle \xi | \xi \rangle = 0 \Longrightarrow \xi = 0 \,,$$

$$(\langle \xi | \xi \rangle + \langle \eta | \eta \rangle \,,\, \langle \eta | \xi \rangle - \langle \xi | \eta \rangle) = (\xi^* \xi + \eta^* \eta \,,\, \xi^* \eta - \eta^* \xi) =$$

$$= (\xi^* \,, -\eta^*)(\xi, \eta) = (\xi, \eta)^*(\xi, \eta) \in \overset{\circ}{E}_+ \,,$$

$$\langle x\xi | \eta \rangle = \eta^* x \xi = (x^* \eta)^* \xi = \langle \xi | x^* \eta \rangle \,,$$

$$\langle x\xi | x\xi \rangle = \xi^* x^* x \xi \leq \|x\|^2 \xi^* \xi = \|x\|^2 \langle \xi | \xi \rangle$$

(Corollary 4.2.2.3)). ■

Proposition 5.6.1.6 (0) *Assume* $\mathbb{K} = \mathbb{R}$. *Let* H *be a (semi-) inner-product right* E*-module and* $\overset{\circ}{E}, \overset{\circ}{H}$ *the complexifications of* E *and of the underlying real vector space of* H, *respectively. Then* $\overset{\circ}{H}$ *endowed with the maps*

$$\overset{\circ}{H} \times \overset{\circ}{E} \longrightarrow \overset{\circ}{H} \,, \quad ((\xi, \eta), (x, y)) \longmapsto (\xi x - \eta y, \xi y + \eta x) \,.$$

$$\overset{\circ}{H} \times \overset{\circ}{H} \longrightarrow \overset{\circ}{E} \,, \quad ((\xi, \eta), (\zeta, \lambda)) \longmapsto (\langle \xi | \zeta \rangle + \langle \eta | \lambda \rangle, \langle \eta | \zeta \rangle - \langle \xi | \lambda \rangle)$$

is a (semi-) inner-product right $\overset{\circ}{E}$*-module called* ***the complexification of*** H. *If* H *is a Hilbert right* E*-module then* $\overset{\circ}{H}$ *is a Hilbert right* $\overset{\circ}{E}$*-module such that*

$$\sup\{\|\xi\|^2, \|\eta\|^2\} \leq \|(\xi, \eta)\|^2 = \|(\langle \xi | \xi \rangle + \langle \eta | \eta \rangle \,,\, \langle \eta | \xi \rangle - \langle \xi | \eta \rangle)\| \leq (\|\xi\| + \|\eta\|)^2$$

for all $(\xi, \eta) \in \overset{\circ}{H}$ *and the canonical map* $H \times H \to \overset{\circ}{H}$ *is an isomorphism of real Banach spaces.*

By Lemma 2.1.5.1 a),b), the map

$$\overset{\circ}{H} \times \overset{\circ}{E} \longrightarrow \overset{\circ}{H}, \quad ((\xi,\eta),(x,y)) \longmapsto (\xi x - \eta y, \xi y + \eta x)$$

is bilinear and

$$((\xi,\eta)(x,y))(u,v) = (\xi,\eta)((x,y)(u,v))$$

for all $(\xi,\eta) \in \overset{\circ}{H}$ and $(x,y),(u,v) \in \overset{\circ}{E}$.

For all $(\xi,\eta),(\zeta,\lambda) \in \overset{\circ}{H}$

$$\langle(\xi,\eta)|(\zeta,\lambda)\rangle^* = (\langle\xi|\zeta\rangle + \langle\eta|\lambda\rangle, \langle\eta|\zeta\rangle - \langle\xi|\lambda\rangle)^* =$$

$$= (\langle\zeta|\xi\rangle) + \langle\lambda|\eta\rangle, \langle\lambda|\xi\rangle - \langle\zeta|\eta\rangle) = \langle(\zeta,\lambda)|(\xi,\eta)\rangle,$$

$$\langle(\xi,\eta)|(\xi,\eta)\rangle = (\langle\xi|\xi\rangle + \langle\eta|\eta\rangle, \langle\eta|\xi\rangle - \langle\xi|\eta\rangle) \geq 0.$$

Take $(\zeta,\lambda) \in \overset{\circ}{H}$. Then $\langle\cdot|\zeta\rangle$ and $-\langle\cdot|\lambda\rangle$ are linear. By Lemma 2.1.5.3, the map

$$\langle\cdot|(\zeta,\lambda)\rangle : \overset{\circ}{H} \longrightarrow \overset{\circ}{E}, \quad (\xi,\eta) \longmapsto (\langle\xi|\zeta\rangle + \langle\eta|\lambda\rangle, \langle\eta|\zeta\rangle - \langle\xi,\lambda\rangle)$$

is linear. It follows, by the above result, that the map $\langle(\zeta,\lambda)|\cdot\rangle$ is conjugate linear. Hence the map

$$\langle\cdot|\cdot\rangle : \overset{\circ}{H} \times \overset{\circ}{H} \longrightarrow \overset{\circ}{E}$$

is sesquilinear.

We have

$$\langle\xi x|\eta\rangle = \langle\xi|\eta\rangle x$$

for all $\xi,\eta \in H$ and $x \in E$. By Lemma 2.1.5.1 b),

$$\langle(\xi,\eta)(x,y)|(\zeta,\lambda)\rangle = \langle(\xi,\eta)|(\zeta,\lambda)\rangle(x,y)$$

for all $(\xi,\eta),(\zeta,\lambda) \in \overset{\circ}{H}$ and $(x,y) \in \overset{\circ}{E}$. By Proposition 5.6.1.2 g), $\overset{\circ}{H}$ is a (semi-) inner-product Hilbert right E-module.

Now suppose that H is a Hilbert right E-module. Then, by the above relation,

$$\|(\xi,\eta)\|^2 = \|(\langle\xi|\xi\rangle + \langle\eta|\eta\rangle, \langle\eta|\xi\rangle - \langle\xi|\eta\rangle)\| \leq$$

$$\leq \|\langle\xi|\xi\rangle\| + \|\langle\eta|\eta\|\rangle + \|\langle\eta|\xi\rangle\| + \|\langle\xi|\eta\rangle\| \leq$$

$$\leq \|\xi\|^2 + \|\eta\|^2 + 2\|\xi\|\,\|\eta\| = (\|\xi\| + \|\eta\|)^2$$

(Proposition 5.6.1.2 d)). By Proposition 4.3.6.1 c),

$$\sup\{\|\xi\|^2, \|\eta\|^2\} \leq \|\langle\xi|\xi\rangle + \langle\eta|\eta\rangle\| \leq \|(\xi,\eta)\|^2 .$$

It follows that the canonical map $H \times H \to \overset{\circ}{H}$ is an isomorphism of real Banach spaces and $\overset{\circ}{H}$ is a Hilbert right E-module. ■

Definition 5.6.1.7 (0) *Let G, H be Hilbert right E-modules. We define $\mathcal{L}_E(G,H)$ to be the set of maps $u : G \to H$ for which there is a map $u^* : H \to G$ such that*

$$\langle u\xi|\eta\rangle = \langle\xi|u^*\eta\rangle$$

for all $(\xi,\eta) \in G \times H$. The elements of $\mathcal{L}_E(G,H)$ are called ***adjointable.*** *We set*

$$\mathcal{L}_E(H) := \mathcal{L}_E(H,H) .$$

G and H are called ***isomorphic*** *if there is a bijective $u \in \mathcal{L}_E(G,H)$ such that*

$$\langle u\xi|u\eta\rangle = \langle\xi|\eta\rangle$$

for all $\xi, \eta \in G$.

Let F be a C^*-subalgebra of E such that

$$\langle\xi|\eta\rangle \in F$$

for all $\xi, \eta \in G$ and $\xi, \eta \in H$. Then G, H are Hilbert right F-modules and

$$\mathcal{L}_F(G,H) = \mathcal{L}_E(G,H) .$$

Proposition 5.6.1.8 (0) *Let F, G, H be Hilbert right E-modules.*

a) *If $u \in \mathcal{L}_E(G,H)$, then there is a unique map $u^* : H \to G$ such that*

$$\langle u\xi|\eta\rangle = \langle\xi|u^*\eta\rangle$$

for all $(\xi,\eta) \in G \times H$. u^* is called ***the adjoint of*** *u. It belongs to $\mathcal{L}_E(H,G)$ and $u^{**} = u$.*

b) *$\mathcal{L}_E(G,H)$ is a closed vector subspace of $\mathcal{L}(G,H)$ and $\|u\| = \|u^*\|$ for every $u \in \mathcal{L}_E(G,H)$.*

c) *The map*

$$\mathcal{L}_E(G,H) \longrightarrow \mathcal{L}_E(H,G), \quad u \longmapsto u^*$$

is conjugate linear.

d) $u \in \mathcal{L}_E(G,H)$, $x \in E$, $\xi \in G \Rightarrow u(\xi x) = (u\xi)x$.

e) $u \in \mathcal{L}_E(F,G)$, $v \in \mathcal{L}_E(G,H) \Rightarrow v \circ u \in \mathcal{L}_E(F,H)$, $(v \circ u)^* = u^* \circ v^*$.

f) $u \in \mathcal{L}_E(G,H) \Rightarrow \|u^* \circ u\| = \|u\|^2 = \|u \circ u^*\|$.

g) *Assume that F and G are (unital) Hilbert E-modules and for $x \in E$ and $u \in \mathcal{L}_E(F,G)$ put*

$$\begin{cases} xu : F \longrightarrow G, & \xi \longmapsto x(u\xi) \\ ux : F \longrightarrow G, & \xi \longmapsto u(x\xi) \end{cases}$$

Then $xu, ux \in \mathcal{L}_E(F,G)$ and

$$(xu)^* = u^*x^*, \quad (ux)^* = x^*u^*$$

for every $(x,u) \in E \times \mathcal{L}_E(F,G)$, and $\mathcal{L}_E(F,G)$ endowed with the maps

$$\begin{cases} E \times \mathcal{L}_E(F,G) \longrightarrow \mathcal{L}_E(F,G), & (x,u) \longmapsto xu \\ \mathcal{L}_E(F,G) \times E \longrightarrow \mathcal{L}_E(F,G), & (u,x) \longmapsto ux \end{cases}$$

is a (unital) E-module. In particular $\mathcal{L}_E(F)$ is an involutive (unital) E-algebra with respect to the involution

$$\mathcal{L}_E(F) \longrightarrow \mathcal{L}_E(F), \quad u \longmapsto u^*.$$

h) *Assume that H is a Hilbert (right) E-module and put*

$$\widetilde{x} : H \longrightarrow H, \quad \xi \longmapsto x\xi$$

$$(\widetilde{x} : H \longrightarrow H, \quad \xi \longmapsto \xi x)$$

for every $x \in E$. Then

$$\widetilde{x} \in \mathcal{L}_E(H), \quad \widetilde{x}^* = \widetilde{x^*}$$

for every $x \in E$ ($x \in E^c$) and the map

$$E \text{ (resp. } E^c) \longrightarrow \mathcal{L}_E(H), \quad x \longmapsto \widetilde{x}$$

is an involutive algebra homomorphism (c) and e)).

a) Take $v : H \to G$ with

$$\langle u\xi|\eta\rangle = \langle \xi|v\eta\rangle$$

for all $(\xi, \eta) \in G \times H$. Then for all $\eta \in H$,

$$\|u^*\eta - v\eta\|^2 = \langle u^*\eta - v\eta|u^*\eta - v\eta\rangle =$$

$$\langle u(u^*\eta - v\eta)|\eta\rangle - \langle u(u^*\eta - v\eta)|\eta\rangle = 0\,,$$

$$u^*\eta = v\eta\,,$$

so $u^* = v$.

For all $(\xi, \eta) \in G \times H$,

$$\langle u^*\eta|\xi\rangle = \langle \xi|u^*\eta\rangle^* = \langle u\xi|\eta\rangle^* = \langle \eta|u\xi\rangle\,.$$

Hence $u^* \in \mathcal{L}_E(H, G)$ and $u^{**} = u$.

b) Take $u \in \mathcal{L}_E(G, H)$, $\alpha, \beta \in \mathbb{K}$, $\xi, \eta \in G$, and $\zeta \in H$. Then

$$\langle u(\alpha\xi + \beta\eta)|\zeta\rangle = \langle \alpha\xi + \beta\eta|u^*\zeta\rangle = \alpha\langle \xi|u^*\zeta\rangle + \beta\langle \eta|u^*\zeta\rangle =$$

$$= \alpha\langle u\xi|\zeta\rangle + \beta\langle u\eta|\zeta\rangle = \langle \alpha u\xi + \beta u\eta|\zeta\rangle\,.$$

Since ζ is arbitrary, we get

$$u(\alpha\xi + \beta\eta) = \alpha u\xi + \beta u\eta\,,$$

i.e. u is linear.

For $\eta \in H^{\#}$, put

$$v_\eta : G \longrightarrow E\,, \quad \xi \longmapsto \langle u\xi|\eta\rangle\,.$$

v_η is linear and continuous and

$$\sup_{\eta \in H^{\#}} \|v_\eta\xi\| \leq \|u\xi\|$$

for all $\xi \in G$ (Proposition 5.6.1.2 d)). By the Theorem of Banach–Steinhaus,

$$\alpha := \sup_{\eta \in H^{\#}} \|v_\eta\| < \infty\,.$$

We get

$$\|\langle u\xi|\eta\rangle\| = \|v_\eta\xi\| \leq \alpha\|\xi\|$$

for all $\xi \in G$ and $\eta \in H^\#$. It follows that

$$\|u\xi\|^2 = \|\langle u\xi|u\xi\rangle\| \leq \alpha\|u\xi\|\,\|\xi\|\,,$$

$$\|u\xi\| \leq \alpha\|\xi\|\,,$$

for all $\xi \in G$, and so $u \in \mathcal{L}(G,H)$. From

$$\|\langle\xi|u^*\eta\rangle\| = \|\langle u\xi|\eta\rangle\| \leq \|u\xi\|\,\|\eta\| \leq \|u\|\,\|\xi\|\,\|\eta\|$$

for all $(\xi,\eta) \in G \times H$ (Proposition 5.6.1.2 d)), we get

$$\|u^*\eta\|^2 = \|\langle u^*\eta|u^*\eta\rangle\| \leq \|u\|\,\|u^*\eta\|\,\|\eta\|\,,$$

$$\|u^*\eta\| \leq \|u\|\,\|\eta\|$$

for all $\eta \in H$, so that $\|u^*\| \leq \|u\|$. By a), $\|u^*\| = \|u\|$.

It is easy to see that $\mathcal{L}_E(G,H)$ is a vector subspace of $\mathcal{L}(G,H)$. Take $u \in \mathcal{L}(G,H)$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}_E(G,H)$ converging to u. Then $(u_n^*)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(H,G)$, so it converges to a $v \in \mathcal{L}(H,G)$. We get

$$\langle u\xi|\eta\rangle = \lim_{n\to\infty}\langle u_n\xi|\eta\rangle = \lim_{n\to\infty}\langle\xi|u_n^*\eta\rangle = \langle\xi|v\eta\rangle$$

for all $(\xi,\eta) \in G \times H$ (Proposition 5.6.1.2 d)). Hence $u \in \mathcal{L}_E(G,H)$ and $\mathcal{L}_E(G,H)$ is a closed vector subspace of $\mathcal{L}(G,H)$.

c) Take $\alpha,\beta \in \mathbb{K}$ and $u,v \in \mathcal{L}_E(G,H)$. Then for all $(\xi,\eta) \in G \times H$,

$$\langle(\alpha u + \beta v)\xi|\eta\rangle = \alpha\langle u\xi|\eta\rangle + \beta\langle v\xi|\eta\rangle = \alpha\langle\xi|u^*\eta\rangle + \beta\langle\xi|v^*\eta\rangle =$$

$$= \langle\xi|\overline{\alpha}u^*\eta + \overline{\beta}v^*\eta\rangle = \langle\xi|(\overline{\alpha}u^* + \overline{\beta}v^*)\eta\rangle\,,$$

so that

$$(\alpha u + \beta v)^* = \overline{\alpha}u^* + \overline{\beta}v^*\,.$$

d) For all $\eta \in H$,

$$\langle u(\xi x)|\eta\rangle = \langle\xi x|u^*\eta\rangle = \langle\xi|u^*\eta\rangle x = \langle u\xi|\eta\rangle x = \langle(u\xi)x|\eta\rangle\,,$$

so that

$$u(\xi x) = (u\xi)x\,.$$

e) For all $(\xi, \eta) \in F \times H$,

$$\langle (v \circ u)\xi | \eta \rangle = \langle v(u\xi) | \eta \rangle = \langle u\xi | v^*\eta \rangle = \langle \xi | u^*(v^*\eta) \rangle = \langle \xi | (u^* \circ v^*)\eta \rangle\,,$$

so that $v \circ u \in \mathcal{L}_E(F, H)$ and

$$(v \circ u)^* = u^* \circ v^*\,.$$

f) We have

$$\|u\xi\|^2 = \|\langle u\xi | u\xi \rangle\| = \|\langle u^*u\xi | \xi \rangle\| \leq \|u^*u\xi\|\,\|\xi\| \leq \|u^* \circ u\|\,\|\xi\|^2\,,$$

for every $\xi \in G$ (Proposition 5.6.1.2 d)), so

$$\|u\|^2 \leq \|u^* \circ u\|\,.$$

Since

$$\|u^* \circ u\| \leq \|u^*\|\,\|u\|\,,$$

we get by b),

$$\|u\|^2 = \|u^* \circ u\|\,.$$

It follows by a) and b),

$$\|u \circ u^*\| = \|u^{**} \circ u^*\| = \|u^*\|^2 = \|u\|^2\,.$$

g) For $(\xi, \eta) \in F \times G$,

$$\langle (xu)\xi | \eta \rangle = \langle x(u\xi) | \eta \rangle = \langle u\xi | x^*\eta \rangle = \langle \xi | u^*(x^*\eta) \rangle = \langle \xi | (u^*x^*)\eta \rangle\,,$$

$$\langle (ux)\xi | \eta \rangle = \langle u(x\xi) | \eta \rangle = \langle x\xi | u^*\eta \rangle = \langle \xi | x^*(u^*\eta) \rangle = \langle \xi | (x^*u^*)\eta \rangle$$

so that xu, $ux \in \mathcal{L}_E(F, G)$ and

$$(xu)^* = u^*x^*\,, \quad (ux)^* = x^*u^*\,.$$

Take $x, y \in E$ and $u \in \mathcal{L}_E(F, G)$. Then for any $\xi \in F$,

$$((xy)u)\xi = (xy)(u\xi) = x(y(u\xi)) = x((yu)\xi) = (x(yu))\xi\,,$$

$$(u(xy))\xi = u((xy)\xi) = u(x(y\xi)) = (ux)(y\xi) = ((ux)y)\xi\,,$$

$$((xu)y)\xi = (xu)(y\xi) = x(u(y\xi)) = x((uy)\xi) = (x(uy))\xi\,,$$

$$\|(xu)\xi\| = \|x(u\xi)\| \le \|x\|\,\|u\xi\| \le \|x\|\,\|u\|\,\|\xi\|\,,$$

$$\|(ux)\xi\| = \|u(x\xi)\| \le \|u\|\,\|x\xi\| \le \|u\|\,\|x\|\,\|\xi\|$$

$$((1u)\xi = 1(u\xi) = u\xi\,, \quad (u1)\xi = u(1\xi) = u\xi)\,,$$

so that

$$(xy)u = x(yu)\,, \quad u(xy) = (ux)y\,, \quad (xu)y = x(uy)\,,$$

$$\|xu\| \le \|x\|\,\|u\|\,, \quad \|ux\| \le \|u\|\,\|x\|$$

$$(1u = u1 = u)\,.$$

Since the above maps are bilinear, $\mathcal{L}_E(F,G)$ is a (unital) E-module.

Similarly it can be proved that

$$x(uv) = (xu)v\,, \quad (uv)x = u(vx)\,, \quad (ux)v = u(xv)$$

for all $x \in E$ and $u, v \in \mathcal{L}_E(F)$.

h) For $\xi, \eta \in H$,

$$\langle \widetilde{x}\xi|\eta\rangle = \langle x\xi|\eta\rangle = \langle \xi|x^*\eta\rangle = \langle \xi|\widetilde{x^*}\eta\rangle$$

$$(\langle \widetilde{x}\xi|\eta\rangle = \langle \xi x|\eta\rangle = \langle \xi|\eta\rangle x = x\langle \xi|\eta\rangle = \langle \xi|\eta x^*\rangle = \langle \xi|\widetilde{x^*}\eta\rangle)$$

so that

$$\widetilde{x} \in \mathcal{L}_E(H)\,, \quad \widetilde{x}^* = \widetilde{x^*}\,.$$

Take $x, y \in E$ (resp. E^c). For every $\xi \in H$,

$$\widetilde{xy}\xi = (xy)\xi = x(y\xi) = \widetilde{x}(\widetilde{y}\xi) = \widetilde{x}\widetilde{y}\xi$$

$$(\widetilde{xy}\xi = \xi(xy) = \xi(yx) = (\xi y)x = \widetilde{x}(\widetilde{y}\xi) = \widetilde{x}\widetilde{y}\xi)\,,$$

so that

$$\widetilde{xy} = \widetilde{x}\widetilde{y}\,.$$

Hence the map

$$E \text{ (resp. } E^c\text{)} \longrightarrow \mathcal{L}_E(H)\,, \quad x \longmapsto \widetilde{x}$$

is an algebra homomorphism. By the above, this map is involutive. ■

Remark. The inclusion map $c_0 \to \ell^\infty$ does not belong to $\mathcal{L}_{\ell^\infty}(c_0, \ell^\infty)$ (Example 5.6.1.5).

Proposition 5.6.1.9 (0) *Assume* $\mathbb{K} = \mathbb{R}$. *Let* G, H *be Hilbert right* E*-modules,* $\overset{\circ}{G}, \overset{\circ}{H}$ *the complexification of* G *and* H, *respectively (Proposition 5.6.1.6) and* $\overset{\circ}{\mathcal{L}}(G,H)$, $\overset{\circ}{\mathcal{L}}(H,G)$ *the complexification of the underlying real vector spaces* $\mathcal{L}(G,H)$ *and* $\mathcal{L}(H,G)$, *respectively. Given* $(u,v) \in \overset{\circ}{\mathcal{L}}(G,H)$ *(resp.* $(u,v) \in \overset{\circ}{\mathcal{L}}(H,G)$*), define*

$$\widetilde{(u,v)} : \overset{\circ}{G} \longrightarrow \overset{\circ}{H}\,, \quad (\xi,\eta) \longmapsto (u\xi - v\eta, u\eta + v\xi)$$

$$\textit{(resp. } \widetilde{(u,v)} : \overset{\circ}{H} \longrightarrow \overset{\circ}{G}\,, \quad (\xi,\eta) \longmapsto (u\xi - v\eta, u\eta + v\xi))\,.$$

Let φ *be the isomorphism of complex vector spaces*

$$\varphi : \overset{\circ}{\mathcal{L}}(G,H) \longrightarrow \mathcal{L}(\overset{\circ}{G}, \overset{\circ}{H})\,, \quad (u,v) \longmapsto \widetilde{(u,v)}$$

(Proposition 2.1.5.6). Then

$$\varphi(\overset{\circ}{\mathcal{L}}_E(G,H)) = \mathcal{L}_{\overset{\circ}{E}}(\overset{\circ}{G}, \overset{\circ}{H})$$

and

$$\widetilde{(u,v)}^* = \widetilde{(u^*, -v^*)}$$

for all $(u,v) \in \overset{\circ}{\mathcal{L}}_E(G,H)$ *(hence* $\mathcal{L}_{\overset{\circ}{E}}(\overset{\circ}{G}, \overset{\circ}{H})$ *may be identified with the complexification of* $\mathcal{L}_E(G,H)$*).*

Take $(u,v) \in \overset{\circ}{\mathcal{L}}(G,H)$.

Step 1 $\quad (u,v) \in \overset{\circ}{\mathcal{L}}_E(G,H) \Rightarrow \begin{cases} \widetilde{(u,v)} \in \mathcal{L}_{\overset{\circ}{E}}(\overset{\circ}{G}, \overset{\circ}{H})\,, \\ \widetilde{(u,v)}^* = \widetilde{(u^*, -v^*)} \end{cases}$

Take $(\xi,\eta) \in \overset{\circ}{G}$ and $(\zeta,\lambda) \in \overset{\circ}{H}$. We have

$$\langle \widetilde{(u,v)}(\xi,\eta)|(\zeta,\lambda)\rangle = \langle (u\xi - v\eta, u\eta + v\xi)|(\zeta,\lambda)\rangle =$$

$$= (\langle u\xi - v\eta|\zeta\rangle + \langle u\eta + v\xi|\lambda\rangle, \langle u\eta + v\xi|\zeta\rangle - \langle u\xi - v\eta|\lambda\rangle) =$$

$$= \begin{pmatrix} \langle \xi|u^*\zeta\rangle - \langle \eta|v^*\zeta\rangle + \langle \eta|u^*\lambda\rangle + \langle \xi|v^*\lambda\rangle, \\ \langle \eta|u^*\zeta\rangle + \langle \xi|v^*\zeta\rangle - \langle \xi|u^*\lambda\rangle + \langle \eta|v^*\lambda\rangle \end{pmatrix} =$$

$$= (\langle \xi|u^*\zeta + v^*\lambda\rangle + \langle \eta|u^*\lambda - v^*\zeta\rangle, \langle \eta|u^*\zeta + v^*\lambda\rangle - \langle \xi|u^*\lambda - v^*\zeta\rangle) =$$

$$= \langle (\xi,\eta)|(u^*\zeta + v^*\lambda, u^*\lambda - v^*\zeta)\rangle =$$

$$= \langle (\xi,\eta)|\widetilde{(u^*,-v^*)}(\zeta,\lambda)\rangle .$$

Hence $\widetilde{(u,v)} \in \mathcal{L}_{\overset{\circ}{E}}(\overset{\circ}{G},\overset{\circ}{H})$ and

$$\widetilde{(u,v)}^* = \widetilde{(u^*,-v^*)} .$$

Step 2 $\quad \widetilde{(u,v)} \in \mathcal{L}_{\overset{\circ}{E}}(\overset{\circ}{G},\overset{\circ}{H}) \Rightarrow (u,v) \in \overset{\circ}{\mathcal{L}}_E(G,H)$

Put

$$\widetilde{(u_0,v_0)} := \widetilde{(u,v)}^* .$$

For $(\xi,\eta) \in G \times H$,

$$(\langle u\xi|\eta\rangle , \langle v\xi|\eta\rangle) = \langle (u\xi, v\xi)|(\eta,0)\rangle = \langle \widetilde{(u,v)}(\xi,0)|(\eta,0)\rangle =$$

$$= \langle (\xi,0)|\widetilde{(u_0,v_0)}(\eta,0)\rangle = \langle (\xi,0)|(u_0\eta, v_0\eta)\rangle =$$

$$= (\langle \xi|u_0\eta\rangle, -\langle \xi|v_0\eta\rangle) ,$$

$$\langle u\xi|\eta\rangle = \langle \xi|u_0\eta\rangle , \quad \langle v\xi|\eta\rangle = \langle \xi| - v_0\eta\rangle .$$

Hence

$$u, v \in \mathcal{L}_E(G,H) .$$

■

Definition 5.6.1.10 (0) *A **(unital)** E-C^*-**algebra** is a (unital) involutive E-algebra which is at the same time a C^*-algebra. An **isomorphism of** E-C^*-**algebras** is an isomorphism of C^*-algebras which is at the same time an isometry of involutive E-modules.*

Theorem 5.6.1.11 (0) *Let H be a Hilbert right E-module.*

a) *$\mathcal{L}_E(H)$ endowed with the Banach algebra structure induced by $\mathcal{L}(H)$ and with the involution*

$$\mathcal{L}_E(H) \longrightarrow \mathcal{L}_E(H), \quad u \longmapsto u^*$$

(Proposition 5.6.1.8 a),b)) is a C^*-algebra.

b) *Let $\mathbb{K} = \mathbb{C}$ and $u \in \mathcal{L}_E(H)$ (let $\mathbb{K} = \mathbb{R}$ and $u \in \operatorname{Re}\mathcal{L}_E(H)$). If*

$$\langle u\xi|\xi\rangle \in \operatorname{Re} E \qquad (\langle u\xi|\xi\rangle) = 0)$$

for every $\xi \in H$, then $u \in \operatorname{Re}\mathcal{L}_E(H)$ $(u = 0)$.

c) *If $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$) then the following are equivalent for every $u \in \mathcal{L}_E(H)$ ($u \in \operatorname{Re}\mathcal{L}_E(H)$):*

c_1) $u \in \mathcal{L}_E(H)_+$.

c_2) $\xi \in H \Rightarrow \langle u\xi|\xi\rangle \in E_+$.

If these conditions are fulfilled then

$$\|u\| = \sup\{\|\langle u\xi|\xi\rangle\| | \xi \in H^\#\}$$

and

$$\langle u\xi|\xi\rangle \leq \|u\|\langle\xi|\xi\rangle$$

for every $\xi \in H$.

d) *If H is a Hilbert E-module then $\mathcal{L}_E(H)$ is an E-C^*-algebra (Proposition 5.6.1.8 g)).*

e) *The following are equivalent for all $p, q \in \operatorname{Pr}\mathcal{L}_E(H)$:*

e_1) $p \leq q$.

e_2) $\operatorname{Im} p \subset \operatorname{Im} q$.

f) *The following are equivalent for every* $p \in \mathcal{L}_E(H)$ *with* $p^2 = p$*:*

f_1) $p \in \Pr \mathcal{L}_E(H)$.

f_2) $(\xi, \eta) \in \operatorname{Ker} p \times \operatorname{Im} p \Rightarrow \langle \xi | \eta \rangle = 0$.

g) *The following are equivalent for all* $p, q \in \Pr \mathcal{L}_E(H)$*:*

g_1) $pq = 0$.

g_2) $(\xi, \eta) \in \operatorname{Im} p \times \operatorname{Im} q \Rightarrow \langle \xi | \eta \rangle = 0$.

a) In the complex case, the assertion follows from Proposition 5.6.1.8 a),b),c),e),f). In the real case, it follows from Proposition 5.6.1.9 that the complexification of $\mathcal{L}_E(H)$ is the C^*-algebra $\mathcal{L}_{\overset{\circ}{E}}(\overset{\circ}{H})$, so $\mathcal{L}_E(H)$ is a real C^*-algebra.

b) The map

$$H \times H \longrightarrow E, \quad (\xi, \eta) \longmapsto \langle u\xi | \eta \rangle$$

is sesquilinear. By Proposition 2.3.3.8 b $\Rightarrow$ a (by Proposition 2.3.3.7 a $\Rightarrow$ b),

$$u \in \operatorname{Re} \mathcal{L}_E(H)$$

$$(\xi, \eta \in H \Longrightarrow \langle u\xi | \eta \rangle = 0$$

which implies

$$\|u\xi\|^2 = \|\langle u\xi | u\xi \rangle\| = 0,$$

$$u\xi = 0$$

for every $\xi = H$, i.e. $u = 0$).

$c_1 \Rightarrow c_2$. $\langle u\xi | \xi \rangle = \langle u^{\frac{1}{2}}\xi | u^{\frac{1}{2}}\xi \rangle \in E_+$.

$c_2 \Rightarrow c_1$. If $\mathbb{K} = \mathbb{C}$ then

$$\langle u\xi | \xi \rangle = \langle u\xi | \xi \rangle^* = \langle \xi | u\xi \rangle = \langle u^*\xi | \xi \rangle.$$

By the Polarization Identity (Proposition 2.3.3.2 d)),

$$\langle u\xi | \eta \rangle = \frac{1}{4} \sum_{k=1}^{4} i^k \langle u(\xi + i^k\eta) | \xi + i^k\eta \rangle =$$

$$= \frac{1}{4}\sum_{k=1}^{4} i^k \langle u^*(\xi + i^k\eta)|\xi + i^k\eta\rangle = \langle u^*\xi|\eta\rangle$$

for all $\xi, \eta \in H$, so that $u = u^*$, i.e. $u \in \mathrm{Re}\,\mathcal{L}_E(H)$.

It follows (for $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$),

$$\langle u^+\xi|\xi\rangle - \langle u^-\xi|\xi\rangle = \langle u\xi|\xi\rangle \geq 0$$

(a) and Theorem 4.2.2.9 a)),

$$\langle u^-\xi|\xi\rangle \leq \langle u^+\xi|\xi\rangle .$$

Hence

$$\langle (u^-)^3\xi|\xi\rangle = \langle (u^-)^2\xi|u^-\xi\rangle \leq \langle u^+u^-\xi|u^-\xi\rangle = 0 .$$

By $c_1 \Rightarrow c_2$,

$$\langle (u^-)^3\xi|\xi\rangle = 0$$

so that by b),

$$u^- = 0 ,$$

$$u = u^+ \in \mathcal{L}_E(H)_+ .$$

Now assume that u is positive. Then

$$\sup\{\|\langle u\xi|\xi\rangle\| | \xi \in H^\#\} =$$

$$= \sup\{\|\langle u^{\frac{1}{2}}\xi|u^{\frac{1}{2}}\xi\rangle\| | \xi \in H^\#\} =$$

$$= \sup\{\|u^{\frac{1}{2}}\xi\|^2 | \xi \in H^\#\} = \|u^{\frac{1}{2}}\|^2 = \|u\| .$$

From c_2) and $u \leq \|u\|1$ (Corollary 4.2.1.17 a $\Rightarrow$ b) it follows

$$\|u\|\langle\xi|\xi\rangle - \langle u\xi|\xi\rangle = \langle(\|u\|1 - u)\xi|\xi\rangle \in E_+ ,$$

$$\langle u\xi|\xi\rangle \leq \|u\|\langle\xi|\xi\rangle .$$

d) follows from a) and Proposition 5.6.1.8 g).

$e_1 \Rightarrow e_2$. For every $\xi \in H$,

$$p\xi = qp\xi$$

(Corollary 4.2.7.6 a $\Rightarrow$ b), so that

$$\operatorname{Im} p \subset \operatorname{Im} q\,.$$

$e_2 \Rightarrow e_1$. For every $\xi \in H$,

$$p\xi \in \operatorname{Im} p \subset \operatorname{Im} q$$

so that

$$p\xi = qp\xi\,.$$

Hence

$$p = qp\,,$$

$$p \leq q$$

(Corollary 4.2.7.6 d $\Rightarrow$ a).

$f_1 \Rightarrow f_2$. We have

$$\langle \xi | \eta \rangle = \langle \xi | p\eta \rangle = \langle p\xi | \eta \rangle = 0\,.$$

$f_2 \Rightarrow f_1$. For all $\xi, \eta \in H$,

$$p\xi - \xi\,,\ \eta - p\eta \in \operatorname{Ker} p$$

so that

$$\langle p\xi | \eta \rangle = \langle p\xi | \eta - p\eta \rangle + \langle p\xi | p\eta \rangle = \langle p\xi | p\eta \rangle =$$

$$= \langle p\xi - \xi | p\eta \rangle + \langle \xi | p\eta \rangle = \langle \xi | p\eta \rangle\,.$$

Hence $p = p^*$ and $p \in \operatorname{Pr} \mathcal{L}_E(H)$.

$g_1 \Rightarrow g_2$. Since

$$\xi = p\xi\,, \quad \eta = q\eta\,,$$

it follows

$$\langle \xi | \eta \rangle = \langle p\xi | q\eta \rangle = \langle \xi | pq\eta \rangle = 0\,.$$

$g_2 \Rightarrow g_1$. For every $\xi \in H$,

$$\langle pq\xi | pq\xi \rangle = \langle q\xi | p^*pq\xi \rangle = \langle q\xi | pq\xi \rangle = 0$$

so that

$$pq\xi = 0 .$$

Hence

$$pq = 0 . \qquad \blacksquare$$

Corollary 5.6.1.12 (0) *Let G, H be Hilbert right E-modules. Then for every $u \in \mathcal{L}_E(G,H)$ and $v \in \mathcal{L}_E(H)_+$,*

$$u^* \circ v \circ u \, , \, u^* \circ u \in \mathcal{L}_E(G)_+ .$$

In particular,

$$\langle u\xi | u\xi \rangle \leq \|u\|^2 \langle \xi | \xi \rangle$$

for every $\xi \in G$.

For every $\xi \in G$,

$$\langle u^*u\xi | \xi \rangle = \langle u\xi | u\xi \rangle \in E_+$$

so that

$$u^* \circ u \in \mathcal{L}_E(G)_+$$

by Theorem 5.6.1.11, $c_2 \Rightarrow c_1$ (and Proposition 5.6.1.8 a),e)). It follows

$$u^* \circ v \circ u = u^* \circ v^{\frac{1}{2}} \circ v^{\frac{1}{2}} \circ u = (v^{\frac{1}{2}} \circ u)^* \circ (v^{\frac{1}{2}} \circ u) \in \mathcal{L}_E(G)_+$$

(Proposition 5.6.1.8 e)).

By Theorem 5.6.1.11 c) and Proposition 5.6.1.8 f),

$$\langle u\xi | u\xi \rangle = \langle (u^* \circ u)\xi | \xi \rangle \leq \|u^* \circ u\| \langle \xi | \xi \rangle = \|u\|^2 \langle \xi | \xi \rangle . \qquad \blacksquare$$

Proposition 5.6.1.13 *Let H be a Hilbert right E-module. For every $x \in E$, define*

$$\widetilde{x} : H \longrightarrow H , \quad \xi \longmapsto \xi x .$$

a) *The following are equivalent for all $x, y \in E$:*

$a_1)$ $\xi, \eta \in H \Rightarrow \langle \xi|\eta\rangle x = y\langle \xi|\eta\rangle$.

$a_2)$ $\widetilde{x} \in \mathcal{L}_E(H)$, $\widetilde{x}^* = \widetilde{y}^*$.

b) *The following are equivalent for every* $x \in E$ *:*

$b_1)$ $x \in \{\langle \xi|\eta\rangle | \xi, \eta \in H\}^c$.

$b_2)$ $\widetilde{x} \in \mathcal{L}_E(H)$, $\widetilde{x}^* = \widetilde{x^*}$.

a) For all $\xi, \eta \in H$,

$$\langle \widetilde{x}\xi|\eta\rangle = \langle \xi x|\eta\rangle = \langle \xi|\eta\rangle x ,$$

$$\langle \xi|\widetilde{y^*}\eta\rangle = \langle \xi|\eta y^*\rangle = y\langle \xi|\eta\rangle$$

(Proposition 5.6.1.2 a)). $a_1 \Leftrightarrow a_2$ follows.

b) follows from a). ■

Example 5.6.1.14 *Let* I *be a finite set,* $a \in (E\backslash\{0\})^I$ *, and* $b \in (E^\#)^I$ *. Then* E^I *endowed with the right and left multiplications*

$$E^I \times E \longrightarrow E^I , \quad (\xi, x) \longmapsto (\xi_\iota x)_{\iota \in I} ,$$

$$E \times E^I \longrightarrow E^I , \quad (x, \xi) \longmapsto (x\xi_\iota)_{\iota \in I} ,$$

and with the inner-product

$$E^I \times E^I \longrightarrow E , \quad (\xi, \eta) \longmapsto \sum_{\iota \in I} \eta_\iota^* a_\iota^* a_\iota \xi_\iota + \sum_{\substack{\iota, \lambda \in I \\ \iota \neq \lambda}} \eta_\iota^* a_\iota^* b_\iota^* b_\lambda a_\lambda \xi_\lambda$$

is a Hilbert E*-module.*

First we remark that

$$\langle \xi|\eta\rangle = \sum_{\iota \in I} \eta_\iota^* (a_\iota^* a_\iota - a_\iota^* b_\iota^* b_\iota a_\iota)\xi_\iota + \sum_{\iota, \lambda \in I} \eta_\iota^* a_\iota^* b_\iota^* b_\lambda a_\lambda \xi_\lambda$$

for every $\xi, \eta \in E^I$. By Corollary 4.2.2.3,

$$a_\iota^* b_\iota^* b_\iota a_\iota \leq a_\iota^* a_\iota , \quad \xi_\iota^* a_\iota^* b_\iota^* b_\iota a_\iota \xi_\iota \leq \xi_\iota^* a_\iota^* a_\iota \xi_\iota$$

for every $\xi \in E^I$ and $\iota \in I$. Put

$$c_\iota := (a_\iota^* a_\iota - a_\iota^* b_\iota^* b_\iota a_\iota)^{\frac{1}{2}}$$

for every $\iota \in I$.

By the above, for $\xi \in E^I$,

$$\langle \xi | \xi \rangle \geq \sum_{\iota,\lambda \in I} \xi_\iota^* a_\iota^* b_\iota^* b_\lambda a_\lambda \xi_\lambda = \left(\sum_{\iota \in I} b_\iota a_\iota \xi_\iota \right)^* \left(\sum_{\iota \in I} b_\iota a_\iota \xi_\iota \right) \geq 0 .$$

Take $\xi, \eta \in E^I$. Then

$$(\langle \xi | \xi \rangle + \langle \eta | \eta \rangle , \langle \eta | \xi \rangle - \langle \xi | \eta \rangle) =$$

$$= \left(\sum_{\iota \in I} \xi^* c_\iota^2 \xi_\iota + \sum_{\iota \in I} \eta_\iota^* c_\iota^2 \eta_\iota + \sum_{\iota,\lambda \in I} \xi_\iota^* a_\iota^* b_\iota^* b_\lambda a_\lambda \xi_\lambda + \sum_{\iota,\lambda \in I} \eta_\iota^* a_\iota^* b_\iota^* b_\lambda a_\lambda \eta_\lambda ,\right.$$

$$\left. \sum_{\iota \in I} \xi_\iota^* c_\iota^2 \eta_\iota + \sum_{\iota,\lambda \in I} \xi_\iota^* a_\iota^* b_\iota^* b_\lambda a_\lambda \eta_\lambda - \sum_{\iota \in I} \eta_\iota^* c_\iota^2 \xi_\iota - \sum_{\iota,\lambda \in I} \eta_\iota^* a_\iota^* b_\iota^* b_\lambda a_\lambda \xi_\lambda \right) .$$

We have

$$0 \leq \left(\sum_{\iota \in I} b_\iota a_\iota \xi_\iota , \sum_{\iota \in I} b_\iota a_\iota \eta_\iota \right)^* \left(\sum_{\iota \in I} b_\iota a_\iota \xi_\iota , \sum_{\iota \in I} b_\iota a_\iota \eta_\iota \right) =$$

$$= \left(\sum_{\iota \in I} \xi_\iota^* a_\iota^* b_\iota^* , - \sum_{\iota \in I} \eta_\iota^* a_\iota^* b_\iota^* \right) \left(\sum_{\iota \in I} b_\iota a_\iota \xi_\iota , \sum_{\iota \in I} b_\iota a_\iota \eta_\iota \right) =$$

$$= \begin{pmatrix} \sum_{\iota,\lambda \in I} \xi_\iota^* a_\iota^* b_\iota^* b_\lambda a_\lambda \xi_\lambda + \sum_{\iota,\lambda \in I} \eta_\iota^* a_\iota^* b_\iota^* b_\lambda a_\lambda \eta_\lambda , \\ \sum_{\iota,\lambda \in I} \xi_\iota^* a_\iota^* b_\iota^* b_\lambda a_\lambda \eta_\lambda - \sum_{\iota,\lambda \in I} \eta_\iota^* a_\iota^* b_\iota^* b_\lambda a_\lambda \xi_\lambda \end{pmatrix}$$

and for $\iota \in I$,

$$0 \leq (c_\iota \xi_\iota, c_\iota \eta_\iota)^* (c_\iota \xi_\iota, c_\iota \eta_\iota) = (\xi_\iota^* c_\iota, -\eta_\iota^* c_\iota)(c_\iota \xi_\iota, c_\iota \eta_\iota) =$$

$$= (\xi_\iota^* c_\iota^2 \xi_\iota + \eta_\iota^* c_\iota^2 \eta_\iota, \xi_\iota^* c_\iota^2 \eta_\iota - \eta_\iota^* c_\iota^2 \xi_\iota) .$$

Hence

$$(\langle \xi | \xi \rangle + \langle \eta | \eta \rangle, \langle \eta | \xi \rangle - \langle \xi | \eta \rangle) \in \mathring{E}_+ .$$

The other axioms are easy to verify. ■

5.6.2 Self–duality

Proposition 5.6.2.1 (8) *Let H be a Hilbert right E-module and for every $\xi \in H$ put*

$$\langle\cdot|\xi\rangle : H \longrightarrow E\,, \quad \eta \longmapsto \langle\eta|\xi\rangle\,,$$

$$\widetilde{\xi} : E \longrightarrow H\,, \quad x \longmapsto \xi x\,.$$

a) $\xi \in H \Rightarrow \langle\cdot|\xi\rangle \in \mathcal{L}_E(H,E)$, $\langle\cdot|\xi\rangle^* = \widetilde{\xi}$, $\|\langle\cdot|\xi\rangle\| = \|\xi\|$ *(Example 5.6.1.5).*

b) *If E is unital then the map*

$$H \longrightarrow \mathcal{L}_E(H,E)\,, \quad \xi \longmapsto \langle\cdot|\xi\rangle$$

is conjugate linear and an isometry of real Banach spaces with inverse map

$$\mathcal{L}_E(H,E) \longrightarrow H\,, \quad u \longmapsto u^*1\,.$$

c) *If H is a Hilbert E-module then for every $x \in E$ and $\xi \in H$,*

$$x(\langle\cdot|\xi\rangle) = \langle\cdot|\xi x^*\rangle\,, \quad (\langle\cdot|\xi\rangle)x = \langle\cdot|x^*\xi\rangle$$

(Proposition 5.6.1.8 g), Example 5.6.1.5).

a) By Proposition 5.6.1.2 d),

$$\|(\langle\cdot|\xi\rangle)\eta\| = \|\langle\eta|\xi\rangle\| \leq \|\xi\|\,\|\eta\|$$

for every $\eta \in H$ and therefore $\langle\cdot|\xi\rangle \in \mathcal{L}(H,E)$ and

$$\|\langle\cdot|\xi\rangle\| \leq \|\xi\|\,.$$

From

$$\|\langle\cdot|\xi\rangle\|\,\|\xi\| \geq \|(\langle\cdot|\xi\rangle)\xi\| = \|\langle\xi|\xi\rangle\| = \|\xi\|^2$$

it follows

$$\|\langle\cdot|\xi\rangle\| \geq \|\xi\|\,, \quad \|\langle\cdot|\xi\rangle\| = \|\xi\|\,.$$

For $(\eta, x) \in H \times E$,

$$\langle(\langle\cdot|\xi\rangle)\eta|x\rangle = \langle\langle\eta|\xi\rangle|x\rangle = x^*\langle\eta|\xi\rangle =$$

$$= \langle \eta | \xi x \rangle = \langle \eta | \widetilde{\xi} x \rangle$$

(Example 5.6.1.5, Proposition 5.6.1.2 a)) so that

$$\langle \cdot | \xi \rangle \in \mathcal{L}_E(H, E), \quad \langle \cdot | \xi \rangle^* = \widetilde{\xi}.$$

b) For $u \in \mathcal{L}_E(H, E)$ and $\xi \in H$,

$$\langle \xi | u^* 1 \rangle = \langle u\xi | 1 \rangle = u\xi$$

(Example 5.6.1.5) so that

$$u = \langle \cdot | u^* 1 \rangle .$$

The assertion now follows from a).

c) By Proposition 5.6.1.2 a),

$$(x(\langle \cdot | \xi \rangle))\eta = x\langle \eta | \xi \rangle = \langle \eta | \xi x^* \rangle ,$$

$$((\langle \cdot | \xi \rangle)x)\eta = \langle x\eta | \xi \rangle = \langle \eta | x^* \xi \rangle$$

for every $\eta \in H$ so that

$$x(\langle \cdot | \xi \rangle) = \langle \cdot | \xi x^* \rangle , \quad (\langle \cdot | \xi \rangle)x = \langle \cdot | x^* \xi \rangle . \qquad \blacksquare$$

Remark. If E is not unital then b) may fail. Indeed, if $H := E := c_0$ then $\mathcal{L}_E(H, E)$ is isomorphic to ℓ^∞.

Definition 5.6.2.2 (8) *For G, H innner-product right E-modules define*

$$\widehat{\mathcal{L}}_E(G, H) := \{u \in \mathcal{L}(G, H) | (\xi, x) \in G \times E \Longrightarrow u(\xi x) = (u\xi)x\} ,$$

$$\widehat{H} := \widehat{\mathcal{L}}_E(H, E) .$$

*H is called **self-dual** (Paschke, 1973) if*

$$\{\langle \cdot | \xi \rangle | \xi \in H\} = \widehat{H}$$

(Proposition 5.6.2.1) (this implies that H is complete).

By Proposition 5.6.1.8 d),

$$\mathcal{L}_E(G, H) \subset \widehat{\mathcal{L}}_E(G, H) .$$

If F, G, H are Hilbert right E-modules then $u \in \widehat{\mathcal{L}}_E(F, G)$ and $v \in \widehat{\mathcal{L}}_E(G, H)$ implies

$$v \circ u \in \widehat{\mathcal{L}}_E(F, H) .$$

By the Theorem of Frèchet-Riesz, every Hilbert space is self-dual.

Proposition 5.6.2.3 (8) *Let F be a closed right ideal of E. The inner-product right E-module associated to F (Example 5.6.1.5) is self-dual iff there is a $p \in \Pr E$ such that $F = pE$.*

First assume F is self-dual. Denote by u the inclusion map $F \to E$. Then there is a $p \in F$ such that

$$u = \langle \cdot | p \rangle .$$

We deduce successively

$$p = up = \langle p|p\rangle = p^*p \in \operatorname{Re} E ,$$

$$p^2 = p^*p = p ,$$

$$p \in \Pr E ,$$

$$F = u(F) = pE .$$

Now assume there is a $p \in \Pr E$ with $F = pE$. Take $u \in \widehat{\mathcal{L}}(F, E)$ and put

$$y := p(up)^* \in pE = F .$$

Then for every $x \in F$,

$$\langle x|y\rangle = \langle x|p(up)^*\rangle = (up)px = u(px) = ux$$

so that

$$u = \langle \cdot | y \rangle .$$

Hence F is self-dual. ■

Proposition 5.6.2.4 (8) *Let G, H be Hilbert right E-modules. If G is self-dual then*

$$\mathcal{L}_E(G, H) = \widehat{\mathcal{L}}_E(G, H) .$$

Let $u \in \widehat{\mathcal{L}}_E(G, H)$. Take $\eta \in H$. Then for all $(\xi, x) \in G \times E$,

$$(\langle \cdot | \eta \rangle \circ u)(\xi x) = (\langle \cdot | \eta \rangle) u(\xi x) = (\langle \cdot | \eta \rangle)(u\xi)x =$$

$$= \langle (u\xi)x|\eta\rangle = \langle u\xi|\eta\rangle x = ((\langle\cdot|\eta\rangle \circ u)\xi)x\,.$$

Since G is self-dual, there is an element $v\eta \in G$ such that

$$\langle\cdot|\eta\rangle \circ u = \langle\cdot|v\eta\rangle\,.$$

It follows

$$\langle u\xi|\eta\rangle = \langle\xi|v\eta\rangle$$

for every $\xi \in G$, i.e. $u \in \mathcal{L}_E(G,H)$. Hence $\widehat{\mathcal{L}}_E(G,H) \subset \mathcal{L}_E(G,H)$.

The reverse inclusion follows from Proposition 5.6.1.8 d). ■

Remark. If G is not self-dual then the above equality may fail. Indeed, the inclusion map $c_0 \to \ell^\infty$ belongs to $\widehat{\mathcal{L}}_{\ell^\infty}(c_0,\ell^\infty)$ but not to $\mathcal{L}_{\ell^\infty}(c_0,\ell^\infty)$.

Proposition 5.6.2.5 (8) *Let F be a C^*-subalgebra of E and $\varphi : E \to F$ an involutive continuous linear map such that:*

1) $(x,y) \in E \times F \Rightarrow \varphi(xy) = (\varphi x)y$.
2) $x \in E_+ \Rightarrow \varphi x \in F_+$.
3) $(x,y) \in \overset{\circ}{E}_+ \Rightarrow (\varphi x, \varphi y) \in \overset{\circ}{F}_+$.
4) $0 < \inf\{\|\varphi x\| \,|\, x \in E_+\,,\ \|x\| = 1\}$.

a) *If H is a Hilbert right E-module, then H endowed with the right multiplication*

$$H \times F \longrightarrow H\,, \quad (\xi,y) \longmapsto \xi y$$

and with the inner product

$$H \times H \longrightarrow F\,, \quad (\xi,\eta) \longmapsto \varphi\langle\xi|\eta\rangle$$

is a Hilbert right F-module, which will be denoted in the sequel by $\widetilde{H}$. The norms of H and $\widetilde{H}$ are equivalent so that $\widetilde{H}$ is reflexive iff H is reflexive (Proposition 1.3.8.8).

b) *If G and H are Hilbert right E-modules then*

$$\mathcal{L}_E(G,H) = \mathcal{L}_F(\widetilde{G},\widetilde{H}) \cap \widehat{\mathcal{L}}_E(G,H)$$

and the adjoint defined in $\mathcal{L}_E(G,H)$ coincides with the adjoint defined in $\mathcal{L}_F(\widetilde{G},\widetilde{H})$.

c) *$\mathcal{L}_E(H)$ is a unital C^*-subalgebra of $\mathcal{L}_F(\widetilde{H})$.*

d) *If $\widetilde{H}$ is self-dual then H is also self-dual.*

e) *If E is the real C^*-algebra $\mathbb{C}$ or $\mathbb{H}$ (Example 4.1.1.31) and $F := \mathbb{R}$ then*

$$\varphi : E \longrightarrow F, \quad x \longmapsto \operatorname{re} x$$

has the above properties 1-4. The same holds for the map

$$\varphi : \mathbb{H} \longrightarrow \mathbb{C}, \quad \alpha 1 + \beta i + \gamma j + \delta k \longmapsto \alpha 1 + \beta i.$$

f) *If E is the real C^*-algebra $\mathbb{C}$ or $\mathbb{H}$ then every Hilbert right E-module H is self-dual and H and $\widetilde{H}$ have the same norm.*

g) *Let G be a C^*-algebra and $n \in \mathbb{N}$. If E is the C^*-direct product of the family $(G)_{j \in \mathbb{N}_n}$ and*

$$F = \{(x)_{j \in \mathbb{N}_n} | x \in G\},$$

then

$$\varphi : E \longrightarrow F, \quad x \longmapsto \sum_{j=1}^{n} x_j$$

has the above properties 1-4, where we identified F and G in a natural way.

h) *Let $(E_\iota)_{\iota \in I}$ be a finite family of C^*-algebras and for each $\iota \in I$ let F_ι be a C^*-subalgebra of E_ι and $\varphi_\iota : E_\iota \to F_\iota$ an involutive continuous linear map with the above properties 1) - 4). If E and F are the C^*-direct product of the families $(E_\iota)_{\iota \in I}$ and $(F_\iota)_{\iota \in I}$, respectively, and if*

$$\varphi : E \longrightarrow F, \quad x \longmapsto (\varphi_\iota x_\iota)_{\iota \in I},$$

then φ is an involutive continuous linear map with the above properties 1) - 4).

i) *If G, H are Hilbert right E-modules and $u \in \mathcal{L}_E(G, H)$ then the norm of u in $\mathcal{L}_E(G, H)$ coincides with the norm of u in $\mathcal{L}_F(\widetilde{G}, \widetilde{H})$.*

j) *For every $x \in E$ put*

$$\|x\|_\varphi := \sup_{y \in E^\#} \|\varphi(xy)\|.$$

Then $\|\cdot\|_\varphi$ is a norm on E equivalent to its initial norm.

a) is an easy verification.

b) Take $u \in \mathcal{L}_F(\widetilde{G}, \widetilde{H}) \cap \widehat{\mathcal{L}}_E(G, H)$. Then for $(\xi, \eta, x) \in G \times H \times E$,

$$\varphi(\langle u\xi|\eta\rangle x) = \varphi(\langle (u\xi)x|\eta\rangle = \varphi\langle u(\xi x)|\eta\rangle =$$

$$= \varphi\langle \xi x|u^*\eta\rangle = \varphi(\langle \xi|u^*\eta\rangle x)\,,$$

$$\varphi((\langle u\xi|\eta\rangle - \langle \xi|u^*\eta\rangle)x) = 0\,.$$

If we put

$$x := (\langle u\xi|\eta\rangle - \langle \xi|u^*\eta\rangle)^*$$

then by 4),

$$(\langle u\xi|\eta\rangle - \langle \xi|u^*\eta\rangle)(\langle u\xi|\eta\rangle - \langle \xi|u^*\eta\rangle)^* = 0\,,$$

$$\langle u\xi|\eta\rangle - \langle \xi|u^*\eta\rangle = 0\,,$$

$$\langle u\xi|\eta\rangle = \langle \xi|u^*\eta\rangle\,,$$

$$u \in \mathcal{L}_E(G, H)\,.$$

The inverse inclusion follows from Proposition 5.6.1.8 d).

Now take $u \in \mathcal{L}_E(G, H)$ and $(\xi, \eta) \in G \times H$. Then

$$\langle u\xi|\eta\rangle = \langle \xi|u^*\eta\rangle$$

so that

$$\varphi\langle u\xi|\eta\rangle = \varphi\langle \xi|u^*\eta\rangle\,.$$

Hence the adjoint of u in $\mathcal{L}_E(G, H)$ and $\mathcal{L}_F(\widetilde{G}, \widetilde{H})$ coincide.

c) follows from a) and b) (and Theorem 5.6.1.11 d), Corollary 4.1.1.21).

d) Take $u \in \widehat{\mathcal{L}}(H, E)$. Then $\varphi \circ u \in \mathcal{L}(H, F)$ and for $(\xi, y) \in H \times F$,

$$\varphi \circ u(\xi y) = \varphi u(\xi y) = \varphi((u\xi)y) = (\varphi u\xi)y = ((\varphi \circ u)\xi)y\,.$$

Since $\widetilde{H}$ is self-dual, there is an $\eta \in H$ such that

$$\varphi \circ u = \varphi \circ \langle \cdot|\eta\rangle\,.$$

For $(\xi, x) \in H \times E$,

$$\varphi((u\xi - \langle \xi|\eta\rangle)x) = \varphi((u\xi)x - \langle \xi|\eta\rangle x) =$$

$$= \varphi(u(\xi x) - \langle \xi x|\eta\rangle) = \varphi \circ u(\xi x) - \varphi\langle \xi x|\eta\rangle = 0\,.$$

Putting

$$x := (u\xi - \langle \xi|\eta\rangle)^*$$

it follows by 4),

$$(u\xi - \langle \xi|\eta\rangle)(u\xi - \langle \xi|\eta\rangle)^* = 0\,,$$

$$u\xi - \langle \xi|\eta\rangle = 0\,,$$

$$u = \langle \cdot|\eta\rangle\,.$$

Hence H is self-dual.

e) Only 3) needs a proof.

First assume $E = \mathbb{C}$. Take $(x, y) \in \overset{\circ}{\mathbb{C}}_+$. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with

$$x = x_1 + ix_2\,, \quad y = y_1 + iy_2\,.$$

By Proposition 4.2.2.24,

$$x \pm iy \in \mathbb{C}_+ = \mathbb{R}_+\,,$$

i.e.

$$(x_1 \mp y_2) + i(x_2 \pm y_1) \in \mathbb{R}_+\,.$$

Hence

$$0 \leq |y_2| \leq x_1\,, \quad x_2 = y_1 = 0$$

so that

$$(\varphi x, \varphi y) = (x_1, y_1) \in \mathbb{C}_+\,.$$

Now assume $E = \mathbb{H}$. We identify $\mathbb{H}$ with a real subalgebra of $\mathbb{C}_{2,2}$ and $\overset{\circ}{\mathbb{H}}$ with $\mathbb{C}_{2,2}$ as in Example 2.3.1.46. Take $(x, y) \in \overset{\circ}{\mathbb{H}}_+$ and put

$$\alpha 1 + \beta i + \gamma j + \delta k := y\,.$$

Then $x \in \mathbb{H}_+ = \mathbb{R}_+$ (Proposition 4.2.2.15 a)) and

$$(x,y) = \begin{bmatrix} x+\alpha i-\beta & \gamma i-\delta \\ -\gamma i-\delta & x+\alpha i+\beta \end{bmatrix} \in (\mathbb{C}_{2,2})_+\,.$$

By Example 4.2.3.5,

$$x \pm \beta \in \mathbb{R}_+\,, \quad \alpha = 0\,.$$

Hence

$$(\varphi x, \varphi y) = (x,0) \in \mathbb{C}_+$$

in the case $F = \mathbb{R}$ and

$$(\varphi x, \varphi y) = (x, i\beta) \in \mathbb{C}_+$$

in the case $F = \mathbb{C}$.

f) Let H be a Hilbert right E-module. By e) and a), $\widetilde{H}$ is a real Hilbert space and so it is self-dual. By d), H is self-dual.

g) 1) and 2) are trivial. 4) follows from Corollary 4.2.1.18. In order to prove 3) we may assume $\mathbb{K} = \mathbb{C}$. Take $(x,y) \in \overset{\circ}{E}_+$. By Proposition 4.2.2.24,

$$x \pm iy \in E_+\,,$$

i.e.

$$x_j \pm iy_j \in G_+$$

for every $j \in \mathbb{N}_n$. By Proposition 4.2.2.24, again, $(x_j, y_j) \in \overset{\circ}{G}_+$ for every $j \in \mathbb{N}_n$ so that

$$(\varphi x, \varphi y) = \left(\sum_{j=1}^n x_j\,, \sum_{j=1}^n y_j \right) = \sum_{j=1}^n (x_j, y_j) \in \overset{\circ}{F}_+\,.$$

h) is easy to see.

i) Denote by $\|u\|$ (by $|||u|||$) the norm of u in $\mathcal{L}_E(G,H)$ (in $\mathcal{L}_F(\widetilde{G},\widetilde{H})$). By c) and Proposition 5.6.1.8 f),

$$\|u\|^2 = \|u^* \circ u\| = |||u^* \circ u||| = |||u|||^2\,.$$

j) $\|\cdot\|_\varphi$ *is obviously a norm on* E *with*

$$\|\cdot\|_\varphi \leq \|\varphi\|\,\|\cdot\|\,.$$

Take $x \in E\backslash\{0\}$ and put

$$y := \frac{1}{\|x\|}x^*\,.$$

Then $y \in E^\#$ and so

$$\|x\|_\varphi \geq \|\varphi(xy)\| = \left\|\varphi\left(\frac{1}{\|x\|}xx^*\right)\right\| \geq \frac{1}{\|x\|}\alpha\|x^*x\| = \alpha\|x\|\,,$$

where

$$\alpha := \inf\{\|\varphi z\| \mid z \in E_+, \|z\| = 1\} > 0\,.$$

Hence $\|\cdot\|_\varphi$ and $\|\cdot\|$ are equivalent. ■

Proposition 5.6.2.6 (8) (W. Paschke, 1973) *Let E be a C^*-algebra, G, H inner-product right E-modules, and $u \in \widehat{\mathcal{L}}_E(G,H)$ Then*

$$\langle u\xi|u\xi\rangle \leq \|u\|^2\langle\xi|\xi\rangle$$

for every $\xi \in G$. In particular, if u is an isometry then

$$\langle u\xi|u\xi\rangle = \langle\xi|\xi\rangle$$

for every $\xi \in G$.

We may assume E unital. For every $n \in \mathbb{N}$ put

$$x_n := (\langle\xi|\xi\rangle + \frac{1}{n}1)^{-\frac{1}{2}}\,, \quad \xi_n := \xi x_n\,.$$

Take $n \in \mathbb{N}$. Then

$$\langle\xi_n|\xi_n\rangle = \langle\xi x_n|\xi x_n\rangle = x_n^*\langle\xi|\xi\rangle x_n \leq 1$$

(Proposition 5.6.1.2 b)), so

$$\|\xi_n\| \leq 1$$

(Corollary 4.2.1.17 b ⇒ a). It follows

$$\|u\|^2 1 \geq \|u\xi_n\|^2 1 \geq \langle u\xi_n|u\xi_n\rangle = \langle u(\xi x_n)|u(\xi x_n)\rangle =$$

$$= \langle(u\xi)x_n|(u\xi)x_n\rangle = x_n^*\langle u\xi|u\xi\rangle x_n$$

(Corollary 4.2.1.17 a ⇒ b, Proposition 5.6.1.2 b)),

$$\langle u\xi|u\xi\rangle \leq \|u\|^2\left(\langle\xi|\xi\rangle + \frac{1}{n}1\right)\,.$$

Hence

$$\langle u\xi|u\xi\rangle \leq \lim_{n\to\infty}\|u\|^2(\langle\xi|\xi\rangle + \frac{1}{n}1) = \|u\|^2\langle\xi|\xi\rangle\,.$$ ■

Proposition 5.6.2.7 (8) *Let H be an inner-product right E-module, $x' \in E'_+$, and*

$$G_{x'} := \{\xi \in H | x'(\langle \xi | \xi \rangle) = 0\}.$$

a) *$G_{x'}$ is a vector subspace of H; denote by*

$$q_{x'} : H \longrightarrow H/G_{x'}$$

the quotient map.

b) $\xi \in G_{x'}$, $\eta \in H \Rightarrow x'(\langle \xi | \eta \rangle) = x'(\langle \eta | \xi \rangle) = 0$.

c) *Take $X, Y \in H/G_{x'}$ and $\xi_1, \xi_2 \in X$, $\eta_1, \eta_2 \in Y$. Then*

$$x'(\langle \xi_1 | \eta_1 \rangle) = x'(\langle \xi_2 | \eta_2 \rangle).$$

Define

$$\langle X | Y \rangle_{x'} := x'(\langle \xi_1 | \eta_1 \rangle).$$

d) *$H/G_{x'}$ endowed with the map*

$$(H/G_{x'}) \times (H/G_{x'}) \longrightarrow \mathbb{K}, \quad (X, Y) \longmapsto \langle X | Y \rangle_{x'}$$

is a pre-Hilbert space and $q_{x'}$ is continuous with

$$\|q_{x'}\| \leq \|x'\|^{\frac{1}{2}}.$$

Denote by $H_{x'}$ the completion of the pre-Hilbert space $H/G_{x'}$.

e) *For every $u \in \widehat{H}$ there is a unique $u_{x'} \in H_{x'}$ such that*

$$x'(u\xi) = \langle q_{x'}\xi | u_{x'} \rangle_{x'}$$

for every $\xi \in H$. Moreover,

$$\|u_{x'}\| \leq \|u\| \, \|x'\|^{\frac{1}{2}}.$$

f) $\xi \in H \Rightarrow \langle \cdot | \xi \rangle_{x'} = q_{x'}\xi$.

a) Take $\xi, \eta \in G_{x'}$. By Proposition 5.6.1.2 b),

$$\langle \xi + \eta | \xi + \eta \rangle \leq 2(\langle \xi | \xi \rangle + \langle \eta | \eta \rangle)$$

so that

$$x'(\langle \xi+\eta|\xi+\eta\rangle) \leq 2(x'(\langle \xi|\xi\rangle)+x'(\langle \eta|\eta\rangle)) = 0\,,$$

$$\xi+\eta \in G_{x'}\,.$$

Hence $G_{x'}$ is a vector subspace of H.

b) By Proposition 5.6.1.2 c),

$$\langle \eta|\xi\rangle\langle \xi|\eta\rangle \leq \|\langle \eta|\eta\rangle\|\langle \xi|\xi\rangle$$

so that

$$x'(\langle \eta|\xi\rangle\langle \xi|\eta\rangle) \leq \|\langle \eta|\eta\rangle\| x'(\langle \xi|\xi\rangle) = 0\,,$$

$$x'(\langle \eta|\xi\rangle\langle \xi|\eta\rangle) = 0\,.$$

By Proposition 2.3.4.10 b),

$$|x'(\langle \xi|\eta\rangle)|^2 \leq \|x'\| x'(\langle \eta|\xi\rangle\langle \xi|\eta\rangle) = 0\,,$$

$$x'(\langle \xi|\eta\rangle) = 0\,,$$

$$x'(\langle \eta|\xi\rangle) = \overline{x'(\langle \xi|\eta\rangle)} = 0\,.$$

c) By b),

$$x'(\langle \xi_1|\eta_1\rangle) - x'(\langle \xi_2|\eta_2\rangle) =$$

$$= x'(\langle \xi_1-\xi_2|\eta_1\rangle) + x'(\langle \xi_2|\eta_1-\eta_2\rangle) = 0\,.$$

d) It is easy to see that $H/G_{x'}$ is a pre-Hilbert space. For every $\xi \in H$,

$$\|q_{x'}\xi\|^2 = \langle q_{x'}\xi|q_{x'}\xi\rangle = x'(\langle \xi|\xi\rangle) \leq \|x'\|\,\|\xi\|^2$$

so that $q_{x'}$ is continuous and

$$\|q_{x'}\| \leq \|x'\|^{\frac{1}{2}}\,.$$

e) Take $\xi \in H$. By Proposition 5.6.2.6 (and Example 5.6.1.5),

$$(u\xi)^*(u\xi) = \langle u\xi|u\xi\rangle \leq \|u\|^2\langle \xi|\xi\rangle$$

so that

$$x'((u\xi)^*(u\xi)) \leq \|u\|^2 x'(\langle\xi|\xi\rangle)\,,$$

$$|x'(u\xi)|^2 \leq \|x'\| x'((u\xi)^*(u\xi)) \leq$$

$$\leq \|x'\|\,\|u\|^2 x'(\langle\xi|\xi\rangle) \leq \|x'\|^2\|u\|^2\|\xi\|^2$$

(Proposition 2.3.4.10 b)). It follows that $x'\circ u$ factorizes through $H/G_{x'}$ to a continuous linear form. By the Frèchet–Riesz Theorem (Theorem 5.2.5.2), there is a $u_{x'} \in H_{x'}$ such that

$$x'(u\xi) = \langle q_{x'}\xi|u_{x'}\rangle_{x'}$$

for every $\xi \in H$. By the above,

$$\|u_{x'}\| = \sup\{|\langle q_{x'}\xi|u_{x'}\rangle| \,|\xi \in H, \|q_{x'}\xi\| \leq 1\} =$$

$$= \sup\{|x'(u\xi)||\xi \in H\,,\, x'(\langle\xi|\xi\rangle) \leq 1\} \leq \|x'\|^{\frac{1}{2}}\|u\|\,.$$

The uniqueness follows from the fact that $q_{x'}(H)$ is dense in $H_{x'}$.

f) By Proposition 5.6.2.1 a) (and Proposition 5.6.1.8 d)), $\langle\cdot|\xi\rangle \in \widehat{H}$. For $\eta \in H$,

$$\langle q_{x'}\eta|\langle\cdot|\xi\rangle_{x'}\rangle_{x'} = x'(\langle\cdot|\xi\rangle\eta) =$$

$$= x'(\langle\eta|\xi\rangle) = \langle q_{x'}\eta|q_{x'}\xi\rangle_{x'}\,.$$

Since $q_{x'}(H)$ is dense in $H_{x'}$,

$$\langle\cdot|\xi\rangle_{x'} = q_{x'}\xi\,. \qquad \blacksquare$$

Proposition 5.6.2.8 (8) *We assume* $\mathbb{K} = \mathbb{R}$, *use the notation of Proposition 5.6.2.7, and extend it in a natural way to the complexification* $\overset{\circ}{H}$ *of* H.

a) $G_{(x',0)} = G_{x'} \times G_{x'}$.

b) *The algebraic isomorphism*

$$\overbrace{\overset{\circ}{H/G_{x'}}} \longrightarrow \overset{\circ}{H}/G_{(x',0)}\,, \quad (q_{x'}\xi, q_{x'}\eta) \longmapsto q_{(x',0)}(\xi,\eta)$$

associated to the quotient map

$$q_{(x',0)} : \overset{\circ}{H} \longrightarrow \overset{\circ}{H}/G_{(x',0)}$$

is an isomorphism of complex pre-Hilbert spaces. We identify the above pre-Hilbert spaces as well as their completions $\overbrace{\overset{\circ}{H_{x'}}}$ *and* $(\overset{\circ}{H})_{(x',0)}$ *using this isomorphism.*

c) $u \in \widehat{H} \Rightarrow (u,0)_{(x',0)} = (u_{x'},0)$.

d) $u,v \in \widehat{H} \Rightarrow \langle (u,0)_{(x',0)} | (v,0)_{(x',0)} \rangle_{(x',0)} = \langle u_{x'} | v_{x'} \rangle_{x'}$.

By Proposition 4.3.6.2 $c_3 \Rightarrow c_1$, $(x',0) \in (\overset{\circ}{E})'_+$.

a) For $(\xi,\eta) \in \overset{\circ}{H}$,

$$\langle \langle (\xi,\eta) | (\xi,\eta) \rangle, (x',0) \rangle =$$

$$\langle (\langle \xi|\xi \rangle + \langle \eta|\eta \rangle, \langle \eta|\xi \rangle - \langle \xi|\eta \rangle), (x',0) \rangle =$$

$$= x'(\langle \xi|\xi \rangle) + x'(\langle \eta|\eta \rangle)$$

so that

$$G_{(x',0)} = G_{x'} \times G_{x'}.$$

b) For $(\xi,\eta), (\zeta,\lambda) \in \overset{\circ}{H}$,

$$\langle (q_{x'}\xi, q_{x'}\eta) \mid (q_{x'}\zeta, q_{x'}\lambda) \rangle_{x'} =$$

$$= \langle q_{x'}\xi | q_{x'}\eta \rangle_{x'} + \langle q_{x'}\eta | q_{x'}\lambda \rangle_{x'} + i\langle q_{x'}\eta | q_{x'}\zeta \rangle_{x'} - i\langle q_{x'}\xi | q_{x'}\lambda \rangle_{x'} =$$

$$= x'(\langle \xi|\zeta \rangle + \langle \eta|\lambda \rangle) + ix'(\langle \eta|\zeta \rangle - \langle \xi|\lambda \rangle) =$$

$$= \langle (\langle \xi|\zeta \rangle + \langle \eta|\lambda \rangle, \langle \eta|\zeta \rangle - \langle \xi|\lambda \rangle), (x',0) \rangle =$$

$$= \langle \langle (\xi,\eta) | (\zeta,\lambda) \rangle, (x',0) \rangle =$$

$$= \langle q_{(x',0)}(\xi,\eta) \mid q_{(x',0)}(\zeta,\lambda) \rangle_{(x',0)}.$$

Hence the map

$$\overbrace{H/G_{x'}}^{\circ} \longrightarrow \overset{\circ}{H}/G_{(x',0)}, \quad (q_{x'}\xi, q_{x'}\eta) \longmapsto q_{(x',0)}(\xi,\eta)$$

is an isomorphism of complex pre–Hilbert spaces.

c) For $(\xi,\eta) \in \overset{\circ}{H}$,

$$\langle (q_{x'}\xi, q_{x'}\eta) | (u_{x'},0) \rangle_{x'} =$$

$$= \langle q_{x'}\xi | u_{x'}\rangle_{x'} + i\langle q_{x'}\eta | u_{x'}\rangle_{x'} =$$

$$= x'(u\xi) + ix'(u\eta) = \langle (u,0)(\xi,\eta), (x',0)\rangle =$$

$$= \langle q_{(x',0)}(\xi,\eta) \mid (u,0)_{(x',0)}\rangle_{(x',0)} .$$

Since (ξ,η) is arbitry

$$(u,0)_{(x',0)} = (u_{x'},0) .$$

d) For $\xi \in H$,

$$\langle q_{x'}\xi | v_{x'}\rangle_{x'} = x'(v\xi) = \langle (v,0)(\xi,0), (x',0)\rangle =$$

$$= \langle q_{(x',0)}(\xi,0) | (v,0)_{(x',0)}\rangle_{(x',0)} .$$

Since $H/G_{x'}$ is dense in $H_{x'}$, there is a sequence $(\xi_n)_{n\in\mathbb{N}}$ in H such that $(q_{x'}\xi_n)_{n\in\mathbb{N}}$ converges to $u_{x'}$. By b) and c),

$$(u,0)_{(x',0)} = (u_{x'},0) = \lim_{n\to\infty}(q_{x'}\xi_n,0) = \lim_{n\to\infty} q_{(x',0)}(\xi_n,0) .$$

It follows

$$\langle u_{x'} | v_{x'}\rangle_{x'} = \lim_{n\to\infty} \langle q_{x'}\xi_n | v_{x'}\rangle_{x'} =$$

$$= \lim_{n\to\infty} \langle q_{(x',0)}(\xi_n,0) | (v,0)_{(x',0)}\rangle_{(x',0)} =$$

$$= \langle (u,0)_{(x',0)} | (v,0)_{(x',0)}\rangle_{(x',0)} . \qquad \blacksquare$$

Proposition 5.6.2.9 (8) *We use the notation of Proposition 5.6.2.7 and take $x', y' \in E'_+$ with $x' \leq y'$.*

a) $G_{y'} \subset G_{x'}$.

b) *There is a unique $\varphi_{x',y'} \in \mathcal{L}(H_{y'}, H_{x'})^{\#}$ such that*

$$\varphi_{x',y'} \circ q_{y'} = q_{x'} .$$

c) *For every $u \in \widehat{H}$,*

$$\varphi_{x',y'} u_{y'} = u_{x'} .$$

d) *For every* $u \in \widehat{H}$ *there is a filter* $\mathfrak{F}$ *on* H *such that*

$$\lim_{\xi,\mathfrak{F}} q_{x'}\xi = u_{x'}$$

for every $x' \in E'_+$ *and*

$$\lim_{\xi,\mathfrak{F}} q_{(x',y')}(\xi,0) = (u,0)_{(x',y')}$$

for every $(x',y') \in (\overset{\circ}{E})'_+$.

a) is obvious.

b) follows from a).

c) Since $H/G_{y'}$ is dense in $H_{y'}$, there is a sequence $(\xi_n)_{n\in\mathbb{N}}$ in H such that $(q_{y'}\xi_n)_{n\in\mathbb{N}}$ converges to $u_{y'}$. We have

$$\varphi_{x',y'}u_{y'} = \lim_{n\to\infty} \varphi_{x',y'}q_{y'}\xi_n = \lim_{n\to\infty} q_{x'}\xi_n .$$

Take $\xi \in H$ and $n \in \mathbb{N}$. By Proposition 2.3.4.10 b),

$$|x'(\langle\xi|\xi_n\rangle - u\xi)|^2 \leq$$

$$\leq \|x'\|x'((\langle\xi|\xi_n\rangle - u\xi)(\langle\xi|\xi_n\rangle - u\xi)^*) \leq$$

$$\leq \|y'\|y'((\langle\xi|\xi_n\rangle - u\xi)(\langle\xi|\xi_n\rangle - u\xi)^*) =$$

$$= \|y'\|y'(\langle\xi|\xi_n\rangle\langle\xi_n|\xi\rangle - (u\xi)\langle\xi_n|\xi\rangle - \langle\xi|\xi_n\rangle(u\xi)^* + (u\xi)(u\xi)^*) .$$

By Proposition 5.6.2.7 e),

$$y'((u\xi)(u\xi)^* - \langle\xi|\xi_n\rangle(u\xi)^*) = y'(u(\xi(u\xi)^* - \langle\xi(u\xi)^*|\xi_n\rangle) =$$

$$= \langle q_{y'}(\xi(u\xi)^*)|u_{y'}\rangle_{y'} - \langle q_{y'}(\xi(u\xi)^*)|q_{y'}\xi_n\rangle_{y'} =$$

$$= \langle q_{y'}(\xi(u\xi)^*)|u_{y'} - q_{y'}\xi_n\rangle_{y'}$$

and so

$$\lim_{n\to\infty} y'((u\xi)(u\xi)^* - \langle\xi|\xi_n\rangle(u\xi)^*) = 0 .$$

Again by Proposition 5.6.2.7 e),

$$y'(\langle\xi|\xi_n\rangle\langle\xi_n|\xi\rangle - (u\xi)\langle\xi_n|\xi\rangle) =$$

$$= y'(\langle\xi\langle\xi_n|\xi\rangle|\xi_n\rangle - u(\xi\langle\xi_n|\xi\rangle)) =$$

$$= \langle q_{y'}(\xi\langle\xi_n|\xi\rangle) \mid q_{y'}\xi_n\rangle_{y'} - \langle q_{y'}(\xi\langle\xi_n|\xi\rangle)|u_{y'}\rangle_{y'} =$$

$$= \langle q_{y'}(\xi\langle\xi_n|\xi\rangle)|q_{y'}\xi_n - u_{y'}\rangle_{y'}$$

so that

$$|y'(\langle\xi|\xi_n\rangle\langle\xi_n|\xi\rangle - (u\xi)\langle\xi_n|\xi\rangle)|^2 \leq$$

$$\leq \|q_{y'}(\xi\langle\xi_n|\xi\rangle)\| \, \|q_{y'}\xi_n - u_{y'}\| =$$

$$= \|q_{y'}\xi_n - u_{y'}\|(y'(\langle\xi\langle\xi_n|\xi\rangle \mid \xi\langle\xi_n|\xi\rangle)))^{\frac{1}{2}} =$$

$$= \|q_{y'}\xi_n - u_{y'}\|(y'(\langle\xi_n|\xi\rangle^*\langle\xi|\xi\rangle\langle\xi_n|\xi\rangle))^{\frac{1}{2}} \leq$$

$$\leq \|q_{y'}\xi_n - u_{y'}\| \, \|\xi\|(y'(\langle\xi_n|\xi\rangle^*\langle\xi_n|\xi\rangle))^{\frac{1}{2}} \leq$$

$$\leq \|q_{y'}\xi_n - u_{y'}\| \, \|\xi\|(y'(\|\xi\|^2\langle\xi_n|\xi_n\rangle))^{\frac{1}{2}} =$$

$$= \|q_{y'}\xi_n - u_{y'}\| \, \|\xi\|^2\|q_{y'}\xi_n\|$$

(Proposition 5.6.2.7 e), Proposition 5.6.1.2 b),c), Corollary 4.2.2.3). It follows

$$\lim_{n\to\infty} y'(\langle\xi|\xi_n\rangle\langle\xi_n|\xi\rangle - (u\xi)\langle\xi_n|\xi\rangle) = 0\,,$$

$$0 = \lim_{n\to\infty} x'(\langle\xi|\xi_n\rangle - u\xi) =$$

$$= \lim_{n\to\infty} (\langle q_{x'}\xi|q_{x'}\xi_n\rangle - \langle q_{x'}\xi|u_{x'}\rangle) =$$

$$= \lim_{n\to\infty} (\langle q_{x'}\xi|q_{x'}\xi_n - u_{x'}\rangle)\,.$$

Since ξ is arbitrary,

$$\varphi_{x',y'}u_{y'} = \lim_{n\to\infty} q_{x'}\xi_n = u_{x'}\,.$$

d) For every $A \in \mathfrak{P}_f(E'_+)$ and $\varepsilon > 0$ put

$$B(A,\varepsilon) := \{\xi \in H | x' \in A \Longrightarrow \|q_{x'}\xi - u_{x'}\| < \varepsilon\}.$$

$B(A,\varepsilon)$ is nonempty. Indeed, put

$$y' := \sum_{x' \in A} x'.$$

Then $y' \in E'_+$ and $x' \leq y'$ for every $x' \in A'$. Since $H/G_{y'}$ is dense in $H_{y'}$, there is a $\xi \in H$ such that

$$\|q_{y'}\xi - u_{y'}\| < \varepsilon.$$

By b) and c),

$$\|q_{x'}\xi - u_{x'}\| = \|\varphi_{x',y'}q_{y'}\xi - \varphi_{x',y'}u_{y'}\| \leq$$

$$\leq \|q_{y'}\xi - u_{y'}\| < \varepsilon$$

for every $x' \in A$. Hence $\xi \in B(A,\varepsilon)$ and $B(A,\varepsilon)$ is nonempty.

Denote by $\mathfrak{F}$ the filter on H generated by the filter basis

$$\{B(A,\varepsilon) | A \in \mathfrak{P}_f(E'_+), \varepsilon > 0\}.$$

It is obvious that

$$\lim_{\xi,\mathfrak{F}} q_{x'}\xi = u_{x'}$$

for every $x' \in E'_+$.

Now take $(x',y') \in (\overset{\circ}{E})'_+$. By Corollary 4.3.6.2 $c_1 \Rightarrow c_4$,

$$(x',y') \leq (2x',0).$$

By b),c), and Proposition 5.6.2.8 b),c),

$$\|q_{(x',y')}(\xi,0) - (u,0)_{(x',y')}\| =$$

$$= \|\varphi_{(x',y'),(2x',0)}q_{(2x',0)}(\xi,0) - \varphi_{(x',y'),(2x',0)}(u,0)_{(2x',0)}\| \leq$$

$$\leq \|q_{(2x',0)}(\xi,0) - (u,0)_{(2x',0)}\| = \|q_{2x'}\xi - u_{2x'}\|.$$

By the above,

$$\lim_{\xi,\mathfrak{F}} q_{(x',y')}(\xi,0) = (u,0)_{(x',y')}.$$

■

Proposition 5.6.2.10 (8) *Let H be an inner-product right E-module and $\overset{\circ}{H}$ its complexification (Proposition 5.6.1.6). We identify $\overset{\circ}{\mathcal{L}}(H,E)$ and $\mathcal{L}(\overset{\circ}{H},\overset{\circ}{E})$ using the isomorphism of complex vector spaces*

$$\overset{\circ}{\mathcal{L}}(H,E) \longrightarrow \mathcal{L}(\overset{\circ}{H},\overset{\circ}{E}), \quad (u,v) \longmapsto \widetilde{(u,v)}$$

defined in Proposition 2.1.5.6 (Proposition 4.3.6.1 f), Proposition 5.6.1.6).

a) $\overset{\circ}{\widehat{H}} = \widehat{\overset{\circ}{H}}$.

b) *H is self-dual iff $\overset{\circ}{H}$ is self-dual.*

a) Take $(u,v) \in \overset{\circ}{\widehat{H}}$. For every $(\xi,\eta) \in \overset{\circ}{H}$ and $(x,y) \in \overset{\circ}{E}$,

$$\widetilde{(u,v)}((\xi,\eta)(x,y)) = \widetilde{(u,v)}(\xi x - \eta y, \xi y + \eta x) =$$

$$= (u(\xi x - \eta y) - v(\xi y + \eta x)\,,\, u(\xi y + \eta x) + v(\xi x - \eta y)) =$$

$$= ((u\xi)x - (u\eta)y - (v\xi)y - (v\eta)x\,,\, (u\xi)y + (u\eta)x + (v\xi)x - (v\eta)y) =$$

$$= (u\xi - v\eta, u\eta + v\xi)(x,y) = (\widetilde{(u,v)}(\xi,\eta))(x,y)\,.$$

Hence

$$\widetilde{(u,v)} \in \widehat{\overset{\circ}{H}}$$

and

$$\overset{\circ}{\widehat{H}} \subset \widehat{\overset{\circ}{H}}.$$

Take $w \in \widehat{\overset{\circ}{H}}$. By the above identification, there is a $(u,v) \in \overset{\circ}{\mathcal{L}}(H,E)$ with

$$\widetilde{(u,v)} = w\,.$$

For every $(\xi,x) \in H \times E$,

$$(u(\xi x), v(\xi x)) = w((\xi x, 0)) = w((\xi,0)(x,0)) =$$

$$= (w((\xi,0)))(x,0)(u\xi, v\xi)(x,0) = ((u\xi)x, (v\xi)x)\,,$$

i.e.

$$u(\xi x) = (u\xi)x\,, \quad v(\xi x) = (v\xi)x\,,$$

$$u, v \in \widehat{H}\,,$$

$$w = \widetilde{(u,v)} \in \overset{\circ}{\widehat{H}}\,,$$

$$\widehat{\overset{\circ}{H}} \subset \overset{\circ}{\widehat{H}}\,,$$

$$\overset{\circ}{\widehat{H}} = \widehat{\overset{\circ}{H}}\,.$$

b) First assume H is self-dual. Let $w \in \widehat{\overset{\circ}{H}}$. By a), there is a $(u,v) \in \overset{\circ}{\widehat{H}}$ such that

$$w = \widetilde{(u,v)}\,.$$

Since H is self-dual, there are $\xi_0, \eta_0 \in H$ with

$$u = \langle \cdot | \xi_0 \rangle\,, \quad v = \langle \cdot | \eta_0 \rangle\,.$$

For every $(\xi, \eta) \in \overset{\circ}{H}$,

$$w((\xi,\eta)) = (u\xi - v\eta\,,\, u\eta + v\xi) =$$

$$= (\langle \xi | \xi_0 \rangle - \langle \eta | \eta_0 \rangle\,,\, \langle \eta | \xi_0 \rangle + \langle \xi | \eta_0 \rangle) =$$

$$= \langle (\xi,\eta) | (\xi_0, -\eta_0) \rangle\,.$$

Hence

$$w = \langle \cdot | (\xi_0, -\eta_0) \rangle$$

and $\overset{\circ}{H}$ is self-dual.

Now assume $\overset{\circ}{H}$ is self-dual. Take $u \in \widehat{H}$ and put

$$w := \widetilde{(u,0)}\,.$$

By a), $w \in \widehat{\overset{\circ}{H}}$. Since $\overset{\circ}{H}$ is self-dual, there is a $(\xi,\eta) \in \overset{\circ}{H}$ with

$$w = \langle \cdot | (\xi,\eta) \rangle\,.$$

For every $\zeta \in H$,

$$(u\zeta, 0) = w((\zeta, 0)) = \langle (\zeta, 0) | (\xi, \eta) \rangle =$$

$$= (\langle \zeta | \xi \rangle, -\langle \zeta, \eta \rangle) .$$

Hence

$$u = \langle \cdot | \xi \rangle$$

and H is self-dual. ■

Theorem 5.6.2.11 (8) (Paschke, 1973) *Let E be a W^*-algebra and H an inner-product right E-module. For every $\xi \in H$ put*

$$\widetilde{\xi} := \langle \cdot | \xi \rangle : H \longrightarrow E , \quad \eta \longmapsto \langle \eta | \xi \rangle$$

and identify H with a subset of $\widehat{H}$ using the injective map

$$H \longrightarrow \widehat{H} , \quad \xi \longmapsto \widetilde{\xi}$$

(Proposition 5.6.2.1 a)). For every $(u, x) \in \widehat{H} \times E$ put

$$ux : H \longrightarrow E , \quad \xi \longmapsto x^* u \xi .$$

We shall use the notation of Proposition 5.6.2.7.

a) $u \in \widehat{H}$, $x, y \in E \Rightarrow ux \in \widehat{H}$, $(ux)y = u(xy)$.

b) $(\xi, x) \in H \times E \Rightarrow \widetilde{\xi} x = \widetilde{\xi x}$.

c) *For $\mathbb{K} = \mathbb{C}$ there is a unique map*

$$\widehat{H} \times \widehat{H} \longrightarrow E , \quad (u, v) \longmapsto \langle u | v \rangle$$

such that

$$\langle \langle u | v \rangle, a \rangle = \langle u_a | v_a \rangle_a$$

for all $u, v \in \widehat{H}$ and $a \in \ddot{E}_+$.

d) *Assume $\mathbb{K} = \mathbb{R}$ and take $u, v \in \widehat{H}$. Then*

$$\langle (u, 0) | (v, 0) \rangle \in E \times \{0\}$$

and we define $\langle u | v \rangle \in E$ by

$$(\langle u|v\rangle, 0) = \langle (u,0)|(v,0)\rangle\,.$$

Then

$$\langle (u,v), a\rangle = \langle u_a|v_a\rangle_a$$

for every $a \in \ddot{E}_+$.

e) $\xi, \eta \in H \Rightarrow \langle \widetilde{\xi}|\widetilde{\eta}\rangle = \langle \xi|\eta\rangle$.

f) $(\xi, u) \in H \times \widehat{H} \Rightarrow \langle \widetilde{\xi}|u\rangle = u\xi$.

g) $\widehat{H}$ *endowed with the right multiplication*

$$\widehat{H} \times E \longrightarrow \widehat{H}\,, \quad (u,x) \longmapsto ux$$

(which contains also a scalar multiplication) and with the inner-product

$$\widehat{H} \times \widehat{H} \longrightarrow E\,, \quad (u,v) \longmapsto \langle u|v\rangle$$

is a self-dual Hilbert right E*-module.*

h) *The norm of the Hilbert right* E*-module* $\widehat{H}$ *defined in g) coincides with the initial norm of* $\widehat{H}$ *(Definition 5.6.2.2).*

a) For every $\xi \in H$,

$$(ux)(\xi y) = x^*u(\xi y) = x^*(u\xi)y = ((ux)\xi)y\,,$$

$$((ux)y)\xi = y^*((ux)\xi) = y^*x^*u\xi = (xy)^*u\xi = (u(xy))\xi\,,$$

so that $ux \in \widehat{H}$ and

$$(ux)y = u(xy)\,.$$

b) For every $\eta \in H$,

$$(\widetilde{\xi}x)\eta = x^*\widetilde{\xi}\eta = x^*\langle \eta|\xi\rangle = \langle \eta|\xi x\rangle = \widetilde{\xi x}\eta$$

(Proposition 5.6.1.2 a)), so that

$$\widetilde{\xi}x = \widetilde{\xi x}\,.$$

c) We use the notation of Proposition 5.6.2.9. Take $u, v \in \widehat{H}$. Define

$$\varphi : \ddot{E}_+ \longrightarrow \mathbb{C}\,, \quad a \longmapsto \langle u_a|v_a\rangle_a\,.$$

Step 1 φ may be extended linearly to $\ddot{E}$

Let $(\alpha_\iota)_{\iota\in I}$ be a finite family in $\mathbb{C}$ and $(a_\iota)_{\iota\in I}$ a family in $\ddot{E}_+$ such that

$$\sum_{\iota\in I}\alpha_\iota a_\iota = 0\,.$$

Put

$$a := \sum_{\iota\in I} a_\iota\,.$$

Then $a\in\ddot{E}_+$ and $a_\iota\le a$ for every $\iota\in I$. For $\xi,\eta\in H$,

$$\sum_{\iota\in I}\alpha_\iota\left\langle\varphi^*_{a_\iota,a}\varphi_{a_\iota,a}q_a\xi|q_a\eta\right\rangle_a =$$

$$=\sum_{\iota\in I}\alpha_\iota\left\langle\varphi_{a_\iota,a}q_a\xi\;|\varphi_{a_\iota,a}q_a\eta\right\rangle_{a_\iota} =$$

$$=\sum_{\iota\in I}\alpha_\iota\langle q_{a_\iota}\xi|q_{a_\iota}\eta\rangle_{a_\iota}=\sum_{\iota\in I}\alpha_\iota a_\iota(\langle\xi|\eta\rangle)=0$$

(Proposition 5.6.2.9 b)). It follows

$$\sum_{\iota\in I}\alpha_\iota\varphi^*_{a_\iota,a}\varphi_{a_\iota,a}=0\,,$$

$$\sum_{\iota\in I}\alpha_\iota\varphi(a_\iota)=\sum_{\iota\in I}\alpha_\iota\langle u_{a_\iota}|v_{a_\iota}\rangle_{a_\iota}=$$

$$=\sum_{\iota\in I}\alpha_\iota\left\langle\varphi_{a_\iota,a}u_a|\varphi_{a_\iota,a}v_a\right\rangle_{a_\iota}=$$

$$=\sum_{\iota\in I}\alpha_\iota\left\langle\varphi^*_{a_\iota,a}\varphi_{a_\iota,a}u_a|v_a\right\rangle_a=0$$

(Proposition 5.6.2.9 b)). Hence φ may be extended linearly to $\ddot{E}$. We denote this extension of φ also by φ.

Step 2 φ is continuous

Take $a\in\ddot{E}$. By Theorem 4.4.3.9 (and Corollary 4.4.2.10), there is a family $(a_j)_{j\in\mathbb{N}_4}$ in $\ddot{E}_+$ such that

$$a = a_1 - a_2 + ia_3 - ia_4 ,$$

$$\sum_{j=1}^{4} \|a_j\| \leq 2\|a\| .$$

We get

$$|\varphi(a)| \leq \sum_{j=1}^{4} |\varphi(a_j)| = \sum_{j=1}^{4} \left| \langle u_{a_j} | v_{a_j} \rangle_{a_i} \right| \leq$$

$$\leq \sum_{j=1}^{4} \|u_{a_j}\| \, \|v_{a_j}\| \leq \sum_{j=1}^{4} \|a_j\| \, \|u\| \, \|v\| \leq 2\|a\| \, \|u\| \, \|v\|$$

(Proposition 5.6.2.7 e)). Hence φ is continuous.

Define

$$\langle u|v \rangle := \varphi \in E .$$

Then

$$a \in \ddot{E}_+ \Longrightarrow \langle \langle u|v \rangle, a \rangle = \langle u_a | v_a \rangle_a .$$

Uniqueness follows from Theorem 4.4.3.9 (and Corollary 4.4.2.10).

d) By Proposition 5.6.2.9 d), there is a filter $\mathfrak{F}$ on H such that

$$\lim_{\xi, \mathfrak{F}} q_a \xi = u_a$$

for every $a \in \ddot{E}_+$ and

$$\lim_{\xi, \mathfrak{F}} q_{(a,b)}(\xi, 0) = (u, 0)_{(a,b)}$$

for every $(a, b) \in \overset{\circ\circ}{E}_+$. Put

$$(x, y) := \langle (u, 0) | (v, 0) \rangle$$

and take $(a, b) \in \overset{\circ\circ}{E}_+$. By c) (and Proposition 5.6.2.7 e)),

$$\langle (x, y), (a, b) \rangle = \langle \langle (u, 0) | (v, 0) \rangle, (a, b) \rangle =$$

$$= \langle (u, 0)_{(a,b)} | (v, 0)_{(a,b)} \rangle_{(a,b)} =$$

$$= \lim_{\xi,\mathfrak{F}} \langle q_{(a,b)}(\xi,0) \mid (v,0)_{(a,b)} \rangle_{(a,b)} =$$

$$= \lim_{\xi,\mathfrak{F}} \langle (v,0)(\xi,0),(a,b) \rangle = \lim_{\xi,\mathfrak{F}} \langle (v\xi,0),(a,b) \rangle .$$

By Proposition 4.4.4.23, $y = 0$. By c) and Proposition 5.6.2.8 d),

$$\langle \langle u|v \rangle, a \rangle = \langle \langle (u,0)|(v,0) \rangle, (a,0) \rangle =$$

$$= \langle (u,0)_{(a,0)}|(v,0)_{(a,0)} \rangle_{(a,0)} = \langle u_a|v_a \rangle_a$$

for every $a \in \ddot{E}_+$.

e) First assume $\mathbb{K} = \mathbb{C}$. For every $a \in \ddot{E}_+$, by c) and Proposition 5.6.2.7 c),f),

$$\langle \langle \widetilde{\xi}|\widetilde{\eta} \rangle, a \rangle = \langle \widetilde{\xi}_a|\widetilde{\eta}_a \rangle_a = \langle q_a\xi|q_a\eta \rangle_a = \langle \langle \xi|\eta \rangle, a \rangle$$

so that

$$\langle \widetilde{\xi}|\widetilde{\eta} \rangle = \langle \xi|\eta \rangle .$$

If $\mathbb{K} = \mathbb{R}$ then, by d) and the above,

$$(\langle \widetilde{\xi}|\widetilde{\eta} \rangle, 0) = \langle (\widetilde{\xi},0)|(\widetilde{\eta},0) \rangle = \langle \widetilde{(\xi,0)}|\widetilde{(\eta,0)} \rangle =$$

$$= \langle (\xi,0)|(\eta,0) \rangle = (\langle \xi|\eta \rangle, 0)$$

and so

$$\langle \widetilde{\xi}|\widetilde{\eta} \rangle = \langle \xi|\eta \rangle .$$

f) First assume $\mathbb{K} = \mathbb{C}$. For every $a \in \ddot{E}_+$, by c) and Proposition 5.6.2.7 e),f),

$$\langle u\xi, a \rangle = \langle q_a\xi|u_a \rangle_a = \langle \widetilde{\xi}_a|u_a \rangle_a = \langle \langle \widetilde{\xi}|u \rangle, a \rangle$$

so that

$$\langle \widetilde{\xi}|u \rangle = u\xi .$$

If $\mathbb{K} = \mathbb{R}$ then, by d) and the above,

$$(\langle \widetilde{\xi}|u \rangle, 0) = \langle (\widetilde{\xi},0)|(u,0) \rangle = (u,0)(\xi,0) = (u\xi,0)$$

and so

$$\langle \tilde{\xi} | u \rangle = u\xi .$$

g) Case 1 $\mathbb{K} = \mathbb{C}$

Step 1 $u, v \in \widehat{H}, x \in E \Rightarrow \langle ux|v\rangle = \langle u|v\rangle x$

Take $a \in \ddot{E}_+$. Since $xa \in \ddot{E}$ (Corollary 4.4.2.10), there is a family $(a_j)_{j\in\mathbb{N}_4}$ in $\ddot{E}_+$ and a family $(\alpha_j)_{j\in\mathbb{N}_4}$ in $\mathbb{C}$ such that

$$xa = \sum_{j=1}^{4} \alpha_j a_j$$

(Theorem 4.4.3.9). Put

$$b := a + \sum_{j=1}^{4} a_j .$$

Then b is an upper bound of a and $(a_j)_{j\in\mathbb{N}_4}$ in $\ddot{E}_+$. By c) and Proposition 5.6.2.9 c),

$$\langle \langle u|v\rangle x, a\rangle = \langle \langle u|v\rangle, xa\rangle = \sum_{j=1}^{4} \alpha_j \langle \langle u|v\rangle, a_j\rangle =$$

$$= \sum_{j=1}^{4} \alpha_j \langle u_{a_j} | v_{a_j} \rangle_{a_j} = \sum_{j=1}^{4} \alpha_j \langle u_{a_j} | \varphi_{a_j,b} v_b \rangle_{a_j} .$$

By a),c), and Propositions 5.6.2.7 e) and 5.6.2.9 b),c), for every $\xi \in H$,

$$\sum_{j=1}^{4} \alpha_j \langle u_{a_j} | \varphi_{a_j,b} q_b \xi \rangle_{a_j} = \sum_{j=1}^{4} \alpha_j \langle u_{a_j} | q_{a_j} \xi \rangle =$$

$$= \sum_{j=1}^{4} \alpha_j \langle (u\xi)^*, a_j \rangle = \langle (u\xi)^*, xa\rangle = \langle (u\xi)^* x, a\rangle =$$

$$= \overline{\langle x^*(u\xi), a\rangle} = \overline{\langle (ux)\xi, a\rangle} = \overline{\langle q_a \xi | (ux)_a \rangle_a} =$$

$$= \langle (ux)_a | q_a \xi \rangle_a = \langle (ux)_a | \varphi_{a,b} q_b \xi \rangle_a .$$

Since $q_b(H)$ is dense in H_b, it follows

$$\langle\langle u|v\rangle x, a\rangle = \sum_{j=1}^{4} \alpha_j \langle u_{a_j} | \varphi_{a_j,b} v_b \rangle_{a_j} =$$

$$= \langle (ux)_a | \varphi_{a,b} v_b \rangle_a = \langle (ux)_a | v_a \rangle_a = \langle \langle ux|v\rangle, a\rangle .$$

Hence

$$\langle u|v\rangle x = \langle ux|v\rangle .$$

Step 2 $\quad u, v \in \widehat{H} \Rightarrow \langle u|v\rangle = \langle v|u\rangle^*$

For every $a \in \ddot{E}_+$,

$$\langle\langle u|v\rangle, a\rangle = \langle u_a|v_a\rangle_a = \overline{\langle v_a|u_a\rangle_a} =$$

$$= \overline{\langle\langle v|u\rangle, a\rangle} = \langle \langle v|u\rangle^*, a\rangle$$

so that

$$\langle u|v\rangle = \langle v|u\rangle^* .$$

Step 3 $\quad u \in \widehat{H} \Rightarrow \langle u|u\rangle \in E_+$

For every $a \in \ddot{E}_+$,

$$\langle\langle u|u\rangle, a\rangle = \langle u_a|u_a\rangle_a \in \mathbb{R}_+$$

so that

$$\langle u|u\rangle \in E_+$$

by Corollary 4.4.1.6 a).

Step 4 $\quad u \in \widehat{H}, \langle u|u\rangle = 0 \Rightarrow u = 0$

For every $a \in \ddot{E}_+$,

$$\langle u_a|u_a\rangle_a = \langle\langle u|u\rangle, a\rangle = 0$$

so that

$$u_a = 0 .$$

By Proposition 5.6.2.7 e),

$$a \circ u = 0$$

so that by Corollary 4.4.1.6 b), $u = 0$.

Step 5 $\widehat{H}$ is self-dual

Let $U \in \mathcal{L}(\widehat{H}, E)$ with

$$(v, x) \in \widehat{H} \times E \Longrightarrow U(vx) = (Uv)x\,.$$

Put

$$u := U|H \in \widehat{H}\,,$$

$$V : \widehat{H} \longrightarrow E\,, \quad v \longmapsto Uv - \langle v|u\rangle\,.$$

By f),

$$V\widetilde{\xi} = 0$$

for every $\xi \in H$. Take $v \in \widehat{H}$ and $a \in \ddot{E}_+$. There is a sequence $(\xi_n)_{n\in\mathbb{N}}$ in H such that $(q_a\xi_n)_{n\in\mathbb{N}}$ converges to v_a. For every $n \in \mathbb{N}$, by c), Proposition 5.6.2.6, and Proposition 5.6.2.7 f),

$$\langle (Vv)^\bullet(Vv), a\rangle = \langle (V(v - \widetilde{\xi}_n))^\bullet(V(v - \widetilde{\xi}_n)), a\rangle \leq$$

$$\leq \|V\|^2\langle\langle v - \widetilde{\xi}_n | v - \widetilde{\xi}_n\rangle, a\rangle =$$

$$= \|V\|^2\langle v_a - q_a\xi_n | v_a - q_a\xi_n\rangle_a\,.$$

It follows successively

$$\langle (Vv)^\bullet(Vv), a\rangle = 0\,,$$

$$(Vv)^\bullet(Vv) = 0$$

(Corollary 4.4.1.6 b)),

$$Vv = 0\,,$$

$$U = \langle \cdot | u \rangle ,$$

$\widetilde{H}$ is self-dual.

Case 2 $\quad \mathbb{K} = \mathbb{R}$

Step 6 $\quad (u, x) \in \widehat{H} \times E \Rightarrow (u, 0)(x, 0) = (ux, 0)$

For $(\xi, \eta) \in \overset{\circ}{H}$,

$$((u,0)(x,0))(\xi,\eta) = (x^*,0)((u,0)(\xi,\eta)) = (x^*,0)(u\xi, u\eta) =$$

$$= (x^*u\xi, x^*u\eta) = ((ux)\xi, (ux)\eta) = (ux)(\xi,\eta)$$

so that

$$(u,0)(x,0) = (ux,0) .$$

Step 7 $\quad u, v \in \widehat{H}$, $x \in E \Rightarrow \langle ux | v \rangle = \langle u | v \rangle x$

By d) and Steps 1 and 6,

$$(\langle ux|v\rangle, 0) = \langle (ux,0)|(v,0)\rangle = \langle (u,0)(x,0)|(v,0)\rangle =$$

$$= \langle (u,0)|(v,0)\rangle (x,0) = (\langle u|v\rangle, 0)(x,0) = (\langle u|v\rangle x, 0)$$

and so

$$\langle ux|v\rangle = \langle u|v\rangle x .$$

Step 8 $\quad u, v \in \widehat{H} \Rightarrow \langle u|v\rangle = \langle v|u\rangle^*$

By d) and Step 2,

$$\langle u|v\rangle = \langle (u,0)|(v,0)\rangle = \langle (v,0)|(u,0)\rangle^* = \langle v|u\rangle^* .$$

Step 9 $\quad u \in \widehat{H} \Rightarrow \langle u|u\rangle \in E_+$

By d) and Step 3,

$$(\langle u|u\rangle, 0) = \langle (u,0)|(u,0)\rangle \in \overset{\circ}{E}_+$$

so that

$$\langle u|u\rangle \in E_+$$

by Proposition 4.2.2.15 b).

Step 10 $\quad u \in \widehat{H}, \langle u|u\rangle = 0 \Rightarrow u = 0$

By d),

$$\langle (u,0)|(u,0)\rangle = (\langle u|u\rangle, 0) = 0$$

and so, by Step 4,

$$(u,0) = 0\,,$$

$$u = 0\,.$$

Step 11 $\quad u, v \in \widehat{H} \Rightarrow (\langle u|u\rangle + \langle v|v\rangle, \langle v|u\rangle - \langle u|v\rangle) \in \overset{\circ}{E}_+$

By d) and Case 1 (and Proposition 5.6.1.2 g)),

$$((\langle u|u\rangle, 0) + (\langle v|v\rangle, 0)\,,\, (\langle v|u\rangle, 0) - (\langle u|v\rangle, 0)) =$$

$$= (\langle (u,0)|(u,0)\rangle + \langle (v,0)|(v,0)\rangle, \langle (v,0)|(u,0)\rangle - \langle (u,0)|(v,0)\rangle) \geq 0$$

so that

$$(\langle u|u\rangle + \langle v|v\rangle\,,\, \langle v|u\rangle - \langle u|v\rangle) \in \overset{\circ}{E}_+$$

by Proposition 4.2.2.15 b).

Step 12 $\quad \widehat{H}$ is self-dual

By Proposition 5.6.2.10 a), $\overset{\circ}{\widehat{H}}$ may be identified with $\widehat{\overset{\circ}{H}}$. By Step 5, $\widehat{\overset{\circ}{H}}$ is self-dual, so by Proposition 5.6.2.10 b), $\widehat{H}$ is self-dual.

h) Take $u \in \widehat{H}$ and denote by $\|u\|$ the norm of u in $\widehat{H}$ in the sense of Definition 5.6.2.2. For every $a \in \ddot{E}_+$, by c),d), and Proposition 5.6.2.7 e),

$$\langle \langle u|u\rangle, a\rangle = \langle u_a|u_a\rangle_a = \|u_a\|^2 \leq \|u\|^2\|a\|\,.$$

By Corollary 4.4.3.16 c),

$$\|\langle u|u\rangle\| \leq \|u\|^2\,.$$

Let $\alpha \in]0, \|u\|[$. There is a $\xi \in H^{\#}$ with

$$\alpha < \|u\xi\| .$$

By Corollary 4.4.3.16 c), there is an $a \in \ddot{E}_+^{\#}$ such that

$$\alpha < \langle u\xi | a \rangle .$$

By c),d), and Proposition 5.6.2.7 d),e),

$$\alpha^2 < \langle q_a \xi | u_a \rangle_a^2 \leq \|q_a \xi\|^2 \|u_a\|^2 \leq$$

$$\leq \|a\|^2 \langle u_a | u_a \rangle_a \leq \langle \langle u|u\rangle, a\rangle \leq \|\langle u|u\rangle\| .$$

Hence

$$\|u\|^2 \leq \|\langle u|u\rangle\| ,$$

$$\|u\|^2 = \|\langle u|u\rangle\| . \qquad \blacksquare$$

Proposition 5.6.2.12 *Let F be a closed ideal of E and H a self-dual Hilbert right F-module.*

a) *For every $(\xi, x) \in H \times E$ there is a unique element of H, denoted by ξx, such that*

$$\langle \xi x | \eta \rangle = \langle \xi | \eta \rangle x$$

for every $\eta \in H$.

b) *H endowed with the right multiplication*

$$H \times E \longrightarrow H , \quad (\xi, x) \longmapsto \xi x$$

and with the inner-product

$$H \times H \longrightarrow E , \quad (\xi, \eta) \longmapsto \langle \xi | \eta \rangle$$

is a Hilbert right E-module.

a) Put

$$u : H \longrightarrow F , \quad \eta \longmapsto x^* \langle \eta | \xi \rangle .$$

u is linear and continuous and for $(\eta, y) \in H \times F$,

$$u(\eta y) = x^* \langle \eta y | \xi \rangle = x^* \langle \eta | \xi \rangle y = u(\eta) y \,.$$

Hence $u \in \widehat{\mathcal{L}}_F(H, F)$. Since H is self-dual, there is a $\xi x \in H$ with

$$u = \langle \cdot | \xi x \rangle \,.$$

For every $\eta \in H$

$$\langle \xi x | \eta \rangle = \langle \eta | \xi x \rangle^* = (u\eta)^* = \langle \xi | \eta \rangle x \,.$$

The uniqueness is easy to see.

b) is easy to see. ■

5.6.3 Von Neumann right W^*-modules

Proposition 5.6.3.1 (8) *Let H, K be Hilbert right E-modules. For all $(x', \xi, \eta) \in E' \times H \times K$, put*

$$\widetilde{(x', \xi, \eta)} : \mathcal{L}_E(H, K) \longrightarrow \mathbb{K}, \quad u \longmapsto x'(\langle u\xi | \eta \rangle).$$

Take $(x', \eta, \xi) \in E' \times H \times K$.

a) $\widetilde{(x', \xi, \eta)} \in (\mathcal{L}_E(H, K))'$, $\|\widetilde{(x', \xi, \eta)}\| \leq \|x'\| \, \|\xi\| \, \|\eta\|$.

b) $H = K \Rightarrow \widetilde{(x', \xi, \eta)}^* = \widetilde{(x'^*, \eta, \xi)}$.

c) $x' \in E'_+ \Rightarrow \widetilde{(x', \xi, \xi)} \in (\mathcal{L}_E(H))'_+$, $\|\widetilde{(x', \xi, \xi)}\| = x'(\langle \xi | \xi \rangle)$.

d) $x, y \in E \Rightarrow \widetilde{(xx'y, \xi, \eta)} = \widetilde{(x', \xi x, \eta y^*)}$.

e) $H = K$, $v \in \mathcal{L}_E(H) \Rightarrow \begin{cases} v\widetilde{(x', \xi, \eta)} = \widetilde{(x', v\xi, \eta)} \\ \widetilde{(x', \xi, \eta)}v = \widetilde{(x', \xi, v^*\eta)} \end{cases}$.

f) *If A' is an involutive set of E' then the closed vector subspace of $(\mathcal{L}_E(H))'$ generated by*

$$\{\widetilde{(x', \xi, \eta)} | (x', \xi, \eta) \in A' \times H \times H\}$$

is an involutive unital $\mathcal{L}_E(H)$-submodule of $(\mathcal{L}_E(H))'$.

Take $u \in \mathcal{L}_E(H, K)$.

a) By Proposition 5.6.1.2 d),

$$\left|\widetilde{(x', \xi, \eta)}(u)\right| = |x'(\langle u\xi | \eta \rangle)| \leq \|x'\| \, \|\langle u\xi | \eta \rangle\| \leq$$

$$\leq \|x'\| \, \|u\xi\| \, \|\eta\| \leq \|x'\| \, \|u\| \, \|\xi\| \, \|\eta\|.$$

Hence $\widetilde{(x', \xi, \eta)} \in (\mathcal{L}_E(H, K))'$ and

$$\|\widetilde{(x', \xi, \eta)}\| \leq \|x'\| \, \|\xi\| \, \|\eta\|.$$

b) By Proposition 5.6.1.8 a),

$$\widetilde{(x', \xi, \eta)}^*(u) = \overline{\widetilde{(x', \xi, \eta)}(u^*)} = \overline{x'(\langle u^*\xi | \eta \rangle)} =$$

$$= \overline{x'(\langle \xi | u\eta \rangle)} = x'^*(\langle u\eta | \xi \rangle) = \widetilde{(x'^*, \eta, \xi)}(u)$$

so that

$$\widetilde{(x', \xi, \eta)}^* = \widetilde{(x'^*, \eta, \xi)}.$$

c) By Theorem 5.6.1.11 $c_1 \Rightarrow c_2$,

$$\widetilde{(x', \xi, \xi)}(u) = x'(\langle u\xi | \xi \rangle) \in \mathbb{R}_+$$

for every $u \in \mathcal{L}_E(H)_+$. Hence by b) (and Corollary 4.2.2.10),

$$\widetilde{(x', \xi, \xi)} \in (\mathcal{L}_E(H))'_+$$

and by Corollary 2.3.4.7,

$$\|\widetilde{(x', \xi, \xi)}\| = \widetilde{(x', \xi, \xi)}(1) = x'(\langle \xi | \xi \rangle).$$

d) We have

$$\widetilde{(xx'y, \xi, \eta)}(u) = \langle \langle u\xi | \eta \rangle, xx'y \rangle = \langle y\langle u\xi | \eta \rangle x, x' \rangle =$$

$$= \langle \langle u(\xi x) | \eta y^* \rangle, x' \rangle = \widetilde{(x', \xi x, \eta y^*)}(u)$$

(Proposition 5.6.1.2 a), Proposition 5.6.1.8. d)), so that

$$\widetilde{(xx'y, \xi, \eta)} = \widetilde{(x', \xi x, \eta y^*)}.$$

e) $\langle u, v\widetilde{(x', \xi, \eta)} \rangle = \langle uv, \widetilde{(x', \xi, \eta)} \rangle = x'(\langle uv\xi | \eta \rangle) =$

$$= \widetilde{(x', v\xi, \eta)}(u),$$

$$\langle u, \widetilde{(x', \xi, \eta)}v \rangle = \langle vu, \widetilde{(x', \xi, \eta)} \rangle = x'(\langle vu\xi | \eta \rangle) =$$

$$= x'(\langle u\xi | v^*\eta \rangle) = \widetilde{(x', \xi, v^*\eta)}(u)$$

so that

$$v\widetilde{(x', \xi, \eta)} = \widetilde{(x', v\xi, \eta)}, \quad \widetilde{(x', \xi, \eta)}v = \widetilde{(x', \xi, v^*\eta)}.$$

f) follows from a),b),e). ■

Definition 5.6.3.2 (8) *Let E be a W^*-algebra and H, K Hilbert (right) E-modules. For every $a \in \ddot{E}$ and $(\xi, \eta) \in H \times K$ define*

$$\widetilde{(a,\xi)} : H \longrightarrow \mathbb{K}\,, \quad \zeta \longmapsto \langle\langle \zeta | \xi \rangle, a\rangle\,,$$

$$\widetilde{(a,\xi,\eta)} : \mathcal{L}_E(H,K) \longrightarrow \mathbb{K}\,, \quad u \longmapsto \langle\langle u\xi | \eta \rangle, a\rangle$$

and denote by $\ddot{H}$ (by $\dddot{H}$) the closed vector subspace of H' (of $\mathcal{L}_E(H \times K)'$) generated by

$$\{\widetilde{(a,\xi)} | a \in \ddot{E}\,, \xi \in H\} \quad (\text{by } \{\widetilde{(a,\xi,\eta)} | a \in \ddot{E}\,, (\xi,\eta) \in H \times K\})$$

(Proposition 5.6.3.1 a)).

H is called a ***von Neumann (right) E-module*** *if it is self dual. If E is not specified then we simply speak of* ***von Neumann (right) W^*-module.***

If $E \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ then every Hilbert (right) E module is a von Neumann (right) E-module (Theorem of Frèchet-Riesz for $\mathbb{R}$ and $\mathbb{C}$ and Proposition 5.6.2.5 f) for the real C^*-algebras $\mathbb{C}$ and $\mathbb{H}$).

Proposition 5.6.3.3 (8) (Paschke, 1973) *Let E be a W^*-algebra and H a Hilbert right E-module. The following are equivalent:*

a) *H is self-dual.*

b) *$H^{\#}_{\ddot{H}}$ is compact.*

If these conditions are fulfilled then the map

$$H \longrightarrow (\ddot{H})'\,, \quad \xi \longmapsto (j_H \xi)|\ddot{H}$$

is an isometry of Banach spaces.

a $\Rightarrow$ b. Let $\mathfrak{F}$ be an ultrafilter on $H^{\#}$. Put

$$\widetilde{\xi} : \ddot{E} \longrightarrow \mathbb{K}\,, \quad a \longmapsto \lim_{\eta,\mathfrak{F}} \langle\langle \xi | \eta \rangle, a\rangle$$

for every $\xi \in H$. For $\xi \in H$ and $a \in \ddot{E}$,

$$|\widetilde{\xi}(a)| = \lim_{\eta,\mathfrak{F}} |\langle\langle \xi | \eta \rangle, a\rangle| \leq$$

$$\leq \lim_{\eta,\mathfrak{F}} \|\langle \xi | \eta \rangle\|\, \|a\| \leq \|\xi\|\, \|a\|\,,$$

(Proposition 5.6.1.2 d)). Hence $\widetilde{\xi} \in (\ddot{E})' = E$ and $\|\widetilde{\xi}\| \leq \|\xi\|$ for every $\xi \in H$. Put

$$u : H \longrightarrow E, \quad \xi \longmapsto \widetilde{\xi}.$$

Then $u \in \mathcal{L}(H, E)^{\#}$. Moreover, for $(\xi, x) \in H \times E$ and $a \in \ddot{E}$,

$$\langle \widetilde{\xi x}, a \rangle = \lim_{\eta, \mathfrak{F}} \langle \langle \xi x | \eta \rangle, a \rangle = \lim_{\eta, \mathfrak{F}} \langle \langle \xi | \eta \rangle x, a \rangle =$$

$$= \lim_{\eta, \mathfrak{F}} \langle \langle \xi | \eta \rangle, xa \rangle = \langle \widetilde{\xi}, xa \rangle = \langle \widetilde{\xi} x, a \rangle,$$

so that

$$u(\xi x) = \widetilde{\xi x} = \widetilde{\xi} x = (u\xi)x.$$

Since H is self-dual, there is a $\zeta \in H^{\#}$ such that $u = \langle \cdot | \zeta \rangle$. For every $(a, \xi) \in \ddot{E} \times H$,

$$\langle \widetilde{(a, \xi)}, \zeta \rangle = \langle \langle \zeta | \xi \rangle, a \rangle = \overline{\langle \langle \xi | \zeta \rangle, a^* \rangle} = \overline{\langle u\xi, a^* \rangle} =$$

$$= \overline{\langle \widetilde{\xi}, a^* \rangle} = \lim_{\eta, \mathfrak{F}} \overline{\langle \langle \xi | \eta \rangle, a^* \rangle} = \lim_{\eta, \mathfrak{F}} \langle \langle \eta | \xi \rangle, a \rangle =$$

$$= \lim_{\eta, \mathfrak{F}} \langle \widetilde{(a, \xi)}, \eta \rangle.$$

Hence $\mathfrak{F}$ converges to ζ in $H^{\#}_{\ddot{H}}$ and $H^{\#}_{\ddot{H}}$ is compact.

b $\Rightarrow$ a and the final assertion. Identify H with a subset of $\widehat{H}$ using the injective map

$$H \longrightarrow \widehat{H}, \quad \xi \longmapsto \langle \cdot | \xi \rangle$$

(Proposition 5.6.2.1 a)) and denote by G the closed vector subspace of $(\widehat{H})'$ generated by

$$\{ \widetilde{(a, u)} | (a, u) \in \ddot{E} \times H \},$$

where $\widetilde{(a, u)}$ is defined with respect to $\widehat{H}$ (Theorem 5.6.2.11 g)).

By a $\Rightarrow$ b (and Theorem 5.6.2.11 g)), $\widehat{H}^{\#}_{\ddot{\widehat{H}}}$ is compact. Since $G \subset \ddot{\widehat{H}}$ and since $\widehat{H}_G$ is Hausdorff (Theorem 5.6.2.11 f)),

$$\widehat{H}^{\#}_{\ddot{\widehat{H}}} = \widehat{H}^{\#}_{G}.$$

Assume $\mathbb{K} = \mathbb{C}$ and let $((a_\iota, \xi_\iota))_{\iota \in I}$ be a finite family in $\ddot{E}_+ \times H$. Put

$$a := \sum_{\iota \in I} a_\iota .$$

Take $u \in \widehat{H}^\#$ and $\varepsilon > 0$. In the sequel we use the notation of Proposition 5.6.2.7. Since $q_a H$ is dense in H_a there is an $\eta \in H^\#$ such that

$$\|u_a - q_a \eta\| < \frac{\varepsilon}{(1 + \operatorname{Card} I)(1 + \sup_{\iota \in I} \|\xi_\iota\|)(1 + \|a\|^{\frac{1}{2}})} .$$

By Proposition 5.6.2.9 b), c), for every $\iota \in I$,

$$\|u_{a_\iota} - q_{a_\iota} \eta\| = \|\varphi_{a_\iota, a} u_a - \varphi_{a_\iota, a} q_a \eta\| \leq \|u_a - q_a \eta\| <$$

$$< \frac{\varepsilon}{(1 + \operatorname{Card} I)(1 + \sup_{\iota \in I} \|\xi_\iota\|)(1 + \|a\|^{\frac{1}{2}})} .$$

By Theorem 5.6.2.11 c),e) and Proposition 5.6.2.7 d) (and Corollary 4.2.1.18),

$$\left| \sum_{\iota \in I} < \widetilde{(a_\iota, \langle \cdot | \xi_\iota \rangle)}, u \rangle - \sum_{\iota \in I} \widetilde{\langle (a_\iota, \xi_\iota)}, \eta \rangle \right| \leq$$

$$\leq \sum_{\iota \in I} |\langle \langle u | \xi_\iota \rangle, a_\iota \rangle - \langle \langle \eta | \xi_\iota \rangle, a_\iota \rangle| =$$

$$= \sum_{\iota \in I} \left| \langle u_{a_\iota} | q_{a_\iota} \xi_\iota \rangle_{a_\iota} - \langle q_{a_\iota} \eta | q_{a_\iota} \xi_\iota \rangle_{a_\iota} \right| =$$

$$= \sum_{\iota \in I} \left| \langle u_{a_\iota} - q_{a_\iota} \eta | q_{a_\iota} \xi_\iota \rangle_{a_\iota} \right| \leq$$

$$\leq \sum_{\iota \in I} \|u_{a_\iota} - q_{a_\iota} \eta\| \, \|q_{a_\iota} \xi_\iota\| \leq$$

$$\leq \sum_{\iota \in I} \frac{\varepsilon \|a_\iota\|^{\frac{1}{2}} \|\xi_\iota\|}{(1 + \operatorname{Card} I)\left(1 + \sup_{\iota \in I} \|\xi_\iota\|\right)\left(1 + \|a\|^{\frac{1}{2}}\right)} < \varepsilon .$$

It follows

$$\left| \left\langle \sum_{\iota \in I} \widetilde{(a_\iota, \langle \cdot | \xi_\iota \rangle)}, u \right\rangle \right| \leq$$

$$\leq \left| \left\langle \sum_{\iota \in I} \widetilde{(a_\iota, \xi_\iota)}, \eta \right\rangle \right| + \varepsilon \leq \left\| \sum_{\iota \in I} \widetilde{(a_\iota, \xi_\iota)} \right\| + \varepsilon .$$

Since ε and u are arbitrary,

$$\left\| \sum_{\iota \in I} \widetilde{(a_\iota, \langle \cdot | \xi_\iota \rangle)} \right\| \leq \left\| \sum_{\iota \in I} \widetilde{(a_\iota, \xi_\iota)} \right\| .$$

The reverse inequality being obvious, we have

$$\left\| \sum_{\iota \in I} \widetilde{(a_\iota, \langle \cdot | \xi_\iota \rangle)} \right\| = \left\| \sum_{\iota \in I} \widetilde{(a_\iota, \xi_\iota)} \right\| .$$

Hence the above Banach spaces $\ddot{H}$ and G may be identified. By complexification, this assertion holds also for $\mathbb{K} = \mathbb{R}$.

By Proposition 1.3.6.27, the maps

$$H \longrightarrow (\ddot{H})', \quad \xi \longmapsto (j_H \xi) | \ddot{H} ,$$

$$\widehat{H} \longrightarrow G', \quad u \longmapsto (j_{\widehat{H}} u) | G$$

are isomorphisms of Banach spaces. By the above identification of $\ddot{H}$ and G, we get $H = \widehat{H}$, i.e. H is self-dual. ■

Proposition 5.6.3.4 *Let E be a W^*-algebra , G, H Hilbert right E-modules, and $u \in \mathcal{L}_E(G, H)$.*

a) $(a, \eta) \in \ddot{E} \times H \Rightarrow u' \widetilde{(a, \eta)} = \widetilde{(a, u^* \eta)}$.

b) $u'(\ddot{H}) \subset \ddot{G}$.

c) *The map*

$$G_E \longrightarrow H_H , \quad \xi \longmapsto u\xi$$

is continuous.

d) *If G and H are self-dual then the map*

$$\ddot{H} \longrightarrow \ddot{G} , \quad y' \longmapsto u' y'$$

is the pretranspose of u (Proposition 5.6.3.3).

a) For every $\xi \in G$,

$$\langle \xi, u'\widetilde{(a,\eta)}\rangle = \langle u\xi, \widetilde{(a,\eta)}\rangle = \langle\langle u\xi|\eta\rangle, a\rangle =$$

$$= \langle\langle \xi|u^*\eta\rangle, a\rangle = \langle \xi, \widetilde{(a,u^*\eta)}\rangle$$

so that

$$u'\widetilde{(a,\eta)} = \widetilde{(a,u^*\eta)}\,.$$

b) follows from a).

c) follows from b).

d) By Proposition 5.6.3.3, G and H may be identified with the duals of $\ddot{G}$ and $\ddot{H}$, respectively, and the assertion follows from c) and Corollary 1.3.4.9 a $\Rightarrow$ b. ■

Theorem 5.6.3.5 (8) (Paschke, 1973) *Let E be a W^*-algebra and H, K von Neumann right E-modules.*

a) $(\mathcal{L}_E(H,K))^{\#}_{\ddot{H}}$ *is compact and the map*

$$\mathcal{L}_E(H,K) \longrightarrow (\ddot{\ddot{H}})', \quad u \longmapsto \langle u, \cdot\rangle|\ddot{\ddot{H}}$$

is an isometry of Banach spaces.

b) $\mathcal{L}_E(H)$ *is a W^*-algebra with $\ddot{\ddot{H}}$ as predual.*

c) *Take $a \in \operatorname{Re}\ddot{E}$ and $\xi \in H$ and let $(u, |\widetilde{(a,\xi,\xi)}|)$ be the polar representation of $\widetilde{(a,\xi,\xi)}$. Then*

$$u \in \operatorname{Re}\mathcal{L}_E(H)\,, \quad u^+, u^- \in \operatorname{Pr}\mathcal{L}_E(H)\,,$$

$$\widetilde{(a,\xi,\xi)}^{+} = \widetilde{(a,u^+\xi,u^+\xi)}\,, \quad \widetilde{(a,\xi,\xi)}^{-} = \widetilde{(a,u^-\xi,u^-\xi)}\,,$$

$$|\widetilde{(a,\xi,\xi)}| = \widetilde{(a,|u|\xi,|u|\xi)}\,.$$

d) *Let F be a W^*-algebra,*

$$\varphi : F \longrightarrow \mathcal{L}_E(H)$$

an involutive algebra homomorphism,

$$L := \{\widetilde{(a,\xi,\eta)} \mid (a,\xi,\eta) \in \ddot{E} \times H \times H\},$$

and

$$\psi : L \longrightarrow \ddot{F}$$

a map such that

$$\langle \varphi y, \widetilde{(a,\xi,\eta)}\rangle = \langle y, \psi\widetilde{(a,\xi,\eta)}\rangle$$

for every $y \in F$ and $(a,\xi,\eta) \in \ddot{E} \times H \times H$. Then φ is a W^*-homomorphism and

$$\ddot{\varphi} \mid L = \psi,$$

where $\ddot{\varphi}$ denotes the pretranspose of φ (Corollary 4.4.4.8 c)).

a) $(\mathcal{L}_E(H,K))^{\#}_{\ddot{H}}$ is obviously Hausdorff. Let $\mathfrak{F}$ be an ultrafilter on $(\mathcal{L}_E(H,K))^{\#}$. For $a \in \ddot{E}$, $(\xi,\eta) \in H \times K$, and $u \in \mathcal{L}_E(H,K)$,

$$\widetilde{(a,\eta)}(u\xi) = \langle\langle u\xi|\eta\rangle, a\rangle = \widetilde{(a,\xi,\eta)}(u)$$

so that the map

$$(\mathcal{L}_E(H,K))^{\#}_{\ddot{H}} \longrightarrow \|\xi\| K^{\#}_K, \quad u \longmapsto u\xi$$

is continuous for every $\xi \in H$. By Proposition 5.6.3.3 a $\Rightarrow$ b, the maps

$$v : H \longrightarrow K, \quad \xi \longmapsto \lim_{u,\mathfrak{F}} u\xi,$$

$$w : K \longrightarrow H, \quad \eta \longmapsto \lim_{u,\mathfrak{F}} u^*\eta,$$

where the limits are taken in K_K and H_H, respectively, are well defined and belong to $\mathcal{L}(K,H)^{\#}$ and $\mathcal{L}(H,K)^{\#}$, respectively. For all $a \in \ddot{E}$ and $(\xi,\eta) \in H \times K$,

$$\langle\langle v\xi|\eta\rangle, a\rangle = \widetilde{(a,\eta)}(v\xi) = \lim_{u,\mathfrak{F}} \widetilde{(a,\eta)}(u\xi) =$$

$$= \lim_{u,\mathfrak{F}} \langle\langle u\xi|\eta\rangle, a\rangle = \lim_{u,\mathfrak{F}} \langle\langle \xi|u^*\eta\rangle, a\rangle =$$

$$= \lim_{u,\mathfrak{F}} \overline{\langle\langle u^*\eta|\xi\rangle, a^*\rangle} = \lim_{u,\mathfrak{F}} \overline{\widetilde{(a^*,\xi)}(u^*\eta)} = \overline{\widetilde{(a^*,\xi)}(w\eta)} =$$

$$= \overline{\langle\langle w\eta|\xi\rangle, a^*\rangle} = \langle\langle \xi|w\eta\rangle, a\rangle .$$

It follows

$$\langle v\xi|\eta\rangle = \langle \xi|w\eta\rangle$$

$$v \in \mathcal{L}_E(H,K)^{\#}, \quad v^* = w .$$

By the above, for $a \in \ddot{E}$ and $(\xi,\eta) \in H \times K$,

$$\widetilde{(a,\xi,\eta)}(v) = \langle\langle v\xi|\eta\rangle, a\rangle = \lim_{u,\mathfrak{F}} \langle\langle u\xi|\eta\rangle, a\rangle = \lim_{u,\mathfrak{F}} \widetilde{(a,\xi,\eta)}(u) .$$

Hence $\mathfrak{F}$ converges to v in $(\mathcal{L}_E(H,K))^{\#}_{\ddot{H}}$ and so $(\mathcal{L}_E(H,K))^{\#}_{\dddot{H}}$ is compact. By Proposition 1.3.6.27, the map

$$\mathcal{L}_E(H,K) \longrightarrow (\ddot{H})', \quad u \longmapsto \langle u,\cdot\rangle|\ddot{H}$$

is an isometry of Banach spaces.

b) By Proposition 5.6.3.1 f), $\ddot{H}$ is an involutive unital $\mathcal{L}_E(H)$-submodule of $(\mathcal{L}_E(H))'$. By a), $\mathcal{L}_E(H)$ is a W^*-algebra with $\ddot{H}$ as predual.

c) By Proposition 5.6.3.1 b), $\widetilde{(a,\xi,\xi)} \in \operatorname{Re}\ddot{H}$. By Theorem 4.4.3.9,

$$u \in \operatorname{Re}\mathcal{L}_E(H), \quad u^+, u^- \in \operatorname{Pr}\mathcal{L}_E(H),$$

$$\widetilde{(a,\xi,\xi)}^{+} = u^+\widetilde{(a,\xi,\xi)}u^+, \quad \widetilde{(a,\xi,\xi)}^{-} = u^-\widetilde{(a,\xi,\xi)}u^-,$$

$$|\widetilde{(a,\xi,\xi)}| = |u|\widetilde{(a,\xi,\xi)}|u|,$$

and the assertion follows from Proposition 5.6.3.1 e).

d) For every $y \in F$ and $(a,\xi,\eta) \in \ddot{E} \times H \times H$,

$$\langle y, \psi\widetilde{(a,\xi,\eta)}\rangle = \langle \varphi y, \widetilde{(a,\xi,\eta)}\rangle =$$

$$= \langle \varphi y, j_{\mathcal{L}_E(H)}\widetilde{(a,\xi,\eta)}\rangle = \langle y, \varphi' j_{\mathcal{L}_E(H)}\widetilde{(a,\xi,\eta)}\rangle ,$$

so that

$$\varphi' \circ j_{\mathcal{L}_E(H)} \mid L = \psi ,$$

$$\varphi' \circ j_{\mathcal{L}_E(H)}(L) \subset \ddot{F} .$$

Since $\varphi' \circ j_{\mathcal{L}_E(H)}$ is continuous,

$$\varphi' \circ j_{\mathcal{L}_E(H)}(\ddot{H}) \subset \ddot{F} .$$

Hence φ is a W^*-homomorphism (Proposition 4.4.4.6) and

$$\ddot{\varphi} | L = \psi . \qquad \blacksquare$$

Definition 5.6.3.6 (8) *Let F be a normed space and A a subset of F' such that F_A is Hausdorff. A family $(\xi_\iota)_{\iota \in I}$ in F is called* ***summable in*** *F_A if there is a ξ in F such that*

$$\langle \xi, a \rangle = \sum_{\iota \in I} \langle \xi_\iota, a \rangle$$

for every $a \in A$. ξ is called ***the sum of the family*** *$(\xi_\iota)_{\iota \in I}$* ***in*** *$F_A$ and is denoted by*

$$\sum_{\iota \in I}^{A} \xi_\iota .$$

Proposition 5.6.3.7 (8) *Let E be a W^*-algebra and $(\xi_\iota)_{\iota \in I}$ a family in E_+. The following are equivalent:*

a) *The set $\{\sum_{\iota \in J} \xi_\iota \mid J \in \mathfrak{P}_f(I)\}$ is upper bounded.*

b) *The family $(\xi_\iota)_{\iota \in I}$ is summable in $E_{\ddot{E}}$.*

If these conditions are fulfilled then

$$\sum_{\iota \in I}^{\ddot{E}} \xi_\iota = \bigvee_{J \in \mathfrak{P}_f(I)} \sum_{\iota \in J} \xi_\iota = \lim_{J, \mathfrak{F}_I} \sum_{\iota \in I} \xi_\iota ,$$

where the limit is taken in $E_{\ddot{E}}$.

The proposition follows from Theorem 4.4.1.8 b). $\blacksquare$

Proposition 5.6.3.8 (8) *Let E be a W^*-algebra, H a von Neumann right E-module, and $(\xi_\iota)_{\iota\in I}$ a family in H such that*

$$\langle \xi_\iota | \xi_\lambda \rangle = 0$$

for all distinct $\iota, \lambda \in I$ and such that

$$\left\{ \sum_{\iota\in J} \langle \xi_\iota | \xi_\iota \rangle \;\middle|\; J \in \mathfrak{P}_f(I) \right\}$$

is upper bounded in E.

a) *$(\xi_\iota)_{\iota\in I}$ is summable in $H_{\ddot{H}}$. Denote by ξ its sum in $H_{\ddot{H}}$.*

b) $\iota \in I \Rightarrow \langle \xi_\iota | \xi \rangle = \langle \xi_\iota | \xi_\iota \rangle$.

c) $\langle \xi | \xi \rangle = \bigvee\limits_{J\in\mathfrak{P}_f(I)} \sum\limits_{\iota\in J} \langle \xi_\iota | \xi_\iota \rangle = \sum\limits_{\iota\in I}^{\ddot{E}} \langle \xi_\iota | \xi_\iota \rangle$.

For every $J \in \mathfrak{P}_f(I)$,

$$\left\langle \sum_{\iota\in J} \xi_\iota \;\middle|\; \sum_{\iota\in J} \xi_\iota \right\rangle = \sum_{\iota\in J} \langle \xi_\iota | \xi_\iota \rangle$$

so that

$$\left\{ \sum_{\iota\in J} \xi_\iota \;\middle|\; J \in \mathfrak{P}_f(I) \right\}$$

is a bounded set in H. Let $\mathfrak{F}, \mathfrak{G}$ be ultrafilters on $\mathfrak{P}_f(I)$ finer than $\mathfrak{F}_I$ and put

$$\xi := \lim_{J,\mathfrak{F}} \sum_{\iota\in J} \xi_\iota, \quad \eta := \lim_{J,\mathfrak{G}} \sum_{\iota\in J} \xi_\iota$$

where the limits are taken in $H_{\ddot{H}}$ (Proposition 5.6.3.3 a $\Rightarrow$ b).

Take $\lambda \in I$ and $a \in \ddot{E}$. Then

$$\langle \langle \xi_\lambda | \xi \rangle, a \rangle = \lim_{J,\mathfrak{F}} \langle \langle \xi_\lambda | \sum_{\iota\in J} \xi_\iota \rangle, a \rangle = \langle \langle \xi_\lambda | \xi_\lambda \rangle, a \rangle$$

so that

$$\langle \xi_\lambda | \xi \rangle = \langle \xi_\lambda | \xi_\lambda \rangle .$$

It follows

$$\langle\langle\xi|\eta\rangle, a\rangle = \lim_{J,\mathfrak{G}} \langle\langle\xi| \sum_{\iota\in J} \xi_\iota\rangle, a\rangle =$$

$$= \lim_{J,\mathfrak{G}} \langle \sum_{\iota\in J} \langle\xi_\iota|\xi_\iota\rangle, a\rangle$$

and so (Proposition 5.6.3.7)

$$\langle\xi|\eta\rangle = \bigvee_{J\in\mathfrak{P}_f(I)} \sum_{\iota\in J} \langle\xi_\iota|\xi_\iota\rangle = \sum_{\iota\in I}^{\ddot{E}} \langle\xi_\iota|\xi_\iota\rangle .$$

We deduce

$$\langle\xi-\eta|\xi-\eta\rangle = \langle\xi|\xi\rangle - \langle\xi|\eta\rangle - \langle\eta|\xi\rangle + \langle\eta|\eta\rangle = 0$$

$$\xi = \eta .$$

Hence $(\xi_\iota)_{\iota\in I}$ is summable in $H_{\ddot{H}}$ and ξ is its sum $H_{\ddot{H}}$ ■

Proposition 5.6.3.9 (8) *Let E be a C^*-algebra and H an inner-product right E-module. Then the following are equivalent for every $\xi \in H$:*

a) $\langle\xi|\xi\rangle \in \Pr E$.

b) $\xi\langle\xi|\xi\rangle = \xi$.

c) $\eta \in H \Rightarrow \langle\xi|\xi\rangle\langle\eta|\xi\rangle = \langle\eta|\xi\rangle$.

a $\Rightarrow$ b . By Proposition 5.6.1.2 a),b),

$$\langle\xi - \xi\langle\xi|\xi\rangle|\xi - \xi\langle\xi|\xi\rangle\rangle =$$

$$= \langle\xi|\xi\rangle - \langle\xi|\xi\rangle^2 - \langle\xi|\xi\rangle^2 + \langle\xi|\xi\rangle^3 = 0 ,$$

so that

$$\xi\langle\xi|\xi\rangle = \xi .$$

b $\Rightarrow$ c . By Proposition 5.6.1.2 a) ,

$$\langle\xi|\xi\rangle\langle\eta|\xi\rangle = \langle\eta|\xi\langle\xi|\xi\rangle\rangle = \langle\eta|\xi\rangle .$$

c $\Rightarrow$ a . Putting $\eta := \xi$, we get

$$\langle\xi|\xi\rangle\langle\xi|\xi\rangle = \langle\xi|\xi\rangle ,$$

so that

$$\langle\xi|\xi\rangle \in \Pr E .$$ ■

Proposition 5.6.3.10 (8) *Let E be a $\mathbb{C}$-order σ-complete C^*-algebra, H an inner-product right E-module, $\xi \in H$, and A a Borel set of $\mathbb{R}_+$ with $0 \notin \overline{A}$. Put*

$$f : \sigma(\langle\xi|\xi\rangle) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \begin{cases} \frac{1}{\sqrt{\alpha}} & \text{if } \alpha \in A \\ 0 & \text{if } \alpha \notin A, \end{cases}$$

$$g : \sigma(\langle\xi|\xi\rangle) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \begin{cases} \alpha & \text{if } \alpha \notin A \\ 0 & \text{if } \alpha \in A, \end{cases}$$

$$x := f(\langle\xi|\xi\rangle).$$

a) $\langle\xi x|\xi x\rangle = e_A(\langle\xi|\xi\rangle) \in \Pr E$.

b) $\langle\xi - \xi\langle\xi x|\xi x\rangle|\xi - \xi\langle\xi x|\xi x\rangle\rangle = \langle\xi|\xi\rangle - \langle\xi|\xi\rangle\langle\xi x|\xi x\rangle = g(\langle\xi|\xi\rangle)$.

c) *If $\sigma(\langle\xi|\xi\rangle) \subset A \cup \{0\}$ then*

$$\xi\langle\xi x|\xi x\rangle = \xi.$$

d) *If we put*

$$f_n : \sigma(\langle\xi|\xi\rangle) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \begin{cases} \frac{1}{\sqrt{\alpha}} & \text{if } \alpha > \frac{1}{n} \\ 0 & \text{if } \alpha \leq \frac{1}{n}, \end{cases}$$

$$x_n := f_n(\langle\xi|\xi\rangle)$$

for every $n \in \mathbb{N}$ then

$$\lim_{n\to\infty} \xi\langle\xi x_n|\xi x_n\rangle = \xi.$$

e) *If E is finite-dimensional and $\xi \neq 0$ then*

$$\langle\xi y|\xi y\rangle \in \Pr E\backslash\{0\}, \quad \xi\langle\xi y|\xi y\rangle = \xi$$

for some $y \in E$.

a) By Proposition 5.6.1.2 b) and Corollary 4.3.2.5 c),

$$\langle\xi x|\xi x\rangle = x\langle\xi|\xi\rangle x = e_A(\langle\xi|\xi\rangle) \in \Pr E.$$

b) By a) and Proposition 5.6.1.2 b),

$$\langle \xi - \xi\langle \xi x|\xi x\rangle|\xi - \xi\langle \xi x|\xi x\rangle\rangle =$$

$$= \langle \xi|\xi\rangle - \langle \xi x|\xi x\rangle\langle \xi|\xi\rangle - \langle \xi|\xi\rangle\langle \xi x|\xi x\rangle + \langle \xi x|\xi x\rangle\langle \xi|\xi\rangle\langle \xi x|\xi x\rangle =$$

$$= \langle \xi|\xi\rangle - \langle \xi|\xi\rangle\langle \xi x|\xi x\rangle = g(\langle \xi|\xi\rangle)\,.$$

c) follows from b).

d) By b),

$$\|\xi - \xi\langle \xi x_n|\xi x_n\rangle\|^2 = \|\langle \xi - \xi\langle \xi x_n|\xi x_n\rangle|\xi - \xi\langle \xi x_n|\xi x_n\rangle\rangle\| \leq \frac{1}{n}$$

for every $n \in \mathbb{N}$ so that

$$\lim_{n\to\infty} \xi\langle \xi x_n|\xi x_n\rangle = \xi\,.$$

e) By Proposition 2.2.1.17, $\sigma(\langle \xi|\xi\rangle)$ is finite and the assertion follows from a) and c). ■

Definition 5.6.3.11 (8) *Let E be a C^*-algebra and H an inner-product right E-module. A subset A of H is called* ***Fourier set*** *if*

$$\langle \xi|\xi\rangle \in \Pr E\backslash\{0\}$$

for every $\xi \in A$ and

$$\langle \xi|\eta\rangle = 0$$

for all distinct $\xi, \eta \in A$. A maximal Fourier set is called a ***Fourier basis of*** *H.*

By Zorn's Lemma, every inner-product right E-module has a Fourier basis.

Proposition 5.6.3.12 (8) *Let H be an inner-product right E-module, A a finite Fourier set of H, and $\eta, \zeta \in H$.*

a) $$\left\langle \sum_{\xi\in A} \xi\langle \eta|\xi\rangle \middle| \sum_{\xi\in A} \xi\langle \zeta|\xi\rangle \right\rangle = \sum_{\xi\in A} \langle \xi|\zeta\rangle\langle \eta|\xi\rangle\,.$$

b) $$\left\langle \eta - \sum_{\xi\in A} \xi\langle \eta|\xi\rangle \middle| \eta - \sum_{\xi\in A} \xi\langle \eta|\xi\rangle \right\rangle = \langle \eta|\eta\rangle - \sum_{\xi\in A} \langle \xi|\eta\rangle\langle \eta|\xi\rangle\,.$$

a) By Proposition 5.6.1.2 b) and Proposition 5.6.3.9 a $\Rightarrow$ c,

$$\left\langle \sum_{\xi\in A} \xi\langle\eta|\xi\rangle \,\middle|\, \sum_{\xi\in A} \xi\langle\zeta|\xi\rangle \right\rangle =$$

$$= \sum_{\xi,\xi'\in A} \langle\xi'|\zeta\rangle\langle\xi|\xi'\rangle\langle\eta|\xi\rangle = \sum_{\xi\in A} \langle\xi|\zeta\rangle\langle\eta|\xi\rangle .$$

b) By a) and Proposition 5.6.1.2 a),

$$\left\langle \eta - \sum_{\xi\in A} \xi\langle\eta|\xi\rangle \,\middle|\, \eta - \sum_{\xi\in A} \xi\langle\eta|\xi\rangle \right\rangle =$$

$$= \langle\eta|\eta\rangle - \sum_{\xi\in A} \langle\xi|\eta\rangle\langle\eta|\xi\rangle - \sum_{\xi\in A} \langle\xi|\eta\rangle\langle\eta|\xi\rangle + \sum_{\xi\in A} \langle\xi|\eta\rangle\langle\eta|\xi\rangle =$$

$$= \langle\eta|\eta\rangle - \sum_{\xi\in A} \langle\xi|\eta\rangle\langle\eta|\xi\rangle . \qquad \blacksquare$$

Theorem 5.6.3.13 (8) *Let H be a Hilbert right E-module, A a Fourier set of H, and $\eta,\zeta \in H$.*

a) *If $(x_\xi)_{\xi\in A} \in \prod_{\xi\in A} \langle\xi|\xi\rangle E$ is such that $(x_\xi^* x_\xi)_{\xi\in A}$ is summable in E then $(\xi x_\xi)_{\xi\in A}$ is summable in H. Moreover,*

$$\sum_{\xi\in A} \xi x_\xi = 0$$

implies $x_\xi = 0$ for every $\xi \in A$ and so A is finite if H is finite-dimensional.

b) *If E is $\mathbb{C}$-order σ-complete, A a Fourier basis of H, and*

$$\langle\eta|\xi\rangle = 0$$

for every $\xi \in A$ then $\eta = 0$.

c) *If E is a W^*-algebra then*

$$\sum_{\xi\in A}^{E} \langle\xi|\eta\rangle\langle\eta|\xi\rangle \leq \langle\eta|\eta\rangle .$$

d) *If E is finite-dimensional then $(\xi\langle\eta|\xi\rangle)_{\xi\in A}$ is summable in H.*

e) *If E is a W^*-algebra and H is self-dual then $(\xi\langle\eta|\xi\rangle)_{\xi\in A}$ is summable in H_H.*

f) *If E is a W^*-algebra, A is a Fourier basis of H, and H is self-dual then*

$$\eta = \sum_{\xi\in A}^{H} \xi\langle\eta|\xi\rangle .$$

$$\langle\eta|\zeta\rangle = \sum_{\xi\in A}^{E} \langle\xi|\zeta\rangle\langle\eta|\xi\rangle .$$

a) Take $\varepsilon > 0$. There is a $B \in \mathfrak{P}_f(A)$ such that

$$\left\| \sum_{\xi\in C} x_\xi^* x_\xi \right\| < \varepsilon$$

for every $C \in \mathfrak{P}_f(A\backslash B)$ (Proposition 1.1.6.6). It follows

$$\left\| \sum_{\xi\in C} \xi x_\xi \right\|^2 = \left\| \left\langle \sum_{\xi\in C} \xi x_\xi \,\middle|\, \sum_{\xi\in C} \xi x_\xi \right\rangle \right\| =$$

$$= \left\| \sum_{\xi,\eta\in C} x_\eta^* \langle\xi|\eta\rangle x_\xi \right\| = \left\| \sum_{\xi\in C} x_\xi^* x_\xi \right\| < \varepsilon$$

for every $C \in \mathfrak{P}_f(A\backslash B)$ (Proposition 5.6.1.2 b)). Since H is complete, the family $(\xi x_\xi)_{\xi\in A}$ is summable (Proposition 1.1.6.6).

From

$$\sum_{\xi\in A} \xi x_\xi = 0$$

it follows

$$0 = \left\langle \sum_{\xi\in A} \xi x_\xi \,\middle|\, \eta \right\rangle = \sum_{\xi\in A} \langle\xi|\eta\rangle x_\xi = x_\eta$$

for every $\eta \in A$.

b) Assume $\eta \neq 0$. By Proposition 5.6.3.10 a),d), there is an $x \in E$ with

$$\langle\eta x|\eta x\rangle \in \Pr E\backslash\{0\} .$$

Since

$$\langle \eta x|\xi\rangle = \langle \eta|\xi\rangle x = 0$$

for every $\xi \in A$, $A \cup \{\eta x\}$ is a Fourier set of H, contradicting the maximality of A.

c) By Proposition 5.6.3.12 b), the set

$$\left\{ \sum_{\xi\in B} \langle \xi|\eta\rangle\langle \eta|\xi\rangle \;\middle|\; B \in \mathfrak{P}_f(A) \right\}$$

is upper bounded by $\langle \eta|\eta\rangle$ so that by Proposition 5.6.3.7,

$$\sum_{\xi\in A}^{\ddot{E}} \langle \xi|\eta\rangle\langle \eta|\xi\rangle \leq \langle \eta|\eta\rangle .$$

d) By c) and Minkowski's Theorem, the family

$$(\langle \xi|\eta\rangle\langle \eta|\xi\rangle)_{\xi\in A}$$

is summable. By a) and Proposition 5.6.3.9 a $\Rightarrow$ c, $(\xi\langle \eta|\xi\rangle)_{\xi\in A}$ is summable.

e) By Proposition 5.6.3.12 b), the set

$$\left\{ \sum_{\xi\in B} \langle \xi|\eta\rangle\langle \eta|\xi\rangle \;\middle|\; B \in \mathfrak{P}_f(A) \right\}$$

is upper bounded so that by Proposition 5.6.3.8 a), the family $(\xi\langle \eta|\xi\rangle)_{\xi\in A}$ is summable in $H_{\ddot{H}}$.

f) Put (by e))

$$\zeta := \eta - \sum_{\xi\in A}^{\ddot{H}} \xi\langle \eta|\xi\rangle .$$

For every $\xi_0 \in A$ and $a \in \ddot{E}$,

$$\langle\langle \zeta|\xi_0\rangle, a\rangle = \langle\langle \eta|\xi_0\rangle, a\rangle - \sum_{\xi\in A} \langle\langle \xi\langle \eta|\xi\rangle|\xi_0\rangle, a\rangle =$$

$$\langle\langle \eta|\xi_0\rangle, a\rangle - \sum_{\xi\in A} \langle\langle \xi|\xi_0\rangle\langle \eta|\xi\rangle, a\rangle =$$

$$= \langle\langle \eta|\xi_0\rangle, a\rangle - \langle\langle \eta|\xi_0\rangle, a\rangle = 0$$

(Proposition 5.6.3.9 a $\Rightarrow$ c). Hence

$$\langle\zeta|\xi\rangle = 0$$

for every $\xi \in A$. By b), $\zeta = 0$, i.e.

$$\eta = \sum_{\xi\in A}^{H} \xi\langle\eta|\xi\rangle .$$

For every $a \in \ddot{E}$,

$$\langle\langle\eta|\zeta\rangle, a\rangle = \langle\eta, \widetilde{(a,\zeta)}\rangle = \sum_{\xi\in A} \langle\xi\langle\eta|\xi\rangle, \widetilde{(a,\zeta)}\rangle =$$

$$= \sum_{\xi\in A} \langle\langle\xi\langle\eta|\xi\rangle|\zeta\rangle, a\rangle = \sum_{\xi\in A} \langle\langle\xi|\zeta\rangle\langle\eta|\xi\rangle, a\rangle .$$

Hence

$$\langle\eta|\zeta\rangle = \sum_{\xi\in A}^{E} \langle\xi|\zeta\rangle\langle\eta|\xi\rangle . \qquad \blacksquare$$

Proposition 5.6.3.14 (8) *Let E be a W^*-algebra, G and H von Neumann right E-modules, and A a Fourier set of G. For each $\xi \in A$ let $\widetilde{\xi}$ be an element of H such that*

$$\langle\xi|\eta\rangle = \langle\widetilde{\xi}|\widetilde{\eta}\rangle$$

for all $\xi, \eta \in A$. For every $x \in \ell^\infty(A)$, put

$$\vec{x} : G \longrightarrow H , \quad \eta \longmapsto \sum_{\xi\in A}^{H} x(\xi)\widetilde{\xi}\langle\eta|\xi\rangle ,$$

$$\overleftarrow{x} : H \longrightarrow G , \quad \zeta \longmapsto \sum_{\xi\in A}^{G} x(\xi)\xi\langle\eta|\widetilde{\xi}\rangle$$

(Theorem 5.6.3.13 a), Proposition 5.6.3.8 a), Proposition 5.6.3.9 a $\Rightarrow$ b).

a) *For $\eta \in G$, $\zeta \in H$, and $x \in \ell^\infty(A)$,*

$$\langle\vec{x}\eta|\zeta\rangle = \sum_{\xi\in A}^{E} x(\xi)\langle\widetilde{\xi}|\zeta\rangle\langle\eta|\xi\rangle ,$$

$$\langle\overleftarrow{x}\zeta|\eta\rangle = \sum_{\xi\in A}^{E} x(\xi)\langle\xi|\eta\rangle\langle\zeta|\widetilde{\xi}\rangle .$$

b) *If* $x \in \ell^\infty(A)$ *then*

$$\vec{x} \in \mathcal{L}_E(G,H)\,, \quad \vec{x}^* = \overleftarrow{\overline{x}}\,, \quad \|\vec{x}\| = \|x\|_\infty\,.$$

a) For $a \in \ddot{E}$,

$$\langle\langle \vec{x}\eta|\zeta\rangle, a\rangle = \langle\widetilde{\langle a,\zeta\rangle}, \vec{x}\eta\rangle =$$

$$= \sum_{\xi\in A} x(\xi)\langle\widetilde{\langle a,\zeta\rangle}\widetilde{\xi}\langle\eta|\xi\rangle\rangle = \sum_{\xi\in A} x(\xi)\langle\langle\widetilde{\xi}\langle\eta|\xi\rangle|\zeta\rangle, a\rangle =$$

$$= \sum_{\xi\in A} x(\xi)\langle\langle\widetilde{\xi}|\zeta\rangle\langle\eta|\xi\rangle, a\rangle\,,$$

$$\langle\langle \overleftarrow{x}\zeta|\eta\rangle, a\rangle = \langle\widetilde{\langle a,\eta\rangle}, \overleftarrow{x}\zeta\rangle =$$

$$= \sum_{\xi\in A} x(\xi)\langle\widetilde{\langle a,\eta\rangle}, \xi\langle\zeta|\widetilde{\xi}\rangle\rangle = \sum_{\xi\in A} x(\xi)\langle\langle\xi\langle\zeta|\widetilde{\xi}\rangle|\eta\rangle, a\rangle =$$

$$= \sum_{\xi\in A} x(\xi)\langle\langle\xi|\eta\rangle\langle\zeta|\widetilde{\xi}\rangle, a\rangle\,,$$

so that

$$\langle\vec{x}\eta|\zeta\rangle = \sum_{\xi\in A}^{\ddot{E}} x(\xi)\langle\widetilde{\xi}|\zeta\rangle\langle\eta|\xi\rangle\,,$$

$$\langle\overleftarrow{x}\zeta|\eta\rangle = \sum_{\xi\in A}^{\ddot{E}} x(\xi)\langle\xi|\eta\rangle\langle\zeta|\widetilde{\xi}\rangle\,.$$

b) By a), for all $(\eta,\zeta) \in G \times H$,

$$\langle\eta|\overleftarrow{\overline{x}}\zeta\rangle = \langle\overleftarrow{\overline{x}}\zeta|\eta\rangle^* =$$

$$= \sum_{\xi\in A}^{\ddot{E}} x(\xi)\langle\widetilde{\xi}|\zeta\rangle\langle\eta|\xi\rangle = \langle\vec{x}\eta|\zeta\rangle$$

so that

$$\vec{x} \in \mathcal{L}_E(G,H)\,, \quad \vec{x}^* = \overleftarrow{\overline{x}}\,.$$

For every $\xi \in A$,

$$\|\vec{x}\| \geq \|\vec{x}\xi\| = |x(\xi)|$$

so that

$$\|\vec{x}\| \geq \|x\|_\infty .$$

For $\eta \in G$,

$$\xi \in A \Longrightarrow \langle \vec{x}\eta | \tilde{\xi} \rangle = x(\xi)\langle \eta | \xi \rangle$$

so that by Theorem 5.6.3.13 e),f)

$$\langle \vec{x}\eta | \vec{x}\eta \rangle = \sum_{\xi \in A}^{\ddot{E}} \langle \tilde{\xi} | \vec{x}\eta \rangle \langle \vec{x}\eta | \tilde{\xi} \rangle =$$

$$= \sum_{\xi \in A}^{\ddot{E}} |x(\xi)|^2 \langle \xi | \eta \rangle \langle \eta | \xi \rangle \leq \|x\|_\infty^2 \sum_{\xi \in A}^{\ddot{E}} \langle \xi | \eta \rangle \langle \eta | \xi \rangle \leq \|x\|_\infty^2 \langle \eta | \eta \rangle$$

$$\|\vec{x}\eta\|^2 \leq \|x\|_\infty^2 \|\eta\|^2$$

(Corollary 4.2.1.18),

$$\|\vec{x}\| \leq \|x\|_\infty ,$$

$$\|\vec{x}\| = \|x\|_\infty . \qquad \blacksquare$$

Proposition 5.6.3.15 (8) *Let E be a W^*-algebra, H a von Neumann right E-module, and A a Fourier basis of H. For every $x \in \ell^\infty(A)$ put*

$$\vec{x} : H \longrightarrow H , \quad \eta \longmapsto \sum_{\xi \in A}^{\ddot{H}} x(\xi)\xi\langle \eta | \xi \rangle$$

(Theorem 5.6.3.13 e), Proposition 5.6.3.8 a), Proposition 5.6.3.9 a $\Rightarrow$ b)).

a) *The map*

$$\varphi : \ell^\infty(A) \longrightarrow \mathcal{L}_E(H) , \quad x \longmapsto \vec{x}$$

is an injective unital W^*-homomorphism (Theorem 5.6.3.5 b)) with

$$\ddot{\varphi}(\widetilde{(a,\eta,\zeta)}) = (\langle \langle \xi | \zeta \rangle \langle \eta | \xi \rangle, a \rangle)_{\xi \in A} \in \ell^1(A)$$

for every $(a, \eta, \zeta) \in A \times H \times H$.

b) *If* $x \in \ell^\infty(A)$ *and* $B := \{x \neq 0\}$ *then* $\vec{e}_B$ *is the carrier of* $\vec{x}$.

a) By Proposition 5.6.3.14 b), φ is well-defined, injective and involutive. Take $x, y \in \ell^\infty(A)$. For $\eta, \zeta \in H$,

$$\langle \vec{x}\vec{y}\eta|\zeta\rangle = \sum_{\xi\in A}^{\ddot{E}} x(\xi)\langle\xi|\zeta\rangle\langle\vec{y}\eta|\xi\rangle =$$

$$= \sum_{\xi\in A}^{\ddot{E}} x(\xi)\langle\xi|\zeta\rangle\langle y(\xi)\eta|\xi\rangle =$$

$$= \sum_{\xi\in A}^{\ddot{E}} x(\xi)y(\xi)\langle\xi|\zeta\rangle\langle\eta|\xi\rangle = \langle\vec{xy}\eta|\zeta\rangle$$

(Proposition 5.6.3.14 a)), so that

$$\vec{xy} = \vec{x}\vec{y}.$$

Hence φ is an algebra homomorphism and by Theorem 5.6.3.13 f), it is unital.

For $x \in \ell^\infty(A)$, by a),

$$\langle\widetilde{\langle a, \eta, \zeta\rangle}, \vec{x}\rangle = \langle\langle\vec{x}\eta|\zeta\rangle, a\rangle =$$

$$= \sum_{\xi\in A} x(\xi)\langle\langle\xi|\zeta\rangle\langle\eta|\xi\rangle, a\rangle$$

so that, by Theorem 5.6.3.5 d), φ is a W^*-homomorphism and

$$\ddot{\varphi}(\widetilde{\langle a, \eta, \zeta\rangle}) = (\langle\langle\xi|\zeta\rangle\langle\eta|\xi\rangle, a\rangle)_{\xi\in A} \in \ell^1(A).$$

b) By a), $\vec{e}_B \in \Pr\mathcal{L}_E(H)$ and

$$\vec{e}_B\vec{x} = \vec{x}.$$

Let $u \in \mathcal{L}_E(H)$ with

$$u\vec{x} = 0.$$

Then for $\xi \in B$,

$$0 = (u\vec{x})\xi = u(\vec{x}\xi) = u(x(\xi)\xi) = x(\xi)u\xi,$$

$$u\xi = 0$$

and so

$$u\vec{e}_B\eta = u\left(\sum_{\xi\in B}^{H}\xi\langle\eta|\xi\rangle\right) = \sum_{\xi\in B}^{H}(u\xi)\langle\eta|\xi\rangle = 0$$

for every $\eta \in H$ (Proposition 5.6.3.4 c)). Hence

$$u\vec{e}_B = 0$$

and $\vec{e}_B$ is the right carrier of $\vec{x}$. By Corollary 4.3.3.9, $\vec{e}_B$ is the carrier of $\vec{x}$. ■

Proposition 5.6.3.16 (8) *Let E be a finite-dimensional C^*-algebra, H a Hilbert right E-module, and $\mathfrak{A}$ the set of Fourier sets A of H such that $\langle\xi|\xi\rangle$ is a minimal element of $\Pr E\backslash\{0\}$ for every $\xi \in A$. Then every maximal element of $\mathfrak{A}$ is a Fourier basis of E. In particular, H has a Fourier basis belonging to $\mathfrak{A}$.*

Let A be a maximal element of $\mathfrak{A}$ and assume A is not a Fourier basis of H. Then there is a $\xi \in H$ such that

$$\langle\xi|\xi\rangle \in \Pr E\backslash\{0\},$$

$$\eta \in A \Longrightarrow \langle\xi|\eta\rangle = 0.$$

By Proposition 4.2.7.22, there is a finite set B of minimal elements of $\Pr E\backslash\{0\}$ such that

$$\sum_{p\in B} p = \langle\xi|\xi\rangle.$$

Take $p \in B$. Then

$$\langle\xi p|\xi p\rangle = p\langle\xi|\xi\rangle p = p$$

(Corollary 4.2.7.6 a ⇒e, Proposition 5.6.1.2 b)) and

$$\langle\xi p|\eta\rangle = \langle\xi|\eta\rangle p = 0$$

for every $\eta \in A$. Hence $A \cup \{\xi p\} \in \mathfrak{A}$ and this contradicts the maximality of A. ■

Proposition 5.6.3.17 *The following are equivalent for every family $(x_\iota)_{\iota\in I}$ in a W^*-algebra E:*

a) *$(x_\iota)_{\iota\in I}$ is summable in $E_{\ddot{E}}$.*

b) *For every $a\in\ddot{E}$ and $\varepsilon>0$ there is a $J\in\mathfrak{P}_f(I)$ such that*

$$\left|\left\langle\sum_{\iota\in K}x_\iota,a\right\rangle\right|<\varepsilon$$

for every $K\in\mathfrak{P}_f(I\backslash J)$.

If these conditions are fulfilled then:

c) $\displaystyle\sum_{\iota\in I}^{\ddot{E}}x_\iota^*=\left(\sum_{\iota\in I}^{\ddot{E}}x_\iota\right)^*$.

d) $\displaystyle x\in E\Rightarrow\sum_{\iota\in I}^{\ddot{E}}(xx_\iota)=x\sum_{\iota\in I}^{\ddot{E}}x_\iota\,,\quad\sum_{\iota\in I}^{\ddot{E}}(x_\iota x)=\left(\sum_{\iota\in I}^{\ddot{E}}x_\iota\right)x$.

e) *$(x_\iota)_{\iota\in J}$ is summable in $E_{\ddot{E}}$ for every J, the map*

$$\mathfrak{P}(I)\longrightarrow E_{\ddot{E}}\,,\quad J\longmapsto\sum_{\iota\in J}^{\ddot{E}}x_\iota$$

is continuous, where $\mathfrak{P}(I)$ is identified with $\{0,1\}^I$, and

$$\left\{\sum_{\iota\in J}^{\ddot{E}}x_\iota\mid J\subset I\right\}$$

is a compact set of $E_{\ddot{E}}$ and a bounded set of E.

a $\Rightarrow$ b. There is a $J\in\mathfrak{P}_f(I)$ such that

$$\left|\left\langle\sum_{\iota\in K}x_\iota,a\right\rangle-\langle x,a\rangle\right|<\frac{\varepsilon}{2}$$

for every $K\in\mathfrak{P}_f(I)$, $J\subset K$. Then for every $K\in\mathfrak{P}_f(I\backslash J)$,

$$J\cup K\in\mathfrak{P}_f(I)\,,\quad J\subset K\,,$$

so that

$$\left|\left\langle\sum_{\iota\in K}x_\iota,a\right\rangle\right|=\left|\left\langle\sum_{\iota\in J\cup K}x_\iota,a\right\rangle-\left\langle\sum_{\iota\in J}x_\iota,a\right\rangle\right|\leq$$

$$\leq \left|\left\langle \sum_{\iota\in J\cup K} x_\iota, a\right\rangle - \langle x, a\rangle\right| + \left|\left\langle \sum_{\iota\in J} x_\iota, a\right\rangle - \langle x, a\rangle\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

b $\Rightarrow$ a. Take $a \in \ddot{E}$. By b), there is a $J \in \mathfrak{P}_f(I)$ such that

$$\left|\left\langle \sum_{\iota\in K} x_\iota, a\right\rangle\right| < 1$$

for every $K \in \mathfrak{P}_f(I\backslash J)$. Put

$$\alpha := \sup_{K\subset J} \left|\left\langle \sum_{\iota\in K} x_\iota, a\right\rangle\right| < \infty .$$

Then for $K \in \mathfrak{P}_f(I)$,

$$\left|\left\langle \sum_{\iota\in K} x_\iota, a\right\rangle\right| \leq \left|\left\langle \sum_{\iota\in J\cap K} x_\iota, a\right\rangle\right| + \left|\left\langle \sum_{\iota\in K\backslash J} x_\iota, a\right\rangle\right| < \alpha + 1 .$$

Since a is arbitrary, it follows from the theorem of Banach–Steinhaus (Theorem 1.4.1.2) that

$$\left\{\sum_{\iota\in K} x_\iota \mid K \in \mathfrak{P}_f(I)\right\}$$

is a bounded set of E. Hence, by the Alaoglu–Bourbaki Theorem, it is a relatively compact set of $E_{\ddot{E}}$.

Let x be a point of adherence in $E_{\ddot{E}}$ of the image of the filter $\mathfrak{F}_I$ with respect to the map

$$\mathfrak{P}_f(I) \longrightarrow E, \quad K \longmapsto \sum_{\iota\in K} x_\iota .$$

Let $a \in \ddot{E}$ and $\varepsilon > 0$. By b), there is a $J_0 \in \mathfrak{P}_f(I)$ such that

$$\left|\left\langle \sum_{\iota\in K} x_\iota, a\right\rangle\right| < \frac{\varepsilon}{2}$$

for every $K \in \mathfrak{P}_f(I)$, $K \subset I\backslash J_0$. Further there is a $J \in \mathfrak{P}_f(I)$, $J_0 \subset J$, such that

$$\left|\left\langle \sum_{\iota\in J} x_\iota, a\right\rangle - \langle x, a\rangle\right| < \frac{\varepsilon}{2} .$$

Then for $K \in \mathfrak{P}_f(I)$, $J \subset K$,

$$\left|\left\langle \sum_{\iota\in K} x_\iota, a \right\rangle - \langle x, a\rangle\right| \leq$$

$$\leq \left|\left\langle \sum_{\iota\in J} x_\iota, a \right\rangle - \langle x, a\rangle\right| + \left|\left\langle \sum_{\iota\in K\setminus J} x_\iota, a \right\rangle\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Hence $(x_\iota)_{\iota\in I}$ is summable in $E_{\ddot{E}}$ and x is its sum.

c) follows from a) and Proposition 4.4.1.3.

d) For every $a \in \ddot{E}$,

$$\left\langle x \sum_{\iota\in I}^{\ddot{E}} x_\iota, a \right\rangle = \left\langle \sum_{\iota\in I}^{\ddot{E}} x_\iota, ax \right\rangle = \sum_{\iota\in I} \langle x_\iota, ax\rangle =$$

$$= \sum_{\iota\in I} \langle xx_\iota, a\rangle .$$

Since a is arbitrary,

$$\sum_{\iota\in I}^{\ddot{E}} (xx_\iota) = x \sum_{\iota\in I}^{\ddot{E}} x_\iota .$$

Similarly,

$$\sum_{\iota\in I}^{\ddot{E}} (x_\iota x) = \left(\sum_{\iota\in I}^{\ddot{E}} x_\iota \right) x .$$

e) By a $\Leftrightarrow$ b, $(x_\iota)_{\iota\in J}$ is summable in $E_{\ddot{E}}$ for every $J \subset I$. Take $a \in \ddot{E}$ and $\varepsilon > 0$. By b), there is a $J \in \mathfrak{P}_f(I)$ such that

$$\left|\left\langle \sum_{\iota\in K} x_\iota, a \right\rangle\right| < \frac{\varepsilon}{2}$$

for every $K \in \mathfrak{P}_f(I\setminus J)$. By continuity,

$$\left|\left\langle \sum_{\iota\in K}^{\ddot{E}} x_\iota, a \right\rangle\right| \leq \frac{\varepsilon}{2}$$

for every $K \subset I\setminus J$. Let $K_1, K_2 \in \mathfrak{P}(I)$ with

$$(K_1\setminus K_2) \cup (K_2\setminus K_1) \subset I\setminus J .$$

Then

$$\left|\left\langle \sum_{\iota\in K_1}^{\ddot{E}} x_\iota, a \right\rangle - \left\langle \sum_{\iota\in K_2}^{\ddot{E}} x_\iota, a \right\rangle\right| =$$

$$= \left|\left\langle \sum_{\iota\in K_1\setminus K_2}^{\ddot{E}} x_\iota, a \right\rangle - \left\langle \sum_{\iota\in K_2\setminus K_1}^{\ddot{E}} x_\iota, a \right\rangle\right| \leq$$

$$\leq \left|\left\langle \sum_{\iota\in K_1\setminus K_2} x_\iota, a \right\rangle\right| + \left|\left\langle \sum_{\iota\in K_2\setminus K_1} x_\iota, a \right\rangle\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Hence the map

$$\mathfrak{P}(I) \longrightarrow \mathbb{K}, \quad J \longmapsto \left\langle \sum_{\iota\in J}^{\ddot{E}} x_\iota, a \right\rangle$$

is continuous. Since a is arbitrary, the map

$$\mathfrak{P}(I) \longrightarrow E_{\ddot{E}}, \quad J \longmapsto \sum_{\iota\in J}^{\ddot{E}} x_\iota$$

is continuous. It follows that

$$\left\{ \sum_{\iota\in J}^{\ddot{E}} x_\iota \mid J \subset I \right\}$$

is a compact set of $E_{\ddot{E}}$. By Banach-Steinhaus Theorem (Theorem 1.4.1.2), it is a bounded set of E. ■

Corollary 5.6.3.18 *Let E be a W^*-algebra, $(x_\iota)_{\iota\in I}$ a summable family in $E_{\ddot{E}}$, and*

$$\gamma := \sup_{J\subset I} \left\| \sum_{\iota\in J}^{\ddot{E}} x_\iota \right\| < \infty$$

(Proposition 5.6.3.17 e)). For every $a \in \ddot{E}$ and $J \subset I$ put

$$\gamma(a, J) := \sup \left\{ \left| \left\langle \sum_{\iota\in K}^{\ddot{E}} x_\iota, a \right\rangle \right| K \subset I\setminus J \right\} \leq \gamma\|a\| .$$

a) *If* $a \in \ddot{E}$, $J \subset I$, $K \in \mathfrak{P}_f(I \backslash J)$, *and* $\alpha \in \ell^\infty(I)$ *then*

$$\left| \sum_{\iota \in K} \langle \alpha_\iota x_\iota, a \rangle \right| \leq \begin{cases} 2\|\alpha\|_\infty \, \gamma(a, J) & \text{if } \mathbb{K} = \mathbb{R} \\ 4\|\alpha\|_\infty \, \gamma(a, J) & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

b) *For every* $a \in \ddot{E}$ *and* $\varepsilon > 0$, *there is a* $J \in \mathfrak{P}_f(I)$ *such that*

$$\left| \left\langle \sum_{\iota \in K} \alpha_\iota x_\iota, a \right\rangle \right| < \varepsilon \|\alpha\|_\infty$$

for every $\alpha \in \ell^\infty(I)$ *and* $K \in \mathfrak{P}_f(I \backslash J)$.

c) *For every* $\alpha \in \ell^\infty(I)$, $(\alpha_\iota x_\iota)_{\iota \in I}$ *is summable in* $E_{\ddot{E}}$ *and*

$$\left\| \sum_{\iota \in K}^{\ddot{E}} \alpha_\iota x_\iota \right\| \leq \begin{cases} 2\gamma\|\alpha\|_\infty & \text{if } \mathbb{K} = \mathbb{R} \\ 4\gamma\|\alpha\|_\infty & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

d) *The maps*

$$\ell^\infty(I) \longrightarrow E, \quad \alpha \longmapsto \sum_{\iota \in I}^{\ddot{E}} \alpha_\iota x_\iota,$$

$$(\mathbb{K}^\#)^I \longrightarrow E_{\ddot{E}}, \quad \alpha \longmapsto \sum_{\iota \in I}^{\ddot{E}} \alpha_\iota x_\iota$$

are continuous and

$$\left\{ \sum_{\iota \in I}^{\ddot{E}} \alpha_\iota x_\iota \,\middle|\, \alpha \in (\mathbb{K}^\#)^I \right\}$$

is a compact set of $E_{\ddot{E}}$.

a) follows from Proposition 1.1.1.5.

b) By Proposition 5.6.3.17 a $\Rightarrow$ b, there is a $J \in \mathfrak{P}_f(I)$ such that

$$\gamma(a, J) < \frac{\varepsilon}{5}.$$

By a),

$$\left| \sum_{\iota \in K} \langle \alpha_\iota x_\iota, a \rangle \right| \leq 4\|\alpha\|_\infty \frac{\varepsilon}{5} < \varepsilon \|\alpha\|_\infty$$

for every $\alpha \in \ell^\infty(I)$ and $K \in \mathfrak{P}_f(I \backslash J)$.

c) By b) and Proposition 5.6.3.17 b $\Rightarrow$ a, $(\alpha_\iota x_\iota)_{\iota \in I}$ is summable in $E_{\ddot{E}}$ for every $\alpha \in \ell^\infty(I)$. The inequality follows from a).

d) By c), the map

$$\ell^\infty(I) \longrightarrow E, \quad \alpha \longmapsto \sum_{\iota \in I}^{\ddot{E}} \alpha_\iota x_\iota$$

is continuous.

Let $a \in \ddot{E}$ and $\varepsilon > 0$. By b) and c), there is a $J \in \mathfrak{P}_f(I)$ such that

$$\left| \left\langle \sum_{\iota \in K}^{\ddot{E}} \alpha_\iota x_\iota, a \right\rangle \right| < \frac{\varepsilon}{3}$$

for every $\alpha \in (\mathbb{K}^\#)^I$ and $K \subset I \backslash J$. Let $\alpha, \beta \in (\mathbb{K}^\#)^I$ such that

$$|\alpha_\iota - \beta_\iota| < \frac{\varepsilon}{3(1+8\gamma)}$$

for every $\iota \in J$. By c),

$$\left| \left\langle \sum_{\iota \in I}^{\ddot{E}} \alpha_\iota x_\iota, a \right\rangle - \left\langle \sum_{\iota \in I}^{\ddot{E}} \beta_\iota x_\iota, a \right\rangle \right| \leq$$

$$\leq \left| \left\langle \sum_{\iota \in J} (\alpha_\iota - \beta_\iota) x_\iota, a \right\rangle \right| + \left| \left\langle \sum_{\iota \in I \backslash J}^{\ddot{E}} \alpha_\iota x_\iota, a \right\rangle \right| + \left| \left\langle \sum_{\iota \in I \backslash J}^{\ddot{E}} \beta_\iota x_\iota, a \right\rangle \right| <$$

$$< \frac{\varepsilon}{3(1+8\gamma)} 8\gamma + \frac{2\varepsilon}{3} < \varepsilon .$$

Hence the map

$$(\mathbb{K}^\#)^I \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \left\langle \sum_{\iota \in I}^{\ddot{E}} \alpha_\iota x_\iota, a \right\rangle$$

is continuous. Since a is arbitrary, the map

$$(\mathbb{K}^\#)^I \longrightarrow E_{\ddot{E}}, \quad \alpha \longmapsto \sum_{\iota \in I}^{\ddot{E}} \alpha_\iota x_\iota$$

is continuous. It follows that

$$\left\{ \sum_{\iota \in I}^{\ddot{E}} \alpha_\iota x_\iota \,\middle|\, \alpha \in (\mathbb{K}^\#)^I \right\}$$

is a compact set of $E_{\ddot{E}}$. ■

Proposition 5.6.3.19 *Let E be a W^*-algebra and let $(x_\iota)_{\iota\in I}, (y_\lambda)_{\lambda\in L}$ be summable families in $E_{\ddot{E}}$.*

a) $$\sum_{\lambda\in L}^{\ddot{E}}\left(\left(\sum_{\iota\in I}^{\ddot{E}} x_\iota\right) y_\lambda\right) = \sum_{\iota\in I}^{\ddot{E}}\left(x_\iota\left(\sum_{\lambda\in L}^{\ddot{E}} y_\lambda\right)\right) = \left(\sum_{\iota\in I}^{\ddot{E}} x_\iota\right)\left(\sum_{\lambda\in L}^{\ddot{E}} y_\lambda\right).$$

b) *The map*

$$\mathfrak{P}(I)\times\mathfrak{P}(L)\longrightarrow E_{\ddot{E}},\quad (J,K)\longmapsto\left(\sum_{\iota\in J}^{\ddot{E}} x_\iota\right)\left(\sum_{\lambda\in K}^{\ddot{E}} y_\lambda\right)$$

(Proposition 5.6.3.17 e)) is continuous, where $\mathfrak{P}(I)$ and $\mathfrak{P}(L)$ are identified with $\{0,1\}^I$ and $\{0,1\}^L$, respectively.

a) For $a\in\ddot{E}$,

$$\left\langle\left(\sum_{\iota\in I}^{\ddot{E}} x_\iota\right)\left(\sum_{\lambda\in L}^{\ddot{E}} y_\lambda\right), a\right\rangle = \left\langle\sum_{\lambda\in L}^{\ddot{E}} y_\lambda, a\sum_{\iota\in I}^{\ddot{E}} x_\iota\right\rangle =$$

$$= \sum_{\lambda\in L}\left\langle y_\lambda, a\sum_{\iota\in I}^{\ddot{E}} x_\iota\right\rangle = \sum_{\lambda\in L}\left\langle\left(\sum_{\lambda\in L}^{\ddot{E}} x_\iota\right) y_\lambda, a\right\rangle,$$

$$\left\langle\left(\sum_{\iota\in I}^{\ddot{E}} x_\iota\right)\left(\sum_{\lambda\in L}^{\ddot{E}} y_\lambda\right), a\right\rangle = \left\langle\sum_{\iota\in I}^{\ddot{E}} x_\iota, \left(\sum_{\lambda\in L}^{\ddot{E}} y_\lambda\right) a\right\rangle =$$

$$= \sum_{\iota\in I}\left\langle x_\iota, \left(\sum_{\lambda\in L}^{\ddot{E}} y_\lambda\right) a\right\rangle = \sum_{\iota\in I}\left\langle x_\iota\left(\sum_{\lambda\in L}^{\ddot{E}} y_\lambda\right), a\right\rangle$$

and the assertion follows from the fact that a is arbitrary.

b) By Proposition 5.6.3.17 e), $(x_\iota)_{\iota\in J}$ and $(y_\lambda)_{\lambda\in K}$ are summable in $E_{\ddot{E}}$ for all $J\subset I$ and $K\subset L$. The assertion follows from a) and *C. Constantinescu, Spaces of Measures, Theorem 3.5.2 c).* ■

Proposition 5.6.3.20 *Let $(x_\iota)_{\iota\in I}$ be a family of the W^*-algebra E such that*

$$x_\iota^* x_\lambda = 0$$

for all distinct $\iota,\lambda\in I$. Then the following are equivalent:

a) $(x_\iota)_{\iota\in I}$ *is summable in* $E_{\ddot{E}}$.

b) $(x_\iota^* x_\iota)_{\iota\in I}$ *is summable in* $E_{\ddot{E}}$.

If these conditions are fulfilled then

c)

$$\left(\sum_{\iota\in I}^{\ddot{E}} x_\iota\right)^* \left(\sum_{\iota\in I}^{\ddot{E}} x_\iota\right) = \sum_{\iota\in I}^{\ddot{E}} x_\iota^* x_\iota .$$

a $\Rightarrow$ b & c. By Propositions 5.6.3.17 c),d) and 5.6.3.19 a),

$$\left(\sum_{\iota\in I}^{\ddot{E}} x_\iota\right)^* \left(\sum_{\iota\in I}^{\ddot{E}} x_\iota\right) = \left(\sum_{\iota\in I}^{\ddot{E}} x_\iota^*\right)\left(\sum_{\iota\in I}^{\ddot{E}} x_\lambda\right) = \sum_{\iota\in I}^{\ddot{E}} \left(x_\iota^* \sum_{\lambda\in I}^{\ddot{E}} x_\lambda\right) =$$

$$= \sum_{\iota\in I}^{\ddot{E}} \sum_{\lambda\in I}^{\ddot{E}} x_\iota^* x_\lambda = \sum_{\iota\in I}^{\ddot{E}} x_\iota^* x_\iota .$$

b $\Rightarrow$ a. Let $a \in \ddot{E}$ and let $(y, |a|)$ be its polar decomposition (Theorem 4.4.3.5 a)). For every $J \in \mathfrak{P}_f(I)$, by Proposition 2.3.4.6 c),

$$\left(\sum_{\iota\in J} x_\iota\right)^* \left(\sum_{\iota\in J} x_\iota\right) = \sum_{\iota,\lambda\in J} x_\iota^* x_\lambda = \sum_{\iota\in J} x_\iota^* x_\iota ,$$

$$\left|\left\langle \sum_{\iota\in J} x_\iota, a\right\rangle\right|^2 = \left|\left\langle \sum_{\iota\in J} x_\iota y, |a|\right\rangle\right|^2 = \left|\left\langle \left(\sum_{\iota\in J} x_\iota\right) y, |a|\right\rangle\right|^2 \leq$$

$$\leq \left\langle y^* \left(\sum_{\iota\in J} x_\iota\right)^* \left(\sum_{\iota\in J} x_\iota\right) y, |a|\right\rangle \langle 1, |a|\rangle =$$

$$= \left\langle \sum_{\iota\in J} x_\iota^* x_\iota, y|a|y^*\right\rangle \langle 1, |a|\rangle .$$

By Proposition 5.6.3.17 a $\Rightarrow$ b, for every $\varepsilon > 0$, there is a $J \in \mathfrak{P}_f(I)$ such that

$$\left\langle \sum_{\iota\in K} x_\iota^* x_\iota, y|a|y^*\right\rangle < \frac{\varepsilon^2}{1 + \langle 1, |a|\rangle}$$

for every $K \in \mathfrak{P}_f(I \backslash J)$. By the above,

$$\left| \left\langle \sum_{\iota \in K} x_\iota, a \right\rangle \right| < \varepsilon$$

for every $K \in \mathfrak{P}_f(I \backslash J)$. By Proposition 5.6.3.17 b $\Rightarrow$ a, $(x_\iota)_{\iota \in I}$ is summable in $E_{\ddot{E}}$. ■

Remark. The above conditions imply that $(|x_\iota|^\alpha)_{\iota \in I}$ is summable in $E_{\ddot{E}}$ for every $\alpha \geq 2$, since

$$\left\{ \iota \in I \,\middle|\, |x_\iota|^\alpha \leq x_\iota^* x \right\}$$

is a cofinite subset of I. But it may happen that $(|x_\iota|^\alpha)_{\iota \in I}$ is not summable in $E_{\ddot{E}}$ for every $\alpha \in]0, 2[$. A counterexample in this sense is given by the sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{L}(\ell^2)$, where

$$x_n : \ell^2 \longrightarrow \ell^2, \quad \xi \longrightarrow \frac{1}{\sqrt{n} \log(n+1)} (\delta_{in} \xi_1)_{i \in \mathbb{N}}$$

for every $n \in \mathbb{N}$.

Corollary 5.6.3.21 *Let E be a W^*-algebra and $(p_\iota)_{\iota \in I}$ a family in $\Pr E$ such that*

$$p_\iota p_\lambda = 0$$

for all distinct $\iota, \lambda \in I$. Then $(p_\iota)_{\iota \in I}$ is summable in $E_{\ddot{E}}$ and

$$\sum_{\iota \in I}^{\ddot{E}} p_\iota \in \Pr E.$$

The set

$$\left\{ \sum_{\iota \in J} p_\iota \,\middle|\, J \in \mathfrak{P}_f(I) \right\}$$

is upward directed and bounded (by 1) so that $(p_\iota)_{\iota \in I}$ is summable in $E_{\ddot{E}}$ (Theorem 4.4.1.8 b)). By Proposition 5.6.3.17 c) and Proposition 5.6.3.20 a $\Rightarrow$ b & c,

$$\sum_{\iota \in I}^{\ddot{E}} p_\iota \in \Pr E.$$ ■

Proposition 5.6.3.22 *Let E be a W^*-algebra and $(p_\iota)_{\iota\in I}$ a family in $\operatorname{Pr} E$ such that*

$$p_\iota p_\lambda = 0$$

for all distinct $\iota, \lambda \in I$ and

$$\sum_{\iota\in I}^{\ddot{E}} p_\iota = 1\,.$$

Then the map

$$\ell^\infty(I) \longrightarrow E\,, \quad \alpha \longmapsto \sum_{\iota\in I}^{\ddot{E}} \alpha_\iota p_\iota$$

(Corollary 5.6.3.18 d)) is a unital injective W^*-homomorphism with pretranspose

$$\ddot{E} \longrightarrow \ell^1(I)\,, \quad a \longmapsto (\langle p_\iota, a\rangle)_{\iota\in I}\,.$$

For $\alpha, \beta \in \ell^\infty(I)$, by Propositions 5.6.3.19 a) and 5.6.3.17 c),

$$\left(\sum_{\iota\in I}^{\ddot{E}} \alpha_\iota p_\iota\right)\left(\sum_{\iota\in I}^{\ddot{E}} \beta_\iota p_\iota\right) = \sum_{\iota\in I}^{\ddot{E}} \alpha_\iota\beta_\iota p_\iota\,,$$

$$\left(\sum_{\iota\in I}^{\ddot{E}} \alpha_\iota p_\iota\right)^* = \sum_{\iota\in I}^{\ddot{E}} \overline{\alpha}_\iota p_\iota\,.$$

Hence the map is an involutive algebra homomorphism. It is obviously unital and injective. Moreover, for $\alpha \in \ell^\infty(I)$ and $a \in \ddot{E}$,

$$\left\langle \sum_{\iota\in I}^{\ddot{E}} \alpha_\iota p_\iota, a \right\rangle = \sum_{\iota\in I} \alpha_\iota \langle p_\iota, a\rangle\,,$$

so that the map is a W^*-homomorphism with pretranspose

$$\ddot{E} \longrightarrow \ell^1(I)\,, \quad a \longmapsto (\langle p_\iota, a\rangle)_{\iota\in I}\,. \qquad \blacksquare$$

5.6.4 Examples

Proposition 5.6.4.1 (0) *Let $(H_\iota)_{\iota\in I}$ be a family of (weak) semi-inner-product right E-modules and*

$$H := \bigoplus_{\iota\in I} H_\iota := \{\xi \in \prod_{\iota\in I} H_\iota |\ \text{the family } (\langle \xi_\iota|\xi_\iota\rangle)_{\iota\in I} \text{ is summable in } E\}\,.$$

a) *H is a vector subspace of $\prod_{\iota\in I} H_\iota$ and H endowed with the maps*

$$H\times E \longrightarrow H\,,\quad (\xi,x)\longmapsto (\xi_\iota x)_{\iota\in I}\,,$$
$$H\times H \longrightarrow E\,,\quad (\xi,\eta)\longmapsto \sum_{\iota\in I}\langle \xi_\iota|\eta_\iota\rangle$$

is a (weak) semi-inner-product right E-module.

b) *If H_ι is an inner-product right E-module for each $\iota\in I$, then H is an inner-product right E-module and*

$$\sup_{\iota\in I}\|\xi_\iota\| \le \|\xi\|\,,\quad \|\xi\|^2 \le \sum_{\iota\in I}\|\xi_\iota\|^2$$

for every $\xi\in H$. If $\xi,\eta\in\prod_{\iota\in I} H_\iota$ such that $\eta\in H$ and

$$\langle \xi_\iota|\xi_\iota\rangle \le \langle \eta_\iota|\eta_\iota\rangle$$

for every $\iota\in I$, then $\xi\in H$ and $\|\xi\|\le\|\eta\|$.

c) *If H_ι is a Hilbert right E-module for each $\iota\in I$, then H is a Hilbert right E-module.*

d) *If H_ι is a (unital) Hilbert E-module for each $\iota\in I$, then the Hilbert right E-module H endowed with the map*

$$E\times H\longrightarrow H\,,\quad (x,\xi)\longmapsto (x\xi_\iota)_{\iota\in I}$$

is a (unital) Hilbert E-module. Moreover if $(x_\iota)_{\iota\in I}$ is a bounded family in E then

$$\sum_{\iota\in I}\langle x_\iota\xi_\iota|x_\iota\xi_\iota\rangle \le \left(\sup_{\iota\in I}\|x_\iota\|^2\right)\langle\xi|\xi\rangle$$

for every $\xi\in H$.

e) *Take* $\xi \in H$ *and for every* $\iota \in I$ *put*

$$\xi_{\{\iota\}} := (\delta_{\iota\lambda}\xi_\iota)_{\lambda\in I} \in H .$$

Then $(\xi_{\{\iota\}})_{\iota\in I}$ *is summable in* H *and*

$$\sum_{\iota\in I} \xi_{\{\iota\}} = \xi .$$

f) *If* $(I_\lambda)_{\lambda\in L}$ *is a partition of* I *then* $\bigoplus_{\lambda\in L} \bigoplus_{\iota\in I_\lambda} H_\iota$ *isomorphic to* $\bigoplus_{\iota\in I} H_\iota$ *in a natural way.*

a) It is easy to see that H is a weak semi inner-product right E-module whenever I is finite.

Take $\xi, \eta \in H$ and $x \in E$. Let $J \in \mathfrak{P}_f(I)$. By Proposition 5.6.1.2 d),e) applied to the above construction with respect to J,

$$\left\|\sum_{\iota\in J}\langle\xi_\iota|\eta_\iota\rangle\right\|^2 \leq \left\|\sum_{\iota\in J}\langle\xi_\iota|\xi_\iota\rangle\right\| \left\|\sum_{\iota\in J}\langle\eta_\iota|\eta_\iota\rangle\right\| ,$$

$$\left\|\sum_{\iota\in J}\langle\xi_\iota x|\xi_\iota x\rangle\right\| \leq \|x\|^2 \left\|\sum_{\iota\in J}\langle\xi_\iota|\xi_\iota\rangle\right\| .$$

By Proposition 1.1.6.6, it follows that the families $(\langle\xi_\iota|\eta_\iota\rangle)_{\iota\in I}$ and $(\langle\xi_\iota x|\xi_\iota x\rangle)_{\iota\in I}$ are summable. In particular $(\xi_\iota x)_{\iota\in I} \in H$. By Proposition 5.6.1.2 b),

$$\langle\xi_\iota + \eta_\iota|\xi_\iota + \eta_\iota\rangle = \langle\xi_\iota|\xi_\iota\rangle + 2\mathrm{re}\,\langle\xi_\iota|\eta_\iota\rangle + \langle\eta_\iota|\eta_\iota\rangle$$

for every $\iota \in I$. By the above, the family $(\langle\xi_\iota + \eta_\iota|\xi_\iota + \eta_\iota\rangle)_{\iota\in I}$ is summable, so $\xi + \eta \in H$ and H is a vector subspace of $\prod_{\iota\in I} H_\iota$. Assume H_ι is a semi-inner-product right E module for each $\iota \in I$. Then for $\xi, \eta \in H$,

$$(\langle\xi|\xi\rangle + \langle\eta|\eta\rangle\,,\, \langle\eta|\xi\rangle - \langle\xi|\eta\rangle) =$$

$$= \left(\sum_{\iota\in I}(\langle\xi_\iota|\xi_\iota\rangle + \langle\eta_\iota|\eta_\iota\rangle)\,,\, \sum_{\iota\in I}(\langle\eta_\iota|\xi_\iota\rangle - \langle\xi_\iota|\eta_\iota\rangle)\right) =$$

$$= \sum_{\iota\in I}(\langle\xi_\iota|\xi_\iota\rangle + \langle\eta_\iota|\eta_\iota\rangle\,,\, \langle\eta_\iota|\xi_\iota\rangle - \langle\xi_\iota|\eta_\iota\rangle) \in \overset{\circ}{E}_+ .$$

Hence H is a semi-inner product right E-module.

b) By Corollary 4.2.1.18,

$$\|\xi_\iota\|^2 = \|\langle\xi_\iota|\xi_\iota\rangle\| \leq \|\sum_{\lambda\in I}\langle\xi_\lambda|\xi_\lambda\rangle\| = \|\langle\xi|\xi\rangle\|$$

for every $\iota \in I$. Hence H is an inner-product right E-module and

$$\sup_{\iota\in I}\|\xi_\iota\| \leq \|\xi\|\,.$$

The other inequality follows from

$$\|\xi\|^2 = \left\|\sum_{\iota\in I}\langle\xi_\iota|\xi_\iota\rangle\right\| \leq \sum_{\iota\in I}\|\langle\xi_\iota|\xi_\iota\rangle\| = \sum_{\iota\in I}\|\xi_\iota\|^2\,.$$

The final assertion follows from Corollary 4.2.1.19.

c) Let $(\xi^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in H. For every $\varepsilon > 0$, there is an $m_\varepsilon \in \mathbb{N}$ such that

$$\|\xi^{(n)} - \xi^{(p)}\| < \varepsilon\,,$$

for all $n, p \in \mathbb{N}$, $n \geq m_\varepsilon$, $p \geq m_\varepsilon$. Let J be a finite subset of I. By Corollary 4.2.1.18,

$$\left\|\sum_{\iota\in J}\langle\xi_\iota|\xi_\iota\rangle\right\| \leq \left\|\sum_{\iota\in I}\langle\xi_\iota|\xi_\iota\rangle\right\| = \|\xi\|^2$$

for every $\xi \in H$. It follows that

$$\left\|\sum_{\iota\in J}\langle\xi_\iota^{(n)} - \xi_\iota^{(p)} \mid \xi_\iota^{(n)} - \xi_\iota^{(p)}\rangle\right\| < \varepsilon^2$$

for every $\varepsilon > 0$ and every $n, p \in \mathbb{N}$, with $n \geq m_\varepsilon$, $p \geq m_\varepsilon$. In particular, $\left(\xi_\iota^{(n)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence for every $\iota \in I$. For each $\iota \in I$, put

$$\xi_\iota := \lim_{n\to\infty}\xi_\iota^{(n)}\,.$$

Then for every finite subset J of I,

$$\left\|\sum_{\iota\in J}\langle\xi_\iota^{(n)} - \xi_\iota \mid \xi_\iota^{(n)} - \xi_\iota\rangle\right\| \leq \varepsilon^2\,,$$

for all $\varepsilon > 0$ and every $n \in \mathbb{N}$ with $n \geq m_\varepsilon$. Hence $\xi^{(n)} - \xi \in H$ and

$$\|\xi^{(n)} - \xi\| \leq \varepsilon$$

for every $\varepsilon > 0$ and $n \in \mathbb{N}$, $n \geq m_\varepsilon$. Thus $\xi \in H$ and $(\xi^{(n)})_{n\in\mathbb{N}}$ converges to ξ in H. In particular, H is a right Hilbert E-module.

d) The map $E \times H \to H$ is obviously bilinear. Take $x, y \in E$ and $\xi, \eta \in H$. Then

$$(xy)\xi = ((xy)\xi_\iota)_{\iota\in I} = (x(y\xi_\iota))_{\iota\in I} = x(y\xi_\iota)_{\iota\in I} = x(y\xi)\,,$$

$$(x\xi)y = (x\xi_\iota)_{\iota\in I}y = ((x\xi_\iota)y)_{\iota\in I} = (x(\xi_\iota y))_{\iota\in I} = x(\xi_\iota y)_{\iota\in I} = x(\xi y)\,,$$

$$\langle x\xi|\eta\rangle = \sum_{\iota\in I}\langle x\xi_\iota|\eta_\iota\rangle = \sum_{\iota\in I}\langle x_\iota|x^*\eta_\iota\rangle = \langle\xi|x^*\eta\rangle\,.$$

By Corollary 4.2.1.19 and Proposition 5.6.1.2 e), $(x_\iota\xi_\iota)_{\iota\in I} \in H$ and

$$\sum_{\iota\in I}\langle x_\iota\xi_\iota|x_\iota\xi_\iota\rangle \leq \sum_{\iota\in I}\|x_\iota\|^2\langle\xi_\iota|\xi_\iota\rangle \leq$$

$$\leq \left(\sup_{\iota\in I}\|x_\iota\|^2\right)\sum_{\iota\in I}\langle\xi_\iota|\xi_\iota\rangle = \left(\sup_{\iota\in I}\|x_\iota\|^2\right)\langle\xi|\xi\rangle\,.$$

e) Let $\varepsilon > 0$. By the definition of H, there is a $J \in \mathfrak{P}_f(I)$, such that

$$\|\sum_{\iota\in L}\langle\xi_\iota|\xi_\iota\rangle\| < \varepsilon$$

for every $L \subset I\backslash J$. Hence, for $K \in \mathfrak{P}_f(I)$, $J \subset K$,

$$\|\sum_{\iota\in K}\xi_{\{\iota\}} - \xi\|^2 = \|\langle\sum_{\iota\in K}\xi_{\{\iota\}} - \xi \mid \sum_{\iota\in K}\xi_{\{\iota\}} - \xi\rangle\| =$$

$$= \|\sum_{\iota\in I\backslash K}\langle\xi_\iota|\xi_\iota\rangle\| < \varepsilon\,.$$

Thus $(\xi_{\{\iota\}})_{\iota\in I}$ is summable in H and

$$\sum_{\iota\in I}\xi_{\{\iota\}} = \xi\,.$$

f) follows from a). ■

Remark. Take $\xi \in \prod_{\iota\in I} H_\iota$. If

$$\sum_{\iota\in I}\|\langle\xi_\iota|\xi_\iota\rangle\| < \infty\,,$$

then the family $(\langle\xi_\iota|\xi_\iota\rangle)_{\iota\in I}$ is absolutely summable, so it is summable. The converse is not true as the following example shows: $E := c_0$, $I := \mathbb{N}$, $H_\iota := E$ for every $\iota \in I$, $\xi_n := \frac{1}{\sqrt{n}}e_n$ for every $n \in \mathbb{N}$.

Example 5.6.4.2 (0) *Let I be a set, F a closed right ideal of E, and*

$$\ell^2(I,F) := \{\xi \in F^I | \textit{the family } (\xi_\iota^*\xi_\iota)_{\iota\in I} \textit{ is summable}\}\,.$$

a) *$\ell^2(I,F)$ is a vector subspace of F^I and $\ell^2(I,F)$ endowed with the maps*

$$\ell^2(I,F)\times E \longrightarrow \ell^2(I,F)\,,\quad (\xi,x)\longmapsto (\xi_\iota x)_{\iota\in I}\,,$$

$$\ell^2(I,F)\times \ell^2(I,F) \longrightarrow E\,,\quad (\xi,\eta)\longmapsto \sum_{\iota\in I}\eta_\iota^*\xi_\iota$$

is a Hilbert right E-module.

b) $\xi \in \ell^2(I,F) \Rightarrow \sup\limits_{\iota\in I}\|\xi_\iota\| \le \|\xi\|\,,\ \|\xi\|^2 \le \sum\limits_{\iota\in I}\|\xi_\iota\|^2\,.$

c) *If F is an ideal of E and $(x_\iota)_{\iota\in I}$ is a bounded family in E then for every $\xi \in \ell^2(I,F)$,*

$$(x_\iota\xi_\iota)_{\iota\in I} \in \ell^2(I,F)\,,\quad \|(x_\iota\xi_\iota)_{\iota\in I}\| \le \|\xi\|\sup_{\iota\in I}\|x_\iota\|\,.$$

In particular, $\ell^2(I,F)$ endowed with the map

$$E\times \ell^2(I,F)\longrightarrow \ell^2(I,F)\,,\quad (x,\xi)\longmapsto (x\xi_\iota)_{\iota\in I}$$

is a (unital) Hilbert E-module.

d) *If F is an ideal of E and*

$$\xi^* := (\xi_\iota^*)_{\iota\in I} \in \ell^2(I,F)$$

whenever $\xi \in \ell^2(I,F)$ (Corollary 4.2.6.2) (this is the case when F is finite-dimensional or I is finite) then there is an $\alpha > 0$ such that

$$\|\xi^*\| \le \alpha\|\xi\|$$

for every $\xi \in \ell^2(I,F)$ and

$$(x\xi)^* = \xi^*x^*\,,\quad (\xi x)^* = x^*\xi^*$$

for all $(x,\xi) \in E\times \ell^2(I,F)$.

e) *Take $\xi_1,\xi_2,\xi_3,\xi_4 \in \mathbb{R}^I$ and put*

$$\xi : I \longrightarrow \mathbb{H}\,,\quad \iota \longmapsto \xi_1(\iota)1 + \xi_2(\iota)i + \xi_3(\iota)j + \xi_4(\iota)k\,.$$

Then the following are equivalent:

e$_1$) $\xi_1, \xi_2, \xi_3, \xi_4 \in \ell^2(I, \mathbb{R})$,

e$_2$) $\xi \in \ell^2(I, \mathbb{H})$,

e$_3$) $\xi^* \in \ell^2(I, \mathbb{H})$,

e$_4$) $(\|\xi(\iota)\|)_{\iota \in I} \in \ell^2(I, \mathbb{R})$.

If these conditions are fulfilled then

$$\|\xi\| = \|\xi^*\| = (\|\xi_1\|^2 + \|\xi_2\|^2 + \|\xi_3\|^2 + \|\xi_4\|^2)^{\frac{1}{2}} =$$

$$= \left(\sum_{\iota \in I} \|\xi(\iota)\|^2 \right)^{\frac{1}{2}}.$$

a), b), and c) follow from Example 5.6.1.5 and Proposition 5.6.4.1.

d) We may assume that I is infinite. Suppose that the map

$$\ell^2(I, F) \longrightarrow \ell^2(I, F), \quad \xi \longmapsto \xi^*$$

is not continuous. Then there is a sequence $(\xi_n)_{n \in \mathbb{N}}$ in $\ell^2(I, F)$ such that

$$\|\xi_n\| < \frac{1}{n}, \quad \|\xi_n^*\| > 1$$

for every $n \in \mathbb{N}$. By continuity, for every $n \in \mathbb{N}$, there is an $I_n \in \mathfrak{P}_f(I)$ such that

$$\left\| \sum_{\iota \in I_n} \xi_{n\iota} \xi_{n\iota}^* \right\| > 1.$$

We may assume that $(I_n)_{n \in \mathbb{N}}$ is a disjoint sequence. Put

$$\xi_\iota := \begin{cases} \xi_{n\iota} & \text{if } \iota \in I_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } \iota \in I \setminus \bigcup_{n \in \mathbb{N}} I_n. \end{cases}$$

By Corollary 4.2.1.18

$$\left\| \sum_{\iota \in I} \xi_\iota^* \xi_\iota \right\| \leq \sum_{n \in \mathbb{N}} \left\| \sum_{\iota \in I_n} \xi_{n\iota}^* \xi_{n\iota} \right\| \leq$$

$$\leq \sum_{n \in \mathbb{N}} \left\| \sum_{\iota \in I} \xi_{n\iota}^* \xi_{n\iota} \right\| = \sum_{n \in \mathbb{N}} \|\xi_n\|^2 < \infty,$$

i.e. $\xi \in \ell^2(I,F)$. On the other hand

$$\left\|\sum_{\iota\in I_n} \xi_\iota \xi_\iota^*\right\| = \left\|\sum_{\iota\in I_n} \xi_{n\iota}\xi_{n\iota}^*\right\| > 1$$

for every $n \in \mathbb{N}$, so that $\xi^* \notin \ell^2(I,F)$, which contradicts the hypothesis. Thus the above map is continuous and there is an $\alpha > 0$ with

$$\|\xi^*\| \leq \alpha\|\xi\|$$

for every $\xi \in \ell^2(I,F)$. The other relations are easy to see.

If F is finite-dimensional, then by Corollary 4.2.1.20, $\xi^* \in \ell^2(I,F)$ whenever $\xi \in \ell^2(I,F)$.

e) By the Examples 2.3.1.46 and 4.1.1.31,

$$\xi^*(\iota)\xi(\iota) = (\xi_1(\iota)^2 + \xi_2(\iota)^2 + \xi_3(\iota)^2 + \xi_4(\iota)^2)1 = \xi(\iota)\xi^*(\iota) = \|\xi(\iota)\|^2 1$$

for every $\iota \in I$. ■

Remark. The hypothesis of d) may fail (see Remark of Proposition 5.3.2.13). Even if it holds, α may be different from 1. Indeed, put

$$\xi_1 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \xi_2 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\xi_1^*\xi_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\xi_1\xi_1^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

so that

$$\|\xi_1^*\xi_1 + \xi_2^*\xi_2\| = \|1\| = 1,$$

$$\|\xi_1\xi_1^* + \xi_2\xi_2^*\| = \|2\xi_2\| = 2.$$

Proposition 5.6.4.3 (0) *Take $n \in \mathbb{N}$ and let $E^n := \ell^2(\mathbb{N}_n, E)$ be the Hilbert E-module defined in Example 5.6.4.2 c). For $\xi' \in E'^n$, put*

$$\widetilde{\xi}' : E^n \longrightarrow \mathbb{K}, \quad \xi \longmapsto \sum_{i=1}^{n} \langle \xi_i, \xi_i' \rangle .$$

Then $\widetilde{\xi}' \in (E^n)'$ and

$$\left(\sum_{i=1}^{n} \|\xi_i'\|^2 \right)^{\frac{1}{2}} \leq \|\widetilde{\xi}'\| \leq \sum_{i=1}^{n} \|\xi_i'\|$$

for every $\xi' \in E'^n$ and the map

$$E'^n \longrightarrow (E^n)', \quad \xi' \longmapsto \widetilde{\xi}'$$

is a homomorphism of E-modules and an involutive isomorphism of Banach spaces (Proposition 2.2.7.2, Example 2.2.7.21).

We have

$$|\widetilde{\xi}'(\xi)| = \left| \sum_{i=1}^{n} \langle \xi_i, \xi_i' \rangle \right| \leq \sum_{i=1}^{n} |\langle \xi_i, \xi_i' \rangle| \leq \sum_{i=1}^{n} \|\xi_i\| \, \|\xi_i'\| \leq \|\xi\| \sum_{i=1}^{n} \|\xi_i'\| ,$$

for every $\xi \in E^n$ (Example 5.6.4.2 b)),so that $\widetilde{\xi}' \in (E^n)'$ and

$$\|\widetilde{\xi}'\| \leq \sum_{i=1}^{n} \|\xi_i'\| .$$

Take $\theta \in]0,1[$. For every $i \in \mathbb{N}_n$, there is a $\xi_i \in E \backslash \{0\}$ such that

$$\langle \xi_i, \xi_i' \rangle \geq \theta \|\xi_i'\| \, \|\xi_i\| .$$

Given $i \in \mathbb{N}_n$, put

$$\eta_i := \frac{\|\xi_i'\|}{\|\xi_i\|} \xi_i .$$

Then

$$\|\eta\|^2 = \left\| \sum_{i=1}^{n} \frac{\|\xi_i'\|^2}{\|\xi_i\|^2} \xi_i^* \xi_i \right\| \leq \sum_{i=1}^{n} \frac{\|\xi_i'\|^2}{\|\xi_i\|^2} \|\xi_i^* \xi_i\| = \sum_{i=1}^{n} \|\xi_i'\|^2$$

and so

$$\|\widetilde{\xi}'\| \left(\sum_{i=1}^{n} \|\xi_i'\|^2 \right)^{\frac{1}{2}} \geq \|\widetilde{\xi}'\| \, \|\eta\| \geq \widetilde{\xi}'(\eta) = \sum_{i=1}^{n} \left\langle \frac{\|\xi_i'\|}{\|\xi_i\|} \xi_i, \xi_i' \right\rangle =$$

$$= \sum_{i=1}^{n} \frac{\|\xi_i'\|}{\|\xi_i\|} \langle \xi_i, \xi_i' \rangle \geq \theta \sum_{i=1}^{n} \|\xi_i'\|^2 ,$$

$$\|\widetilde{\xi'}\| \geq \theta \left(\sum_{i=1}^{n} \|\xi_i'\|^2 \right)^{\frac{1}{2}} .$$

Since θ is arbitrary, we see that

$$\|\widetilde{\xi'}\| \geq \left(\sum_{i=1}^{n} \|\xi_i'\|^2 \right)^{\frac{1}{2}} .$$

By Example 5.6.4.2 b), the norm of E^n is equivalent to the Euclidean norm. By Proposition 1.2.2.13, the map

$$E'^n \longrightarrow (E^n)' , \quad \xi' \longmapsto \widetilde{\xi'}$$

is an isomorphism of Banach spaces.

Take $\xi' \in E'^n$. Then

$$\widetilde{\xi'}^*(\xi) = \overline{\widetilde{\xi'}(\xi^*)} = \sum_{i=1}^{n} \overline{\langle \xi_i^*, \xi_i' \rangle} = \sum_{i=1}^{n} \langle \xi_i, \xi_i'^* \rangle = \widetilde{\xi'^*}(\xi)$$

for every $\xi \in E^n$, so

$$\widetilde{\xi'}^* = \widetilde{\xi'^*}$$

and the above map is involutive. Moreover, for every $x \in E$,

$$\langle \xi, \widetilde{x\xi'} \rangle = \sum_{i=1}^{n} \langle \xi_i, x\xi_i' \rangle = \sum_{i=1}^{n} \langle \xi_i x, \xi_i' \rangle = \langle \xi x, \widetilde{\xi'} \rangle = \langle \xi, x\widetilde{\xi'} \rangle ,$$

$$\langle \xi, \widetilde{\xi' x} \rangle = \sum_{i=1}^{n} \langle \xi_i, \xi_i' x \rangle = \sum_{i=1}^{n} \langle x\xi_i, \xi_i' \rangle = \langle x\xi, \widetilde{\xi'} \rangle = \langle \xi, \widetilde{\xi'} x \rangle ,$$

for every $\xi \in E$, so that

$$\widetilde{x\xi'} = x\widetilde{\xi'} , \quad \widetilde{\xi' x} = \widetilde{\xi'} x . \qquad \blacksquare$$

Proposition 5.6.4.4 *Let R, S, T be sets and let*

$$E \longrightarrow E , \quad x \longmapsto \widetilde{x}$$

be a conjugate involution. For each $u \in \mathcal{L}_E(\ell^2(S,E), \ell^2(T,E))$ define

$$\widetilde{u} : \ell^2(S,E) \longrightarrow \ell^2(T,E) , \quad \xi \longmapsto \widetilde{u\widetilde{\xi}} .$$

a) $\xi, \eta \in \ell^2(T,E) \Rightarrow \langle\widetilde{\xi}|\widetilde{\eta}\rangle = \widetilde{\langle\xi|\eta\rangle}$.

b) *The Banach space* $\ell^2(T,E)$ *endowed with the involution*

$$\ell^2(T,E) \longrightarrow \ell^2(T,E), \quad \xi \longmapsto \widetilde{\xi}$$

is an involutive Banach space.

c) *If* $u \in \mathcal{L}_E(\ell^2(S,E), \ell^2(T,E))$ *and* $x \in E$ *then*

$$\widetilde{u} \in \mathcal{L}_E(\ell^2(S,E), \ell^2(T,E)), \quad \widetilde{u}^* = \widetilde{u^*}, \quad \widetilde{\widetilde{u}} = u, \quad \|\widetilde{u}\| = \|u\|,$$

$$\widetilde{x}\widetilde{u} = \widetilde{x}\widetilde{u}, \quad \widetilde{u}\widetilde{x} = \widetilde{u}\widetilde{x}$$

(Example 5.6.4.2 c), Proposition 5.6.1.8 g)).

d) *The Banach space* $\mathcal{L}_E(\ell^2(S,E), \ell^2(T,E))$ *endowed with the involution*

$$\mathcal{L}_E(\ell^2(S,E), \ell^2(T,E)) \longrightarrow \mathcal{L}_E(\ell^2(S,E), \ell^2(T,E)), \quad u \longmapsto \widetilde{u}$$

is an involutive Banach space.

e) $u \in \mathcal{L}_E(\ell^2(S,E), \ell^2(T,E))$, $v \in \mathcal{L}_E(\ell^2(R,E), \ell^2(S,E)) \Rightarrow \widetilde{u \circ v} = \widetilde{u} \circ \widetilde{v}$.

f) *The map*

$$\mathcal{L}_E(\ell^2(T,E)) \longrightarrow \mathcal{L}_E(\ell^2(T,E)), \quad u \longmapsto \widetilde{u}$$

is a conjugate involution of the C^**-algebra* $\mathcal{L}_E(\ell^2(T,E))$ *(Theorem 5.6.1.11 a)).*

g) *Assume* $\mathbb{K} = \mathbb{C}$. *Let* F *be a* C^**-subalgebra of* $\mathcal{L}_E(\ell^2(I,E))$ *such that* $\widetilde{u} \in F$ *whenever* $u \in F$ *and let* $\overset{\circ}{F}$ *denote the complexification of the underlying real* C^**-algebra of* F. *Then the map*

$$\overset{\circ}{F} \longrightarrow F \times F, \quad (u,v) \longmapsto (u + iv, \widetilde{u} + i\widetilde{v})$$

is an isomorphism of complex C^**-algebras.*

a) $\langle\widetilde{\xi}|\widetilde{\eta}\rangle = \sum_{t\in T} \widetilde{\eta}_t^* \widetilde{\xi}_t = \sum_{t\in T} \widetilde{\eta_t^*}\widetilde{\xi}_t = \sum_{t\in T} \widetilde{\eta_t^* \xi_t} = \widetilde{\sum_{t\in T} \eta_t^* \xi_t} = \widetilde{\langle\xi|\eta\rangle}$.

b) follows from a).

c) $\widetilde{u}$ is linear and

$$\sup\{\|\widetilde{u}\xi\| \mid \xi \in \ell^2(S,E)^\#\} = \sup\{\|u\widetilde{\xi}\| \mid \xi \in \ell^2(S,E)^\#\} = \|u\|$$

so that

$$\widetilde{u} \in \mathcal{L}(\ell^2(S,E)\,,\,\ell^2(T,E))\,,\quad \|\widetilde{u}\| = \|u\|\,.$$

Take $\xi \in \ell^2(S,E)$ and $\eta \in \ell^2(T,E)$. By a),

$$\langle \widetilde{u}\xi|\eta\rangle = \langle \widetilde{u\xi}|\eta\rangle = \widetilde{\langle u\xi|\eta\rangle} = \widetilde{\langle \xi|u^*\eta\rangle} = \langle \xi|\widetilde{u^*\eta}\rangle = \langle \xi|\widetilde{u}^*\eta\rangle$$

so that

$$\widetilde{u} \in \mathcal{L}_E(\ell^2(S,E)\,,\,\ell^2(T,E))\,,\quad \widetilde{u}^* = \widetilde{u^*}\,.$$

Moreover,

$$\widetilde{x}\widetilde{u}\xi = \widetilde{(xu)}\widetilde{\xi} = \widetilde{x(u\xi)} = \widetilde{x}\widetilde{u\xi} = \widetilde{x}(\widetilde{u}\xi) = (\widetilde{x}\widetilde{u})\xi\,,$$

$$\widetilde{u}\widetilde{x}\xi = \widetilde{(ux)}\widetilde{\xi} = \widetilde{u(x\xi)} = \widetilde{u}\widetilde{x\xi} = \widetilde{u}(\widetilde{x}\xi) = (\widetilde{u}\widetilde{x})\xi\,,$$

$$\widetilde{\widetilde{u}}\xi = \widetilde{\widetilde{u}\widetilde{\xi}} = u\xi$$

so that

$$\widetilde{xu} = \widetilde{x}\widetilde{u}\,,\quad \widetilde{ux} = \widetilde{u}\widetilde{x}\,,\quad \widetilde{\widetilde{u}} = u\,.$$

d) The involution is conjugate linear, so that the assertion follows from c).

e) For all $\xi \in \ell^2(R,E)$,

$$\widetilde{u\circ v}\xi = \widetilde{\widetilde{(u\circ v)}\xi} = \widetilde{u(v\xi)} = \widetilde{u}\widetilde{v\xi} = \widetilde{u}\widetilde{v}\xi$$

so that

$$\widetilde{u\circ v} = \widetilde{u}\circ\widetilde{v}\,.$$

f) follows from c),d), and e).

g) follows from f) and Theorem 4.1.1.8 e). ■

Proposition 5.6.4.5 (0) *Let E be a finite-dimensional C^*-algebra and $(e_i)_{i\in\mathbb{N}_n}$ an algebraic basis of E consisting of invertible elements of E. For every $x \in E$ define $(x_i)_{i\in\mathbb{N}_n} \in \mathbb{K}^n$ such that*

$$x = \sum_{i=1}^{n} x_i e_i$$

and put

$$a := [(e_i^*)_j]_{i,j\in\mathbb{N}_n} \in \mathbb{K}_{n,n}\,.$$

For S,T sets and

$$u := (u_i)_{i\in\mathbb{N}_n} \in (\mathcal{L}(\ell^2(S)\,,\,\ell^2(T)))^n$$

put

$$u^* := \left(\sum_{j=1}^n a_{ji}u_j^*\right)_{i\in\mathbb{N}_n} \in (\mathcal{L}(\ell^2(T),\ell^2(S)))^n\,,$$

$$\widetilde{u} : \ell^2(S,E) \longrightarrow \ell^2(T,E)\,, \quad \xi \longmapsto \sum_{i,j=1}^n (u_i\xi_j)e_ie_j\,,$$

where for $\xi \in \ell^2(S,E)$ and $j \in \mathbb{N}_n$, $\xi_j \in \ell^2(S)$ is defined by

$$\xi_j : S \longrightarrow \mathbb{K}\,, \quad s \longmapsto (\xi(s))_j\,.$$

Then

$$\widetilde{u} \in \mathcal{L}_E(\ell^2(S,E),\ell^2(T,E))\,, \quad \widetilde{u}^* = \widetilde{u^*}$$

for every $u \in (\mathcal{L}(\ell^2(S),\ell^2(T)))^n$ (Definition 5.6.1.7) and the map

$$(\mathcal{L}(\ell^2(S),\ell^2(T)))^n \longrightarrow \mathcal{L}_E(\ell^2(S,E),\ell^2(T,E))\,, \quad u \longmapsto \widetilde{u}$$

is bijective and linear.

For every $i \in \mathbb{N}_n$

$$e_i^* = \sum_{j=1}^n a_{ij}e_j$$

so that

$$e_i = e_i^{**} = \sum_{j=1}^n \overline{a_{ij}}e_j^*\,.$$

For $\xi \in \ell^2(S,E)$ and $\eta \in \ell^2(T,E)$,

$$\langle\widetilde{u}\xi|\eta\rangle = \sum_{i,j,k=1}^n \langle(u_i\xi_j)e_ie_j|\eta_ke_k\rangle =$$

$$= \sum_{i,j,k=1}^{n} \sum_{t\in T} (u_i\xi_j)(t)\overline{\eta_k(t)} e_k^* e_i e_j =$$

$$= \sum_{i,j,k=1}^{n} \langle u_i\xi_j | \eta_k\rangle e_k^* e_i e_j =$$

$$= \sum_{i,j,k=1}^{n} \langle \xi_j | u_i^*\eta_k\rangle e_k^* \left(\sum_{h=1}^{n} \overline{a}_{ih} e_h^* \right) e_j =$$

$$= \sum_{h,j,k=1}^{n} \left\langle \xi_j \,\middle|\, \left(\sum_{i=1}^{n} a_{ih} u_i^* \right) \eta_k \right\rangle e_k^* e_h^* e_j =$$

$$= \sum_{h,j,k=1}^{n} \left\langle \xi_j \,\middle|\, (u^*)_h \eta_k \right\rangle (e_h e_k)^* e_j =$$

$$= \sum_{h,j,k=1}^{n} \sum_{s\in S} \xi_j(s)\overline{((u^*)_h\eta_k)(s)}\, (e_h e_k)^* e_j =$$

$$= \sum_{h,j,k=1}^{n} \left\langle \xi_j e_j \,\middle|\, ((u^*)_h\eta_k) e_h e_k \right\rangle = \langle \xi | \widetilde{u^*}\eta \rangle .$$

Hence

$$\widetilde{u} \in \mathcal{L}_E(\ell^2(S,E)\,,\, \ell^2(T,E))\,,\ \widetilde{u}^* = \widetilde{u^*}\,.$$

It is obvious that the map is linear. Take

$$u \in (\mathcal{L}(\ell^2(S), \ell^2(T)))^n$$

and assume $\widetilde{u} = 0$. Fix $k \in \mathbb{N}_n$, take $\xi \in \ell^2(S)$, and define $\eta \in \ell^2(S,E)$ by

$$\eta : S \longrightarrow E\,, \quad s \longmapsto \xi(s) e_k\,.$$

Then

$$0 = \left(\sum_{i,j=1}^{n} (u_i\eta_j) e_i e_j \right) e_k^{-1} = \sum_{i,j=1}^{n} (u_i \delta_{jk}\xi) e_i e_j e_k^{-1} = \sum_{i=1}^{n} (u_i\xi) e_i\,,$$

and we obtain successively

$$i \in \mathbb{N}_n \Longrightarrow u_i \xi = 0\,,$$

$$i \in \mathbb{N}_n \Longrightarrow u_i = 0\,,$$

$$u = 0\,,$$

the map is injective. Take

$$v \in \mathcal{L}_E(\ell^2(S,E)\,,\ \ell^2(T,E))\,.$$

Since E is finite-dimensional, it is unital. Denote by 1 the unit of E. For every $i \in \mathbb{N}_n$, put

$$u_i : \ell^2(S) \longrightarrow \ell^2(T)\,, \quad \xi \longmapsto (v(\xi 1))_i\,,$$

$$u := (u_i)_{i \in \mathbb{N}_n} \in (\mathcal{L}(\ell^2(S), \ell^2(T)))^n\,.$$

For every $\xi \in \ell^2(S,E)$,

$$\tilde{u}\xi = \sum_{i,j=1}^{n} (u_i \xi_j) e_i e_j = \sum_{i,j=1}^{n} (v(\xi_j 1))_i e_i e_j =$$

$$= \sum_{j=1}^{n} v(\xi_j 1) e_j = v\left(\sum_{j=1}^{n} \xi_j e_j\right) = v\xi$$

(Proposition 5.6.1.8 d)), so that $\tilde{u} = v$ and the map is surjective. ■

Proposition 5.6.4.6 (8) *Suppose E is a W^*-algebra and let $(H_\iota)_{\iota \in I}$ be a family of Hilbert right E-modules. Put*

$$H := \overset{W}{\bigoplus_{\iota \in I}} H_\iota := \left\{ (\xi_\iota)_{\iota \in I} \in \prod_{\iota \in I} H_\iota \;\middle|\; (\langle \xi_\iota | \xi_\iota \rangle)_{\iota \in I} \text{ is summable in } E_{\bar{E}} \right\}$$

and

$$\xi_J : I \longrightarrow \bigcup_{\iota \in I} H_\iota\,, \quad \iota \longmapsto \begin{cases} \xi_\iota & \text{if } \iota \in J \\ 0 & \text{if } \iota \in I \backslash J \end{cases}$$

for every $\xi \in \prod_{\iota \in I} H_\iota$ and $J \subset I$.

a) *If* $\xi, \eta \in H$ *then* $\xi + \eta \in H$ *and the family* $(\langle \xi_\iota | \eta_\iota \rangle)_{\iota \in I}$ *is summable in* $E_{\ddot{E}}$. *Put*

$$\langle \xi | \eta \rangle := \sum_{\iota \in I}^{\ddot{E}} \langle \xi_\iota | \eta_\iota \rangle .$$

b) H *endowed with the right multiplication*

$$H \times E \longrightarrow H , \quad (\xi, x) \longmapsto \xi x := (\xi_\iota x)_{\iota \in I}$$

and with the inner-product

$$H \times H \longrightarrow E , \quad (\xi, \eta) \longmapsto \langle \xi | \eta \rangle$$

is a Hilbert right E*-module.*

c) *For every* $a \in \ddot{E}$, $\xi \in H$, *and* $\varepsilon > 0$, *there is a* $J \in \mathfrak{P}_f(I)$ *such that*

$$\left| \langle \langle \eta | \xi_{I \setminus K} \rangle, a \rangle \right| < \varepsilon$$

for every $\eta \in H^\#$ *and* $K \in \mathfrak{P}_f(I)$, $J \subset K$. *In particular,*

$$\xi = \sum_{\iota \in I}^{\ddot{H}} \xi_\iota$$

for every $\xi \in H$, *where for every* $\iota \in I$ *we identified* H_ι *with a subset of* H *in a natural way.*

d) *If* H_ι *is a (unital) Hilbert* E*-module for each* $\iota \in I$ *then the Hilbert right* E*-module* H *endowed with the left multiplication*

$$E \times H \longrightarrow H , \quad (x, \xi) \longmapsto x\xi := (x\xi_\iota)_{\iota \in I}$$

is a (unital) Hilbert E*-module.*

e) *If* E *is finite-dimensional then*

$$\bigoplus_{\iota \in I}^{W} H_\iota = \bigoplus_{\iota \in I} H_\iota .$$

f) *Assume there is a family* $(p_\iota)_{\iota \in I}$ *in* $\operatorname{Pr} E \setminus \{0\}$ *such that*

$$H_\iota = p_\iota E$$

for every $\iota \in I$. *Put*

$$f_\iota := (\delta_{\iota\lambda} p_\iota)_{\lambda \in I}$$

for every $\iota \in I$ *Then* $(f_\iota)_{\iota \in I}$ *is a Fourier basis of* H.

a) We may assume $\mathbb{K} = \mathbb{C}$ (Theorem 4.4.1.2, Proposition 5.6.1.6). By Proposition 5.6.1.2 b),

$$\langle \xi_\iota + \eta_\iota | \xi_\iota + \eta_\iota \rangle \leq 2(\langle \xi_\iota | \xi_\iota \rangle + \langle \eta_\iota | \eta_\iota \rangle)$$

for each $\iota \in I$ so that $\xi + \eta \in H$ (Proposition 5.6.3.7 a $\Rightarrow$ b). The family $(\langle \xi_\iota | \eta_\iota \rangle)_{\iota \in I}$ is summable in E_E by the Polarization Identity (Proposition 2.3.3.2 d)).

b) Take $\xi, \eta \in H$ and $x \in E$. Then

$$\left\| \sum_{\iota \in J} \langle \xi_\iota x | \xi_\iota x \rangle \right\| = \left\| \sum_{\iota \in J} x^* \langle \xi_\iota | \xi_\iota \rangle x \right\| = \left\| x^* \left(\sum_{\iota \in J} \langle \xi_\iota | \xi_\iota \rangle \right) x \right\| \leq \|x\|^2 \left\| \sum_{\iota \in J} \langle \xi_\iota | \xi_\iota \rangle \right\|$$

for every $J \in \mathfrak{P}_f(I)$ (Proposition 5.6.1.2 b)) so that $\xi x \in H$ (Proposition 5.6.3.7 a $\Rightarrow$ b). Moreover,

$$\langle \xi x | \eta \rangle = \sum_{\iota \in I}^{E} \langle \xi_\iota x | \eta_\iota \rangle = \sum_{\iota \in I}^{E} \langle \xi_\iota | \eta_\iota \rangle x = \langle \xi | \eta \rangle x \,,$$

$$\langle \xi | \eta \rangle = \sum_{\iota \in I}^{E} \langle \xi_\iota | \eta_\iota \rangle = \sum_{\iota \in I}^{E} \langle \eta_\iota | \xi_\iota \rangle^* =$$

$$= \left(\sum_{\iota \in I}^{E} \langle \eta_\iota | \xi_\iota \rangle \right)^* = \langle \eta | \xi \rangle^* \,,$$

$$\langle \xi | \xi \rangle = \sum_{\iota \in I}^{E} \langle \xi_\iota | \xi_\iota \rangle \in E_+$$

(Proposition 4.4.1.3),

$$(\langle \xi | \xi \rangle + \langle \eta | \eta \rangle \,,\, \langle \eta | \xi \rangle - \langle \xi | \eta \rangle) =$$

$$= \sum_{\iota \in I}^{E} (\langle \xi_\iota | \xi_\iota \rangle + \langle \eta_\iota | \eta_\iota \rangle \,,\, \langle \eta_\iota | \xi_\iota \rangle - \langle \xi_\iota | \eta_\iota \rangle) \in \overset{\circ}{E}_+$$

(Proposition 4.4.1.3),

$$\xi 1 = (\xi_\iota 1)_{\iota \in I} = (\xi_\iota)_{\iota \in I} = \xi \,,$$

$$\langle\xi|\xi\rangle = 0 \Longrightarrow (\iota \in I \Longrightarrow \langle\xi_\iota|\xi_\iota\rangle = 0) \Longrightarrow (\iota \in I \Longrightarrow \xi_\iota = 0) \Longrightarrow \xi = 0\,.$$

Let $(\xi^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in H. For every $\varepsilon > 0$, there is a $p_\varepsilon \in \mathbb{N}$ such that

$$\|\xi^{(m)} - \xi^{(n)}\| < \varepsilon$$

for all $m, n \in \mathbb{N}$, $m \geq p_\varepsilon$, $n \geq p_\varepsilon$. Since for every $\iota \in I$, the map

$$H \longrightarrow H_\iota\,, \quad \xi \longmapsto \xi_\iota$$

is linear and continuous, $(\xi_\iota^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore a convergent sequence in H_ι. Put

$$\xi : I \longrightarrow \bigcup_{\iota\in I} H_\iota\,, \quad \iota \longmapsto \lim_{n\to\infty} \xi_\iota^{(n)}\,.$$

For every $\varepsilon > 0$, $m \in \mathbb{N}$, $m \geq p_\varepsilon$, and $J \in \mathfrak{P}_f(I)$,

$$\left\|\xi_J^{(m)} - \xi_J\right\| = \lim_{n\to\infty} \left\|\xi_J^{(m)} - \xi_J^{(n)}\right\| \leq \lim_{n\to\infty} \|\xi^{(m)} - \xi^{(n)}\| \leq \varepsilon\,.$$

Hence $\xi^{(m)} - \xi \in H$ and from

$$\langle\xi^{(m)} - \xi|\xi^{(m)} - \xi\rangle = \lim_{J\in\mathfrak{F}_I} \langle\xi_J^{(m)} - \xi_J|\xi_J^{(m)} - \xi_J\rangle$$

it follows

$$\|\xi^{(m)} - \xi\| \leq \varepsilon$$

(Proposition 5.6.3.7 and Corollary 4.2.1.16 a $\Leftrightarrow$ b). By a), $\xi \in H$ and

$$\lim_{m\to\infty} \xi^{(m)} = \xi\,.$$

Thus H is a Hilbert right E-module.

c) Assume the contrary. By recursion, we construct an increasing sequence $(J_n)_{n\in\mathbb{N}}$ in $\mathfrak{P}_f(I)$ and a sequence $(\eta^{(n)})_{n\in\mathbb{N}}$ in $H^\#$ such that

$$\langle\langle\eta^{(n)}|\xi_{J_{2n}\setminus J_{2n-1}}\rangle, a\rangle > \frac{\varepsilon}{2}$$

for every $n \in \mathbb{N}$. Take $n \in \mathbb{N}$ and assume that the sequences were constructed up to $n-1$. By the above assumption, there is a $J_{2n-1} \in \mathfrak{P}_f(I)$ and an $\eta^{(n)} \in H^\#$ such that $J_{2n-2} \subset J_{2n-1}$ $(J_0 := \emptyset)$ and

$$\langle\langle\eta^{(n)}|\xi_{I\setminus J_{2n-1}}\rangle, a\rangle \geq \varepsilon\,.$$

By a), there is a $J_{2n} \in \mathfrak{P}_f(I)$, $J_{2n-1} \subset J_{2n}$ such that

$$\langle\langle \eta^{(n)} | \xi_{J_{2n} \setminus J_{2n-1}} \rangle, a \rangle > \frac{\varepsilon}{2}.$$

This finishes the recursive construction. Put

$$K_n := J_{2n} \setminus J_{2n-1}, \quad L_n := \bigcup_{k=1}^{n} K_k$$

for every $n \in \mathbb{N}$.

Define

$$\eta : I \longrightarrow E, \quad \iota \longmapsto \begin{cases} \frac{1}{n}\eta^{(n)} & \text{if } \iota \in K_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 4.2.1.16 a $\Rightarrow$ b,

$$\left(\sum_{n \in \mathbb{N}} \frac{1}{n^2} \right) 1 \geq \sum_{n \in \mathbb{N}} \left\langle \frac{1}{n}\eta^{(n)} \,\middle|\, \frac{1}{n}\eta^{(n)} \right\rangle \geq$$

$$\geq \sum_{n \in \mathbb{N}} \sum_{\iota \in K_n} \langle \eta_\iota | \eta_\iota \rangle \geq \sum_{\iota \in J} \langle \eta_\iota | \eta_\iota \rangle$$

for every $J \in \mathfrak{P}_f(I)$ so that $\eta \in H$. For every $n \in \mathbb{N}$,

$$\left\langle \sum_{\iota \in K_n} \langle \eta_\iota | \xi_\iota \rangle, a \right\rangle = \langle \langle \eta | \xi_{K_n} \rangle, a \rangle =$$

$$= \frac{1}{n} \left\langle \sum_{\iota \in K_n} \langle \eta_\iota^{(n)} | \xi_\iota \rangle, a \right\rangle > \frac{\varepsilon}{2n},$$

$$\left\langle \sum_{\iota \in L_n} \langle \eta_\iota | \xi_\iota \rangle, a \right\rangle = \sum_{k=1}^{n} \langle \sum_{\iota \in K_n} \langle \eta_\iota | \xi_\iota \rangle, a \rangle > \frac{\varepsilon}{2} \sum_{k=1}^{n} \frac{1}{k}.$$

Hence

$$\lim_{n \to \infty} \left\langle \sum_{\iota \in L_n} \langle \eta_\iota | \xi_\iota \rangle, a \right\rangle = \infty,$$

which is a contradiction.

d) Take $\xi, \eta \in H$ and $x, y \in E$. Then

$$\langle x\xi_\iota | x\xi_\iota \rangle \leq \|x\|^2 \langle \xi_\iota | \xi_\iota \rangle$$

for every $\iota \in I$ so that

$$\sum_{\iota \in J} \langle x\xi_\iota | x\xi_\iota \rangle \leq \|x\|^2 \sum_{\iota \in J} \langle \xi_\iota | \xi_\iota \rangle$$

for every $J \in \mathfrak{P}_f(I)$. It follows $x\xi \in H$ and

$$\langle x\xi | x\xi \rangle \leq \|x\|^2 \langle \xi | \xi \rangle .$$

The relations

$$(xy)\xi = x(y\xi) , \quad (x\xi)y = x(\xi y) ,$$

$$\langle x\xi | \eta \rangle = \langle \xi | x^*\eta \rangle , \quad (1\xi = \xi)$$

are easy to verify.

e) follows from Minkowski's Theorem. ■

f) For $\iota, \lambda \in I$,

$$\langle f_\iota | f_\lambda \rangle = \sum_{\mu \in I}^{\ddot{E}} \delta_{\lambda\mu} p_\lambda \delta_{\iota\mu} p_\mu = \delta_{\iota\lambda} p_\lambda .$$

Let $\xi \in H$ such that

$$\langle \xi | f_\iota \rangle = 0$$

for every $\iota \in I$. Then for every $\iota \in I$,

$$0 = \langle \xi | f_\iota \rangle = \sum_{\lambda \in I}^{\ddot{E}} \delta_{\iota\lambda} p_\iota \xi_\lambda = p_\iota \xi_\iota = \xi_\iota .$$

Hence $\xi = 0$ and $f_\iota)_{\iota \in I}$ is a Fourier basis of H. ■

Theorem 5.6.4.7 (8) *We use the notation and the hypotheses of Proposition 5.6.4.6 and assume in addition that H_ι is self-dual for each $\iota \in I$.*

a) *H is self-dual.*

b) *$\bigoplus_{\iota \in I} H_\iota$ is a Banach subspace of H dense in $H_{\ddot{H}}$.*

c) *For every $J \subset I$, the map*

$$\pi_J : H \longrightarrow H , \quad \xi \longmapsto \xi_J$$

belongs to $\Pr \mathcal{L}_E(H)$ and for every $u \in \mathcal{L}_E(H)$

$$u = \lim_{J, \mathfrak{F}_I} \pi_J u \pi_J ,$$

where the limit is taken in $(\mathcal{L}_E(H))_{\ddot{H}}$.

d) *Put*

$$\mathcal{G} := \left\{ u \in \mathcal{L}_E(H) \,\middle|\, u\left(\bigoplus_{\iota\in I} H_\iota\right) \subset \bigoplus_{\iota\in I} H_\iota\,,\quad u^*\left(\bigoplus_{\iota\in I} H_\iota\right) \subset \bigoplus_{\iota\in I} H_\iota \right\}$$

and

$$\widetilde{u} : \bigoplus_{\iota\in I} H_\iota \longrightarrow \bigoplus_{\iota\in I} H_\iota\,,\quad \xi \longmapsto u\xi$$

for every $u \in \mathcal{G}$. *Then* $\mathcal{G}$ *is a unital* C^**-subalgebra of* $\mathcal{L}_E(H)$, *dense in* $(\mathcal{L}_E(H))_{\ddot{H}}$, *generating* $\mathcal{L}_E(H)$ *as* W^**-algebra (a) and Theorem 5.6.3.5 b)),* $\widetilde{u} \in \mathcal{L}_E\left(\bigoplus_{\iota\in I} H_\iota\right)$ *for every* $u \in \mathcal{G}$, *and the map*

$$\varphi : \mathcal{G} \longrightarrow \mathcal{L}_E\left(\bigoplus_{\iota\in I} H_\iota\right)\,,\quad u \longmapsto \widetilde{u}$$

is an isometry of C^**-algebras.*

a) By Proposition 5.6.3.3 b $\Rightarrow$ a, it is sufficient to prove that $H_{\ddot{H}}^{\#}$ is compact. Let $\mathfrak{F}$ be an ultrafilter on $H^{\#}$. Put

$$\eta : I \longrightarrow \bigcup_{\iota\in I} H_\iota\,,\quad \iota \longmapsto \lim_{\zeta,\mathfrak{F}} \zeta_\iota\,,$$

where the limit is taken in $(H_\iota)_{H_\iota}$ (Proposition 5.6.3.3 a $\Rightarrow$ b). Let $a \in \ddot{E}$ and $J \in \mathfrak{P}_f(I)$. Then

$$|\langle\langle \eta_J | \eta_J \rangle, a\rangle| = \lim_{\zeta,\mathfrak{F}} |\langle\langle \zeta | \eta_J \rangle, a\rangle| \leq \|a\|\,\|\eta_J\|$$

(Proposition 5.6.1.2 d)) so that

$$\|\eta_J\|^2 = \|\langle \eta_J | \eta_J \rangle\| = \sup_{a\in E^{\#}} |\langle\langle \eta_J | \eta_J \rangle, a\rangle| \leq \|\eta_J\|\,,$$

$$\|\eta_J\| \leq 1\,.$$

Since J is arbitrary, $\eta \in H^{\#}$ (Proposition 5.6.3.7, Corollary 4.2.1.17 a $\Leftrightarrow$ b).

Take $a \in \ddot{E}$, $\xi \in H$, and $\varepsilon > 0$. By Proposition 5.6.4.6 c), there is a $J \in \mathfrak{P}_f(I)$ such that

$$|\langle\langle \zeta | \xi_{I\setminus J} \rangle, a\rangle| < \varepsilon$$

for every $\zeta \in H^{\#}$. It follows

$$|\widetilde{(a,\xi)}(\eta) - \widetilde{(a,\xi)}(\zeta)| = |\langle\langle\eta|\xi\rangle, a\rangle - \langle\langle\zeta|\xi\rangle, a\rangle| \leq$$

$$\leq |\langle\langle\eta - \zeta|\xi_J\rangle, a\rangle| + |\langle\langle\eta|\xi_{I\setminus J}\rangle, a\rangle| + |\langle\langle\zeta|\xi_{I\setminus J}\rangle, a\rangle| \leq$$

$$\leq |\langle\langle\eta - \zeta|\xi_J\rangle, a\rangle| + 2\varepsilon$$

for every $\zeta \in H^{\#}$ and so

$$\limsup_{\zeta,\mathfrak{F}} \left|\widetilde{(a,\xi)}(\eta) - \widetilde{(a,\xi)}(\zeta)\right| \leq 2\varepsilon\,.$$

Since ε is arbitrary,

$$\lim_{\zeta,\mathfrak{F}} \widetilde{(a,\xi)}(\zeta) = \widetilde{(a,\xi)}(\eta)\,.$$

Since a and ξ are also arbitrary, $\mathfrak{F}$ converges to η in $H^{\#}_{\ddot{H}}$. Hence $H^{\#}_{\ddot{H}}$ is compact.

b) follows from Proposition 5.6.4.6 c).

c) It is easy to see that $\pi_J \in \Pr\mathcal{L}_{\ddot{E}}(H)$ for every $J \subset I$. Take $(a,\xi,\eta) \in \ddot{E} \times H \times H$ and $\varepsilon > 0$. By Proposition 5.6.4.6 c), there is a $J \in \mathfrak{P}_f(I)$ such that for every $\zeta \in H^{\#}$ and $K \in \mathfrak{P}_f(I)$, $J \subset K$,

$$|\langle\langle\zeta|\eta_{I\setminus K}\rangle, a\rangle| < \frac{\varepsilon}{2(1 + \|u\|\,\|\xi\|)}\,,$$

$$|\langle\langle\zeta|(u^*\eta)_{I\setminus K}\rangle, a\rangle| < \frac{\varepsilon}{2(1 + \|\xi\|)}\,.$$

Hence for every $K \in \mathfrak{P}_f(I)$, $J \subset K$,

$$\left|\widetilde{(a,\xi,\eta)}(u\pi_K - \pi_K u\pi_K)\right| = |\langle\langle u\pi_K\xi - \pi_K u\pi_K\xi|\eta\rangle, a\rangle| =$$

$$= |\langle\langle(u\pi_K\xi)_{I\setminus K}|\eta\rangle, a\rangle| = |\langle\langle u\pi_K\xi|\eta_{I\setminus K}\rangle, a\rangle| <$$

$$< \frac{\varepsilon\|u\|\,\|\xi\|}{2(1 + \|u\|\,\|\xi\|)} < \frac{\varepsilon}{2}\,,$$

$$|\widetilde{(a,\xi,\eta)}(u - u\pi_K)| = |\langle\langle u(\xi - \pi_K\xi)|\eta\rangle, a\rangle| =$$

$$= |\langle\langle\xi|(u^*\eta)_{I\setminus K}\rangle, a\rangle| < \frac{\varepsilon\|\xi\|}{2(1 + \|\xi\|)} < \frac{\varepsilon}{2}\,,$$

so that

$$\left|\widetilde{(a,\xi,\eta)}(u) - \widetilde{(a,\xi,\eta)}(\pi_K u \pi_K)\right| \leq$$

$$\leq \left|\widetilde{(a,\xi,\eta)}(u - u\pi_K)\right| + \left|\widetilde{(a,\xi,\eta)}(u\pi_K - \pi_K u \pi_K\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Hence

$$u = \lim_{J,\mathfrak{F}_I} \pi_J u \pi_J$$

in $(\mathcal{L}_E(H))_{\ddot{H}}$.

d) It is easy to see that $\mathcal{G}$ is a unital C^*-subalgebra of $\mathcal{L}_E(H)$. By c), $\mathcal{G}$ is dense in $(\mathcal{L}_E(H))_{\ddot{H}}$, so that $\mathcal{G}$ generates $\mathcal{L}_E(H)$ as W^*-algebra (Theorem 5.6.3.5 b) and Corollary 4.4.4.12 a)). Let $u \in \mathcal{G}$. For $\xi, \eta \in \bigoplus_{\iota \in I} H_\iota$,

$$\langle \tilde{u}\xi | \eta \rangle = \langle u\xi | \eta \rangle = \langle \xi | u^* \eta \rangle = \langle \xi | \widetilde{u^*} \eta \rangle$$

so that $\tilde{u} \in \mathcal{L}_E(\bigoplus_{\iota \in I} H_\iota)$ and $\tilde{u}^* = \widetilde{u^*}$. In particular, φ is involutive. It is easy to see that φ is an injective unital algebra homomorphism. We want to show that φ is surjective.

Let $u \in \mathcal{L}_E(\bigoplus_{\iota \in I} H_\iota)$ and let $\mathfrak{F}$ be an ultrafilter on $\mathfrak{P}_f(I)$ finer than $\mathfrak{F}_I$. Then

$$\|u\xi_J\| \leq \|u\| \, \|\xi_J\| \leq \|u\| \, \|\xi\|$$

for every $\xi \in H$ and $J \in \mathfrak{P}_f(I)$. By a) and Proposition 5.6.3.3 a $\Rightarrow$ b, we may define

$$v : H \longrightarrow H, \quad \xi \longmapsto \lim_{J,\mathfrak{F}} u\xi_J ,$$

$$w : H \longrightarrow H, \quad \xi \longmapsto \lim_{J,\mathfrak{F}} u^*\xi_J ,$$

where the limits are considered in $H_{\ddot{H}}$. Take $\xi, \eta \in H$. Then for every $a \in \ddot{E}$,

$$\langle \langle \xi | w\eta \rangle, a \rangle = \overline{\langle \langle w\eta | \xi \rangle, a^* \rangle} =$$

$$= \overline{\widetilde{(a^*, \xi)}(w\eta)} = \lim_{J,\mathfrak{F}} \overline{\widetilde{(a^*, \xi)}(u^*\eta_J)} =$$

$$\lim_{J,\mathfrak{F}} \overline{\langle\langle u^*\eta_J|\xi\rangle, a^*\rangle} = \lim_{J,\mathfrak{F}} \langle\langle \xi|u^*\eta_J\rangle, a\rangle\,,$$

$$\langle\langle v\xi|\eta\rangle, a\rangle = \widetilde{(a,\eta)}\,(v\xi) = \lim_{J,\mathfrak{F}} \widetilde{(a,\eta)}\,(u\xi_J) =$$

$$= \lim_{J,\mathfrak{F}} \langle\langle u\xi_J|\eta\rangle, a\rangle = \lim_{J,\mathfrak{F}} \lim_{K,\mathfrak{F}} \langle\langle u\xi_J|\eta_K\rangle, a\rangle =$$

$$= \lim_{J,\mathfrak{F}} \lim_{K,\mathfrak{F}} \langle\langle \xi_J|u^*\eta_K\rangle, a\rangle = \lim_{J,\mathfrak{F}} \langle\langle \xi_J|w\eta\rangle, a\rangle =$$

$$= \langle\langle \xi|w\eta\rangle, a\rangle\,.$$

Hence

$$\langle v\xi|\eta\rangle = \langle \xi|w\eta\rangle$$

and so $v \in \mathcal{L}_E(H)$ and $v^* = w$. For every $\zeta \in \bigoplus_{\iota\in I} H_\iota$,

$$\zeta = \lim_{J,\mathfrak{F}} \zeta_J$$

so that

$$v\zeta = \lim_{J,\mathfrak{F}} u\zeta_J = u\zeta\,,$$

$$w\zeta = \lim_{J,\mathfrak{F}} u^*\zeta_J = u^*\zeta\,.$$

Thus $v \in \mathcal{G}$ and

$$u = \widetilde{v} \in \varphi(\mathcal{G})\,.$$

Therefore φ is surjective and so it is an isometry of C^*-algebras. ■

Proposition 5.6.4.8 *Let E be a W^*-algebra and $(p_\iota)_{\iota\in I}$ a family in $E^c \cap \operatorname{Pr} E$ (in $\operatorname{Pr} E$). Put*

$$H_\iota := p_\iota E$$

for every $\iota \in I$.

a) *H_ι is a von Neumann (right) unital E-module for every $\iota \in I$.*

b) $H := \overset{W}{\bigoplus_{\iota \in I}} H_\iota$ *is a von Neumann (right) unital* E*-module. For every* $x \in E$ *(resp.* $x \in E^c$*) put*

$$\widetilde{x} : H \longrightarrow H\,, \quad \xi \longmapsto x\xi \ \ (\textit{resp.}\ \ \xi x)\,.$$

c) $\widetilde{x} \in \mathcal{L}_E(H)$ *for every* $x \in E$ *(resp.* $x \in E^c$*). Put*

$$\varphi : E \ \ (\textit{resp.}\ \ E^c) \longrightarrow \mathcal{L}_E(H)\,, \quad x \longmapsto \widetilde{x}\,.$$

d) φ *is a* W^**-homomorphism (Theorem 5.6.3.5 b), Corollary 4.4.4.12 e)) and*

$$\widetilde{\ddot{\varphi}(a,\xi,\eta)} = \sum_{\iota \in I}^{E} \xi_\iota a \eta_\iota^* \quad \left(\textit{resp.}\ \ i'\left(\sum_{\iota \in I}^{E} \xi_\iota a \eta_\iota^* \right) \right)$$

for every $(a,\xi,\eta) \in \ddot{E} \times H \times H$*, where* $i : E^c \to E$ *denotes the inclusion map and* $\ddot{\varphi}$ *the pretranspose of* φ*.*

e) *Let* $\xi, \eta_l \in H$ *such that*

$$\xi_\iota^* \xi_\lambda = \eta_\iota^* \eta_\lambda = \eta_\iota^* \xi_\lambda = 0$$

for all distinct $\iota, \lambda \in I$*. Then* $(\xi_\iota)_{\iota \in I}$ *and* $(\eta_\iota)_{\iota \in I}$ *are summable in* $E_{\ddot{E}}$ *and*

$$\langle \xi | \eta \rangle = \left(\sum_{\iota \in I}^{\ddot{E}} \eta_\iota \right)^* \left(\sum_{\iota \in I}^{\ddot{E}} \xi_\iota \right)\,.$$

f) *Assume* $p_\iota p_\lambda = 0$ *for all distinct* $\iota, \lambda \in I$ *and put*

$$p := \sum_{\iota \in I}^{\ddot{E}} p_\iota \in \operatorname{Pr} E$$

(Corollary 5.6.3.21). Then the map

$$H \longrightarrow pE\,, \quad \xi \longmapsto \sum_{\iota \in I}^{E} \xi_\iota$$

is an isomorphism of von Neumann (right) E*-modules.*

a) By Example 5.6.1.5, H_ι is a Hilbert (right) unital E-module and by Proposition 5.6.2.3, it is self-dual.

b) By a) and Proposition 5.6.4.6 b),d), H is a Hilbert (right) unital E-module and by Theorem 5.6.4.7 a), it is self-dual.

c) was proved in Proposition 5.6.1.8 h).

d) By Proposition 5.6.1.8 h), φ is an involutive algebra homomorphism. For every $x \in E$, (resp. $x \in E^c$),

$$\langle \varphi x, \widetilde{(a,\xi,\eta)} \rangle = \langle \widetilde{x}, \widetilde{(a,\xi,\eta)} \rangle = \langle \langle \widetilde{x}\xi | \eta \rangle, a \rangle =$$

$$= \langle \langle x\xi | \eta \rangle, a \rangle \quad (\text{resp. } \langle \langle \xi x | \eta \rangle, a \rangle) =$$

$$= \left\langle \sum_{\iota \in I}^{E} \langle x\xi_\iota | \eta_\iota \rangle, a \right\rangle \quad \left(\text{resp. } \left\langle \sum_{\iota \in I}^{\ddot{E}} \langle \xi_\iota x | \eta_\iota \rangle, a \right\rangle \right) =$$

$$= \left\langle \sum_{\iota \in I}^{\ddot{E}} \eta_\iota^* x \xi_\iota, a \right\rangle = \sum_{\iota \in I} \langle \eta_\iota^* x \xi_\iota, a \rangle =$$

$$= \sum_{\iota \in I} \langle x, \xi_\iota a \eta_\iota^* \rangle = \left\langle x, \sum_{\iota \in I}^{E} \xi_\iota a \eta_\iota^* \right\rangle .$$

By Theorem 5.6.3.5 d), φ is a W^*-homomorphism and

$$\ddot{\varphi}\widetilde{(a,\xi,\eta)} = \sum_{\iota \in I}^{E} \xi_\iota a \eta_\iota^* \quad \left(\text{resp. } i' \left(\sum_{\iota \in I}^{E} \xi_\iota a \eta_\iota^* \right) \right) .$$

e) By Proposition 5.6.3.20 b $\Rightarrow$ a, $(\xi_\iota)_{\iota \in I}$ and $(\eta_\iota)_{\iota \in I}$ are summable in $E_{\ddot{E}}$. For $J \in \mathfrak{P}_f(I)$,

$$\left(\sum_{\iota \in J} \eta_\iota \right)^* \left(\sum_{\iota \in J} \xi_\iota \right) = \sum_{\iota, \lambda \in J} \eta_\iota^* \xi_\lambda = \sum_{\iota \in J} \eta_\iota^* \xi_\iota .$$

By Proposition 5.6.3.19 a) (and Proposition 4.4.1.3),

$$\left(\sum_{\iota \in I}^{\ddot{E}} \eta_\iota \right)^* \left(\sum_{\iota \in I}^{\ddot{E}} \xi_\iota \right) = \sum_{\iota \in I}^{\ddot{E}} \eta_\iota^* \xi_\iota = \langle \xi | \eta \rangle .$$

f) For every $\xi, \eta \in H$ and distinct $\iota, \lambda \in I$,

$$\eta_\iota^* \xi_\lambda = \eta_\iota^* p_\iota p_\lambda \xi_\lambda = 0 .$$

By e), $(\xi_\iota)_{\iota \in I}$ and $(\eta_\iota)_{\iota \in I}$ are summable in $E_{\ddot{E}}$ and

$$\left(\sum_{\iota \in I}^{\ddot{E}} \eta_\iota \right)^* \left(\sum_{\iota \in I}^{\ddot{E}} \xi_\iota \right) = \langle \xi | \eta \rangle .$$

Let $\zeta \in pE$. Then $p_\iota \zeta \in H_\iota$ for every $\iota \in I$ and

$$\sum_{\iota \in J} (p_\iota \zeta)^* (p_\iota \zeta) = \sum_{\iota \in J} \zeta^* p_\iota \zeta \leq \zeta^* p \zeta = \zeta^* \zeta$$

for every $J \in \mathfrak{P}_f(I)$. Hence $(p_\iota \zeta)_{\iota \in I} \in H$. Moreover

$$\sum_{\iota \in I}^{\ddot{E}} (p_\iota \zeta) = p\zeta = \zeta$$

(Proposition 5.6.3.17 d)), so that the map

$$H \longrightarrow pE , \quad \xi \longmapsto \sum_{\iota \in I}^{\ddot{E}} \xi_\iota$$

is surjective. ■

Example 5.6.4.9 *Let I be an infinite set. Put*

$$E := \ell^\infty(I) , \quad H := \bigoplus_{\iota \in I}^{W} E$$

(Proposition 5.6.4.8 a),b)),

$$\mathcal{G} := \left\{ u \in \mathcal{L}_E(H) \mid u(\ell^2(I,E)) \subset \ell^2(I,E) , \quad u^*(\ell^2(I,E)) \subset \ell^2(I,E) \right\} ,$$

and for every $\xi \in H$ and $\iota, \lambda \in I$,

$$\xi_{\iota\lambda} := \xi_\iota(\lambda) \in \mathbb{K} ,$$

i.e.

$$\xi := ((\xi_{\iota\lambda})_{\lambda \in I})_{\iota \in I} .$$

Fix a point ι_0 in I and define for every $\iota \in I$ a map $p_\iota : H \to H$ by putting

$$(p_\iota \xi)_{\lambda,\mu} := \frac{1}{2} (\xi_{\iota_0 \iota} + \xi_{\iota\iota}) \, \delta_{\iota\mu} \, (\delta_{\lambda\iota_0} + \delta_{\lambda\iota})$$

for every $\xi \in H$ and $\lambda, \mu \in I$.

a) *If* $\iota, \lambda \in I \backslash \{\iota_0\}$ *then* $p_\iota \in \Pr \mathcal{G}$ *and*

$$p_\iota p_\lambda = \delta_{\iota\lambda} p_\lambda .$$

b) *The family* $(p_\iota)_{\iota \in I}$ *is unbounded in* $\mathcal{G}$.

c) $\mathcal{L}_E(\ell^2(I, E))$ *is not a* W^**-algebra.*

a) For $\xi, \eta \in H$ and $\mu \in I$,

$$\langle p_\iota \xi | \eta \rangle_\mu = \sum_{\nu \in I} (p_\iota \xi)_{\nu\mu} \overline{\eta}_{\nu\mu} =$$

$$= \frac{1}{2} \sum_{\nu \in I} (\xi_{\iota_0 \iota} + \xi_{\iota\iota}) \, \delta_{\iota\mu} \, (\delta_{\nu\iota_0} + \delta_{\nu\iota}) \, \overline{\eta}_{\nu\mu} =$$

$$= \frac{1}{2} \, (\xi_{\iota_0 \iota} + \xi_{\iota\iota}) \, \delta_{\iota\mu} \, (\overline{\eta}_{\iota_0 \mu} + \overline{\eta}_{\iota\mu}) =$$

$$= \frac{1}{2} \, (\xi_{\iota_0 \mu} + \xi_{\iota\mu}) \, \delta_{\iota\mu} (\overline{\eta}_{\iota_0 \iota} + \overline{\eta}_{\iota\iota}) =$$

$$= \frac{1}{2} \sum_{\nu \in I} \xi_{\nu\mu} \, (\delta_{\nu\iota_0} + \delta_{\nu\iota}) \, \delta_{\iota\mu} \, (\overline{\eta}_{\iota_0 \iota} + \overline{\eta}_{\iota\iota}) =$$

$$= \sum_{\nu \in I} \xi_{\nu\mu} \overline{(p_\iota \eta)}_{\nu\mu} = \langle \xi | p_\iota \eta \rangle_\mu$$

so that

$$\langle p_\iota \xi | \eta \rangle = \langle \xi | p_\iota \eta \rangle .$$

Hence $p_\iota \in \mathcal{L}_E(H)$ and $p_\iota = p_\iota^*$. It is easy to see that $p \in \mathcal{G}$ for every $\iota \in I$.

For $\xi \in H$ and $\mu, \nu \in I$,

$$(p_\iota p_\lambda \xi)_{\mu\nu} = \frac{1}{2} ((p_\lambda \xi)_{\iota_0 \iota} + (p_\lambda \xi)_{\iota\iota}) \delta_{\iota\nu} (\delta_{\mu\iota_0} + \delta_{\mu\iota}) =$$

$$= \frac{1}{2} \begin{pmatrix} \frac{1}{2}(\xi_{\iota_0 \lambda} + \xi_{\lambda\lambda}) \delta_{\lambda\iota} (\delta_{\iota_0 \iota_0} + \delta_{\iota_0 \lambda}) + \\ + \frac{1}{2} (\xi_{\iota_0 \lambda} + \xi_{\lambda\lambda}) \delta_{\lambda\iota} (\delta_{\iota\iota_0} + \delta_{\iota\lambda}) \end{pmatrix} \delta_{\iota\nu} (\delta_{\mu\iota_0} + \delta_{\mu\iota}) =$$

$$= \frac{1}{4}(\xi_{\iota_0\iota} + \xi_{\iota\iota})\delta_{\lambda\iota}(1 + \delta_{\iota\lambda})\delta_{\iota\nu}(\delta_{\mu\iota_0} + \delta_{\mu\iota}) =$$

$$= \delta_{\iota\lambda}\frac{1}{2}(\xi_{\iota_0\iota} + \xi_{\iota\iota})\delta_{\iota\nu}(\delta_{\mu\iota_0} + \delta_{\mu\iota}) = \delta_{\iota\lambda}(p_\iota\xi)_{\mu\nu}$$

so that

$$p_\iota p_\lambda \xi = \delta_{\iota\lambda} p_\iota \xi\,,$$

$$p_\iota p_\lambda = \delta_{\iota\lambda} p_\iota\,,$$

$$p_\iota \in \Pr\mathcal{G}\,.$$

b) Let u be an upper bound of the family $(p_\iota)_{\iota\in I}$ in $\mathcal{G}$. Put

$$\xi := (\delta_{\iota_0\iota} e_I)_{\iota\in I} \in \ell^2(I, E)\,.$$

Then for $\iota \in I\backslash\{\iota_0\}$ and $\lambda, \mu \in I$,

$$(p_\iota\xi)_{\lambda\mu} = \frac{1}{2}(\xi_{\iota_0\iota} + \xi_{\iota\iota})\delta_{\iota\mu}(\delta_{\lambda\iota_0} + \delta_{\lambda\iota}) = \frac{1}{2}\delta_{\iota\mu}(\delta_{\lambda\iota_0} + \delta_{\lambda\iota})$$

so that

$$\langle p_\iota\xi|\xi\rangle_\mu = \sum_{\lambda\in I}(p_\iota\xi)_{\lambda\mu}\bar{\xi}_{\lambda\mu} = \frac{1}{2}\delta_{\iota\mu}\left(\bar{\xi}_{\iota_0\mu} + \bar{\xi}_{\iota\mu}\right) = \frac{\delta_{\iota\mu}}{2}\,.$$

By Theorem 5.6.1.11 $c_1 \Rightarrow c_2$,

$$\langle u\xi|\xi\rangle_\iota \geq \langle p_\iota\xi|\xi\rangle_\iota \geq \frac{1}{2}\,,$$

$$(\langle u\xi|\xi\rangle^*\langle u\xi|\xi\rangle)_\iota \geq \frac{1}{4}\,.$$

Hence the family $(\langle u\xi|\xi\rangle^*\langle u\xi|\xi\rangle)_{\iota\in I}$ is not summable in E, i.e.

$$\langle u\xi|\xi\rangle \notin \ell^2(I, E)$$

and this is a contradiction.

c) By a), $(p_\iota)_{\iota\in I\backslash\{\iota_0\}}$ is a family in $\Pr\mathcal{G}$. By b) and Theorem 4.4.1.8 i), $\mathcal{G}$ is not a W^*-algebra. By Theorem 5.6.4.7 d), $\mathcal{L}_E(\ell^2(I, E))$ is not a W^*-algebra. ■

Proposition 5.6.4.10 (8) *Let E be a W^*-algebra, H a von Neumann right E module, and A a Fourier basis of H. For every $\xi \in A$ denote by*

$$H_\xi := \langle \xi | \xi \rangle E$$

the von Neumann right E-module (Proposition 5.6.2.3, Proposition 5.6.4.8 a)) and put

$$G := \overset{W}{\bigoplus_{\xi \in A}} H_\xi .$$

a) *For every $\eta \in H$,*

$$(\langle \eta | \xi \rangle)_{\xi \in A} \in G$$

and the map

$$\varphi : H \longrightarrow G , \quad \eta \longmapsto (\langle \eta | \xi \rangle)_{\xi \in A}$$

is an isomorphism of von Neumann right E-modules (Theorem 5.6.4.7 a)).

b) *If $\ddot{\varphi}$ denotes the pretranspose of φ then*

$$\ddot{\varphi}\widetilde{(a, \zeta)} = \widetilde{(a, \varphi^{-1}\zeta)}$$

for every $(a, \zeta) \in \ddot{E} \times G$.

c) *For every $p \in \Pr E$ put*

$$T_p := \{\xi \in A | \langle \xi | \xi \rangle = p\}$$

and define

$$B := \{p \in \Pr E | T_p \neq \emptyset\} .$$

If E is finite-dimensional then H is isomorphic to

$$\bigoplus_{p \in B} \ell^2(T_p, pE) .$$

a) By Proposition 5.6.3.9 a $\Rightarrow$ b, for every $\xi \in A$,

$$\langle \eta | \xi \rangle = \langle \eta | \xi \langle \xi | \xi \rangle \rangle = \langle \xi | \xi \rangle \langle \eta | \xi \rangle \in H_\xi$$

(Proposition 5.6.1.2 a)). By Theorem 5.6.3.13 f), the family

$$(\langle \xi|\eta\rangle\langle\eta|\xi\rangle)_{\xi\in A}$$

is summable in $E_{\ddot{E}}$, so that

$$(\langle\eta|\xi\rangle)_{\xi\in A} \in G\,.$$

φ is obviously linear and injective (Theorem 5.6.3.13 b).

Take $\zeta \in G$. By Corollary 4.2.2.3 (and Proposition 5.6.1.2 b)), for every $B \in \mathfrak{P}_f(A)$,

$$\sum_{\xi\in B}\langle \xi\zeta_\xi|\xi\zeta_\xi\rangle = \sum_{\xi\in B}\zeta_\xi^*\langle\xi|\xi\rangle\zeta_\xi \leq$$

$$\leq \sum_{\xi\in B}\zeta_\xi^*\zeta_\xi \leq \sum_{\xi\in A}^{\ddot{E}}\zeta_\xi^*\zeta_\xi\,.$$

By Proposition 5.6.3.8 a), the family $(\xi\zeta_\xi)_{\xi\in A}$ is summable in $H_{\ddot{H}}$. Put

$$\eta := \sum_{\xi\in A}^{\ddot{H}}\xi\zeta_\xi\,.$$

Then

$$\langle\eta|\xi_0\rangle = \sum_{\xi\in A}^{\ddot{E}}\langle\xi\zeta_\xi|\xi_0\rangle = \sum_{\xi\in A}^{\ddot{E}}\langle\xi|\xi_0\rangle\zeta_\xi = \zeta_{\xi_0}$$

for every $\xi_0 \in A$. Hence

$$\varphi\eta = \zeta$$

and φ is surjective.

Take $\eta, \zeta \in H$. By Theorem 5.6.3.13 f),

$$\langle\eta|\zeta\rangle = \sum_{\xi\in A}^{\ddot{E}}\langle\xi|\zeta\rangle\langle\eta|\xi\rangle = \sum_{\xi\in A}^{\ddot{E}}\langle\langle\eta|\xi\rangle|\langle\zeta|\xi\rangle\rangle =$$

$$= \langle\varphi\eta|\varphi\zeta\rangle\,.$$

Hence φ is an isomorphism of von Neumann right E-modules.

b) For every $\eta \in H$,

$$\langle\varphi\eta, \widetilde{(a,\zeta)}\rangle = \langle\langle\varphi\eta|\zeta\rangle, a\rangle =$$

$$= \sum_{\xi \in A} \langle \zeta_\xi^* \langle \eta | \xi \rangle, a \rangle = \sum_{\xi \in A} \langle \langle \eta | \xi \zeta_\xi \rangle, a \rangle =$$

$$= \langle \langle \eta | \varphi^{-1} \zeta \rangle, a \rangle = \langle \eta, \widetilde{(a, \varphi^{-1} \zeta)} \rangle$$

so that

$$\ddot{\varphi}\widetilde{(a, \zeta)} = \widetilde{(a, \varphi^{-1} \zeta)} .$$

c) Take $x \in \bigoplus_{p \in B} \ell^2(T_p, pE)$. By Theorem 5.6.3.13 a), $(\xi x_\xi)_{\xi \in A}$ is summable in H. Put

$$\widetilde{x} := \sum_{\xi \in A} \xi x_\xi .$$

Let $\eta \in H$ and put

$$y := (\langle \eta | \xi \rangle)_{\xi \in A} .$$

By Theorem 5.6.3.13 d),f), $y \in \bigoplus_{p \in \Pr \mathcal{U}} \ell^2(T_p, pE)$ and

$$\widetilde{y} = \eta ,$$

so that the map

$$\bigoplus_{p \in \Pr E} \ell^2(T_p, pE) \longrightarrow H , \quad x \longmapsto \widetilde{x}$$

is surjective. By Theorem 5.6.3.13 f),

$$\langle \widetilde{x} | \widetilde{y} \rangle = \sum_{\xi \in A} \langle \xi | \widetilde{y} \rangle \langle \widetilde{x} | \xi \rangle = \sum_{\xi \in A} y_\xi^* x_\xi = \langle x | y \rangle$$

for all $x, y \in \bigoplus_{p \in \Pr E} \ell^2(T_p, pE)$. Hence the above map is an isomorphism of Hilbert right E-modules. ■

Corollary 5.6.4.11 (8) *Let H be a Hilbert right $\mathbb{H}$-module and A a Fourier basis of H.*

a) *For every $\eta \in H$,*

$$\eta = \sum_{\xi \in A} \xi \langle \eta | \xi \rangle .$$

b) *The map*

$$H \longrightarrow \ell^2(A, \mathbb{H}), \quad \eta \longmapsto (\langle \eta | \xi \rangle)_{\xi \in A}$$

is an isomorphism of Hilbert right $\mathbb{H}$*-modules.*

c) *All Fourier bases of* H *have the same cardinality and* H *may be endowed with the structure of a unital Hilbert* $\mathbb{H}$*-module.*

d) *Two Hilbert right* $\mathbb{H}$*-modules are isomorphic iff their Fourier bases have the same cardinality.*

e) *If* $\widetilde{H}$ *denotes the real Hilbert space obtained by endowing the underlying real vector space of* H *with the scalar product*

$$H \times H \longrightarrow \mathbb{R}, \quad (\xi, \eta) \longmapsto \mathrm{re}\, \langle \xi | \eta \rangle$$

(Proposition 5.6.2.5 a),e)) then $A \cup (Ai) \cup (Aj) \cup (Ak)$ *is an orthonormal basis of* $\widetilde{H}$.

a) By Proposition 5.6.2.5 f), H is self-dual so that the assertion follows from Theorem 5.6.3.13 d),f).

b) follows from a) and Proposition 5.6.4.10 c).

c) and d) follow from b).

e) is easy to see. ■

Proposition 5.6.4.12 (8) *Take* $E \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $n \in \mathbb{N}$, *and* H *a Hilbert right* $E_{n,n}$ *module. For every* $i \in \mathbb{N}_n$, *put*

$$p_i := [\delta_{ij}\delta_{ik}]_{j,k \in \mathbb{N}_n} \in E_{n,n}.$$

a) p_i *is a minimal element of* $\mathrm{Pr}\, E_{n,n} \backslash \{0\}$ *for every* $i \in \mathbb{N}_n$.

b) *If* $\mathfrak{A}$ *denotes the set of Fourier sets* A *of* H *such that*

$$\langle \xi | \xi \rangle = p_1$$

for every $\xi \in A$, *then every maximal element of* $\mathfrak{A}$ *(*$\mathfrak{A}$ *ordered by inclusion) is a Fourier basis of* H.

c) *There is a unique cardinal number* $\aleph$ *such that* H *is isomorphic to* $\ell^2(\aleph, p_1 E_{n,n})$.

a) It is easy to see that $p_i \in \Pr E_{n,n}\backslash\{0\}$. Let $q \in \Pr E_{n,n}\backslash\{0\}$ with $q \leq p_i$. Then

$$q = p_i q p_i$$

(Corollary 4.2.7.6 a $\Rightarrow$ f) so that

$$q_{jk} = \sum_{\ell,m=1}^{n} \delta_{ij}\delta_{i\ell}q_{\ell m}\delta_{im}\delta_{ik} = \delta_{ij}q_{ii}\delta_{ik}$$

for all $j,k \in \mathbb{N}_n$. Hence $q = p$ and p is a minimal element of $\Pr E_{n,n}\backslash\{0\}$.

b) Let A be a maximal element of $\mathfrak{A}$. Assume A is not a Fourier basis of H. Then there is an $\eta \in H$ such that

$$\langle\eta|\eta\rangle \in \Pr E_{n,n}\backslash\{0\},$$

and

$$\xi \in A \Longrightarrow \langle\xi|\eta\rangle = 0.$$

Since

$$\eta = \eta 1 = \eta\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \eta p_i,$$

there is an $i \in \mathbb{N}_n$ with $\eta p_i \neq 0$. It follows

$$p_i\langle\eta|\eta\rangle p_i = \langle\eta p_i|\eta p_i\rangle \neq 0,$$

so that there is an $\alpha > 0$ with

$$p_i\langle\eta|\eta\rangle p_i = \alpha p_i.$$

Put

$$x := [\delta_{ij}\delta_{1k}]_{j,k\in\mathbb{N}_n} \in E_{n,n}$$

and

$$\zeta := \frac{1}{\sqrt{\alpha}}\eta p_i x.$$

For $j,k \in \mathbb{N}_n$,

$$(p_i x)_{jk} = \sum_{\ell=1}^{n} \delta_{ij}\delta_{i\ell}\delta_{i\ell}\delta_{1k} = \delta_{ij}\delta_{1k},$$

$$(x^* p_i x)_{jk} = \sum_{\ell=1}^{n} \delta_{1j}\delta_{i\ell}\delta_{i\ell}\delta_{1k} = \delta_{1j}\delta_{1k}$$

so that

$$x^* p_i x = p_1 ,$$

$$\langle \zeta | \zeta \rangle = \left\langle \frac{1}{\sqrt{\alpha}} \eta p_i x \,\middle|\, \frac{1}{\sqrt{\alpha}} \eta p_i x \right\rangle = \frac{1}{\alpha} x^* p_i \langle \eta | \eta \rangle p_i x = x^* p_i x = p_1 ,$$

and

$$\langle \zeta | \xi \rangle = \left\langle \frac{1}{\sqrt{\alpha}} \eta p_i x \,\middle|\, \xi \right\rangle = \langle \eta | \xi \rangle \frac{1}{\sqrt{\alpha}} p_i x = 0$$

for every $\xi \in A$. This contradicts the maximality of A.

c) The existence follows from b) and Proposition 5.6.4.10 c). The uniqueness is obvious. ■

Remark. $\aleph$ is a kind of Hilbert dimension for H.

Proposition 5.6.4.13 (8) *Assume E is the C^*-direct product of a finite family $(E_\iota)_{\iota \in I}$ of unital C^*-algebras and for every $\iota \in I$, denote by e_ι the unit of E_ι. Further let G, H be Hilbert right E-modules.*

a) *For every $\iota \in I$, He_ι is a Hilbert right E-module and a Hilbert right E_ι-module in a natural way.*

b) *The map*

$$\bigoplus_{\iota \in I} He_\iota \longrightarrow H , \quad (\xi_\iota)_{\iota \in I} \longmapsto \sum_{\iota \in I} \xi_\iota$$

is an isomorphism of Hilbert right E-modules and

$$H \longrightarrow \bigoplus_{\iota \in I} He_\iota , \quad \xi \longmapsto (\xi e_\iota)_{\iota \in I}$$

is its inverse.

c) *For every*

$$u := (u_\iota)_{\iota \in I} \in \prod_{\iota \in I} \mathcal{L}_{E_\iota}(Ge_\iota, He_\iota)$$

put

$$u^* := (u_\iota^*)_{\iota \in I},$$

$$\widetilde{u} : G \longrightarrow H, \quad \xi \longmapsto \sum_{\iota \in I} u_\iota(\xi e_\iota).$$

Then $\widetilde{u} \in \mathcal{L}_E(G,H)$ *and* $\widetilde{u}^* = \widetilde{u^*}$ *for every*

$$u \in \prod_{\iota \in I} \mathcal{L}_{E_\iota}(Ge_\iota, He_\iota).$$

d) *The map*

$$\prod_{\iota \in I} \mathcal{L}_{E_\iota}(Ge_\iota, He_\iota) \longrightarrow \mathcal{L}_E(G,H), \quad u \longmapsto \widetilde{u}$$

is an isomorphism of Banach spaces.

e) *The map of d) is an isomorphism of the* C^**-direct product of the family* $(\mathcal{L}_E(He_\iota))_{\iota \in I}$ *of* C^**-algebras (Theorem 5.6.1.11 a)) onto* $\mathcal{L}_E(H)$.

a) and b) are obvious.

c) For $(\xi, \eta) \in G \times H$,

$$\langle \widetilde{u}\xi | \eta \rangle = \sum_{\iota \in I} \langle u_\iota \xi e_\iota | \eta e_\iota \rangle = \sum_{\iota \in I} \langle \xi e_\iota | u_\iota^* \eta e_\iota \rangle = \langle \xi | \widetilde{u^*} \eta \rangle$$

so that

$$\widetilde{u} \in \mathcal{L}_E(G,H), \quad \widetilde{u}^* = \widetilde{u^*}.$$

d) For every $v \in \mathcal{L}_E(G,H)$ and $\iota \in I$, put

$$v_\iota : Ge_\iota \longrightarrow He_\iota, \quad \xi \longmapsto (v\xi)e_\iota.$$

Then for $(\xi, \eta) \in (Ge_\iota) \times (He_\iota)$,

$$\langle v_\iota \xi | \eta \rangle = \langle (v\xi)e_\iota | \eta \rangle = \langle v\xi | \eta \rangle e_\iota =$$

$$= \langle \xi | v^* \eta \rangle e_\iota = \langle \xi | (v^* \eta) e_\iota \rangle = \langle \xi | (v^*)_\iota \eta \rangle$$

so that $v_\iota \in \mathcal{L}_{E_\iota}(Ge_\iota, He_\iota)$. Putting

$$u := (v_\iota)_{\iota \in I},$$

it follows

$$\widetilde{u}\xi = \sum_{\iota\in I} v_\iota(\xi e_\iota) = \sum_{\iota\in I}(v\xi)e_\iota = v\xi$$

for every $\xi \in G$, i.e.

$$\widetilde{u} = v$$

and the map is surjective.

The other assertions are easy to see.

e) follows from a),c),d). ■

Example 5.6.4.14 (0) *Let G, H be Hilbert right E-modules. Then $\mathcal{L}_E(G,H)$ endowed with the bilinear map*

$$\mathcal{L}_E(G,H) \times \mathcal{L}_E(G) \longrightarrow \mathcal{L}_E(G,H)\,, \quad (u,w) \longmapsto uw := u\circ w$$

and with the sesquilinear map

$$\mathcal{L}_E(G,H) \times \mathcal{L}_E(G,H) \longrightarrow \mathcal{L}_E(G)\,, \quad (u,v) \longmapsto \langle u|v\rangle := v^*\circ u$$

is a Hilbert right $\mathcal{L}_E(G)$-module (Theorem 5.6.1.11 a)).

Take $u, v \in \mathcal{L}_E(G,H)$ and $w, w_1, w_2 \in \mathcal{L}_E(G)$. Then

$$(uw_1)w_2 = u(w_1w_2)\,,$$

$$\langle uw|v\rangle = v^*\circ(u\circ w) = (v^*\circ u)\circ w = \langle u|v\rangle w\,,$$

$$\langle v|u\rangle^* = (u^*\circ v)^* = v^*\circ u = \langle u|v\rangle$$

(Proposition 5.6.1.8 a),e)),

$$\langle u|u\rangle = u^*\circ u \in \mathcal{L}_E(G)_+$$

(Corollary 5.6.1.12),

$$u1 = u\,,$$

and

$$(\langle u|u\rangle = 0) \Longrightarrow (u^*\circ u = 0) \Longrightarrow (\|u\|^2 = \|u^*\circ u\| = 0) \Longrightarrow (u = 0)$$

(Proposition 5.6.1.8 f)). By Proposition 5.6.1.2 g), $\mathcal{L}_E(G,H)$ is a Hilbert right E-module in the complex case.

Assume now $\mathbb{K} = \mathbb{R}$. In the sequel we use the notation of Proposition 5.6.1.9. By this proposition, $\mathcal{L}_{\overset{\circ}{E}}(\overset{\circ}{G}, \overset{\circ}{H})$ and $\mathcal{L}_{\overset{\circ}{E}}(\overset{\circ}{G})$ may be identified with the complexifications of $\mathcal{L}_E(G,H)$ and $\mathcal{L}_E(G)$, respectively. Let $u, v \in \mathcal{L}_E(G,H)$. By the above,

$$0 \leq \widetilde{(u,v)}^* \circ \widetilde{(u,v)} = \widetilde{(u^*, -v^*)} \circ \widetilde{(u,v)} =$$

$$= (u^* \circ u + v^* \circ v\,,\; u^* \circ v - v^* \circ u) =$$

$$= (\langle u|u\rangle + \langle v|v\rangle\,,\; \langle v|u\rangle - \langle u|v\rangle)$$

(Proposition 5.6.1.9, Lemma 2.1.5.3). Hence $\mathcal{L}_E(G,H)$ is a Hilbert right $\mathcal{L}_E(G)$-module. ■

Definition 5.6.4.15 (0) *Let $F := (F_t)_{t\in T}$ and $G := (G_s)_{s\in S}$ be families of Hilbert right E-modules. We denote by $\mathcal{N}_{G,F}$ the vector space of families $(u_{s,t})_{(s,t)\in S\times T}$ such that*

$$u_{s,t} \in \mathcal{L}_E(F_t, G_s)$$

for every $(s,t) \in S \times T$ and such that

$$\{(s,t) \in S \times T | u_{s,t} \neq 0\}$$

is finite. For every $u \in \mathcal{N}_{G,F}$, define $u^* \in \mathcal{N}_{F,G}$ by

$$(u^*)_{t,s} = u^*_{s,t}$$

for all $(t,s) \in T \times S$. If $H := (H_r)_{r\in R}$ is a third family of Hilbert right E-modules then for all $u \in \mathcal{N}_{G,F}$ and $v \in \mathcal{N}_{H,G}$ define $vu \in \mathcal{N}_{H,F}$ by

$$(vu)_{r,t} := \sum_{s\in S} v_{rs} \circ u_{st}$$

for all $(r,t) \in R \times T$.

It is easy to see that the above u^* and vu are well-defined. If $S := \mathbb{N}_m$, $T := \mathbb{N}_n$ for some $m, n \in \mathbb{N}$ and

$$F_t = G_s =: H$$

for all $(s,t) \in S \times T$ then

$$\mathcal{N}_{G,F} := (\mathcal{L}_E(H))_{m,n}$$

and u^* and vu coincide with the corresponding notation in the matrix case (Definitions 2.1.4.23, 2.3.1.30). Hence $\mathcal{N}_{G,F}$ has to be seen as a generalized matrix. The more natural notation $\mathcal{M}_{G,F}$ will be reserved for the completion of $\mathcal{N}_{G,F}$ (Definition 5.6.4.18).

Proposition 5.6.4.16 (0) *Let $F := (F_t)_{t\in T}$ and $G := (G_s)_{s\in S}$ be families of Hilbert right E-modules and for every $u \in \mathcal{N}_{G,F}$ define*

$$\widetilde{u} : \bigoplus_{t\in T} F_t \longrightarrow \bigoplus_{s\in S} G_s\,, \quad \xi \longmapsto \left(\sum_{t\in T} u_{st}\xi_t\right)_{s\in S}\,.$$

a) *For every $u \in \mathcal{N}_{G,F}$,*

$$\widetilde{u} \in \mathcal{L}_E\left(\bigoplus_{t\in T} F_t\,,\ \bigoplus_{s\in S} G_s\right)\,, \quad \widetilde{u}^* = \widetilde{u^*}$$

(Proposition 5.6.4.1 c)), and

$$\sup_{(s,t)\in S\times T} \|u_{s,t}\| \le \sup_{s\in S} \|\widetilde{u}_{s,\cdot}\| \le \|\widetilde{u}\| \le$$

$$\le \left(\sum_{s\in S}\left(\sum_{t\in T}\|u_{s,t}\|\right)^2\right)^{\frac{1}{2}} \le \sum_{t\in T}\left(\sum_{s\in S}\|u_{s,t}\|^2\right)^{\frac{1}{2}}\,,$$

where for every $s \in S$, $u_{s,\cdot}$ is the element of $\mathcal{N}_{G,F}$ defined by

$$(u_{s,\cdot})_{s',t} := \begin{cases} u_{s,t} & \text{if } s' = s \\ 0 & \text{if } s' \neq s \end{cases}$$

for all $(s',t) \in S \times T$.

b) *The map*

$$\mathcal{N}_{G,F} \longrightarrow \mathcal{L}_E\left(\bigoplus_{t\in T} F_t\,,\ \bigoplus_{s\in S} G_s\right)\,, \quad u \longmapsto \widetilde{u}$$

is linear and injective (bijective if S and T are finite).

c) *If $H := (H_r)_{r\in R}$ is a family of Hilbert right E-modules then for all $u \in \mathcal{N}_{G,F}$ and $v \in \mathcal{N}_{H,G}$,*

$$\widetilde{v}\circ\widetilde{u} = \widetilde{vu}\,.$$

d) *If T is finite then $\mathcal{N}_{F,F}$ endowed with the norm*

$$\mathcal{N}_{F,F} \longrightarrow \mathbb{R}_+ , \quad u \longmapsto \|\widetilde{u}\|$$

is a C^*-algebra and the map

$$\mathcal{N}_{F,F} \longrightarrow \mathcal{L}_E \left(\bigoplus_{t\in T} F_t \right) , \quad u \longmapsto \widetilde{u}$$

is an isomorphism of C^*-algebras.

a) Take $\xi \in \bigoplus_{t\in T} F_t$ and $\eta \in \bigoplus_{s\in S} G_s$. Then

$$\langle \widetilde{u}\xi | \eta \rangle = \left\langle \left(\sum_{t\in T} u_{st}\xi_t \right)_{s\in S} \middle| (\eta_s)_{s\in S} \right\rangle =$$

$$= \sum_{t\in T} \sum_{s\in S} \langle u_{st}\xi_t | \eta_s \rangle = \sum_{t\in T} \sum_{s\in S} \langle \xi_t | u_{st}^* \eta_s \rangle =$$

$$= \sum_{t\in T} \left\langle \xi_t \middle| \left(\sum_{s\in S} (u^*)_{t,s}\eta_s \right)_t \right\rangle = \sum_{t\in T} \langle \xi_t | (\widetilde{u^*}\eta)_t \rangle =$$

$$= \langle \xi | \widetilde{u^*}\eta \rangle .$$

Hence

$$\widetilde{u} \in \mathcal{L}_E \left(\bigoplus_{t\in T} F_t , \bigoplus_{s\in S} G_s \right) , \quad \widetilde{u}^* = \widetilde{u^*} .$$

Moreover,

$$\|\widetilde{u}\xi\|^2 = \|\langle \widetilde{u}\xi | \widetilde{u}\xi \rangle\| =$$

$$= \left\| \sum_{s\in S} \left\langle \sum_{t\in T} u_{st}\xi_t \middle| \sum_{t'\in T} u_{st'}\xi_{t'} \right\rangle \right\| \leq$$

$$\leq \sum_{s\in S} \sum_{t,t'\in T} \|u_{st}\xi_t\| \, \|u_{st'}\xi_{t'}\| \leq$$

$$\leq \sum_{s\in S} \sum_{t,t'\in T} \|u_{st}\| \, \|\xi_t\| \, \|u_{st'}\| \, \|\xi_{t'}\| \leq \|\xi\|^2 \sum_{s\in S} \left(\sum_{t,t'\in T} \|u_{st}\| \, \|u_{st'}\| \right)$$

(Proposition 5.6.1.2 d), Proposition 5.6.4.1 b)). Hence

$$\|\widetilde{u}\|^2 \leq \sum_{s\in S}\sum_{t,t'\in T}\|u_{st}\|\,\|u_{st'}\| = \sum_{s\in S}\left(\sum_{t\in T}\|u_{st}\|\right)^2 .$$

From

$$\sum_{s\in S}\sum_{t,t'\in T}\|u_{st}\|\,\|u_{st'}\| = \sum_{t,t'\in T}\sum_{s\in S}\|u_{st}\|\,\|u_{st'}\| \leq$$

$$\leq \sum_{t,t'\in T}\left(\sum_{s\in S}\|u_{st}\|^2\right)^{\frac{1}{2}}\left(\sum_{s\in S}\|u_{st'}\|^2\right)^{\frac{1}{2}} =$$

$$= \left(\sum_{t\in T}\left(\sum_{s\in S}\|u_{st}\|^2\right)^{\frac{1}{2}}\right)^2$$

it follows

$$\|\widetilde{u}\| \leq \left(\sum_{s\in S}\left(\sum_{t\in T}\|u_{st}\|\right)^2\right)^{\frac{1}{2}} \leq \sum_{t\in T}\left(\sum_{s\in S}\|u_{st}\|^2\right)^{\frac{1}{2}} .$$

Fix $(s,t)\in S\times T$ and take $\xi_t \in F_t^{\#}$. Define $\xi' \in \bigoplus_{t'\in T} F_{t'}$ by

$$\xi'_{t'} := \begin{cases} \xi_t & \text{if } t' = t \\ 0 & \text{if } t' \neq t \end{cases}$$

for every $t'\in T$. By Proposition 5.6.4.1 b), $\xi' \in \left(\bigoplus_{t'\in T} F_{t'}\right)^{\#}$ and so

$$\|\widetilde{u}_{s,\cdot}\| \geq \|\widetilde{u}_{s,\cdot}\xi'\| = \left\|\sum_{t'\in T} u_{s,t'}\xi_{t'}\right\| = \|u_{s,t}\xi_t\| .$$

It follows

$$\|\widetilde{u}_{s,\cdot}\| \geq \|u_{s,t}\| ,$$

$$\|\widetilde{u}_{s,\cdot}\| \geq \sup_{t\in T}\|u_{s,t}\| .$$

Now fix $s\in S$ and take $\xi \in \left(\bigoplus_{t\in T} F_t\right)^{\#}$. By Proposition 5.6.4.1 b),

$$\|\widetilde{u}\xi\| \geq \|\sum_{t\in T} u_{s,t}\xi_t\| = \|\widetilde{u}_{s,\cdot}\xi\|$$

so that

$$\|\widetilde{u}\| \geq \|\widetilde{u}_{s,\cdot}\| .$$

It follows

$$\sup_{(s,t)\in S\times T} \|u_{s,t}\| \leq \sup_{s\in S} \|\widetilde{u}_{s,\cdot}\| \leq \|\widetilde{u}\| .$$

b) It is obvious that the map is linear. By a), it is injective. Assume S and T finite and take

$$v \in \mathcal{L}_E\left(\bigoplus_{t\in T} F_t\,,\ \bigoplus_{s\in S} G_s\right) .$$

Fix $(s,t) \in S \times T$. For every $\xi \in F_t$ and $\eta \in G_s$ denote by ξ' the element of $\bigoplus_{t'\in T} F_{t'}$ and by η' the element of $\bigoplus_{s'\in S} G_{s'}$ defined by

$$\xi'_{t'} := \begin{cases} \xi & \text{if } t' = t \\ 0 & \text{if } t' \neq t \end{cases} , \quad \eta'_{s'} := \begin{cases} \eta & \text{if } s' = s \\ 0 & \text{if } s' \neq s \end{cases}$$

for every $t' \in T$ and $s' \in S$. Define

$$u_{st} : F_t \longrightarrow G_s\,, \quad \xi \longmapsto (v\xi')_s .$$

Then for $\xi \in F_t$ and $\eta \in G_s$,

$$\langle u_{st}\xi|\eta\rangle = \langle v\xi'|\eta'\rangle = \langle \xi'|v^*\eta'\rangle = \langle \xi|(v^*\eta')_t\rangle .$$

Hence $u_{st} \in \mathcal{L}_E(F_t, G_s)$ and it is easy to see that $\widetilde{u} = v$. Therefore the map

$$\mathcal{N}_{G,F} \longrightarrow \mathcal{L}_E\left(\bigoplus_{t\in T} F_t\,,\ \bigoplus_{s\in S} G_s\right) , \quad u \longmapsto \widetilde{u}$$

is surjective.

c) Take $\xi \in \bigoplus_{t\in T} F_t$. Then

$$\widetilde{v}\widetilde{u}\xi = \left(\sum_{s\in S} v_{r,s}(\widetilde{u}\xi)_s\right)_{r\in R} = \left(\sum_{s\in S}\sum_{t\in T} v_{rs}u_{st}\xi_t\right)_{r\in R} =$$

$$= \left(\sum_{t\in T}\left(\sum_{s\in S} v_{rs}\circ u_{st}\right)\xi_t\right)_{r\in R} = \left(\sum_{t\in T}(vu)_{r,t}\xi_t\right)_{r\in R}$$

so that

$$\widetilde{v}\widetilde{u}\xi = \widetilde{vu}\xi\,,$$

$$\widetilde{v}\circ\widetilde{u} = \widetilde{vu}\,.$$

d) follows from a),b),c). ■

Corollary 5.6.4.17 *Let H be an infinite-dimensional Hilbert space and $n \in \mathbb{N}$. Then the C^*-algebras $\mathcal{L}(H)$ and $\mathcal{K}(H)$ are isomorphic to the C^*-algebras $\mathcal{L}(H)_{n,n}$ and $\mathcal{K}(H)_{n,n}$, respectively (Proposition 5.6.4.16 d)).*

Let A be an orthonormal basis of H. Since A is infinite, there is a bijection $A \to A \times \mathbb{N}_n$, so H is isomorphic to $\bigoplus_{k\in\mathbb{N}_n} H$ (Proposition 5.5.1.22). Hence $\mathcal{L}(H)$ (resp. $\mathcal{K}(H)$) is isomorphic to $\mathcal{L}\left(\bigoplus_{k\in\mathbb{N}_n} H\right)$ $\left(\text{resp. } \mathcal{K}\left(\bigoplus_{k\in\mathbb{N}_n} H\right)\right)$ which is isomorphic to $\mathcal{L}(H)_{n,n}$ (resp. $\mathcal{K}(H)_{n,n}$) (Proposition 5.6.4.16 d)). ■

Proposition 5.6.4.18 (0) *In this proposition we use the notation of Proposition 5.5.7.10. For every $\xi \in H$, put*

$$\begin{cases} \xi_1 : T \longrightarrow \mathbb{C}\,, & t \longmapsto \xi(t,0) \\ \xi_2 : T \longrightarrow \mathbb{C}\,, & t \longmapsto \xi(t,1) \end{cases}$$

and identify H and $\ell^2(T) \times \ell^2(T)$ via the map

$$H \longrightarrow \ell^2(T) \times \ell^2(T)\,, \quad \xi \longmapsto (\xi_1, \xi_2)\,.$$

For each $u \in \mathcal{L}(\ell^2(T))_{2,2}$ put

$$\widetilde{u} : H \longrightarrow H\,, \quad \xi \longmapsto (u_{11}\xi_1 + u_{12}\xi_2,\quad u_{21}\xi_1 + u_{22}\xi_2)\,.$$

a) *For every $u \in \mathcal{L}(\ell^2(T))_{2,2}$,*

$$\widetilde{u} \in \mathcal{L}(H)\,,$$

$$\widetilde{u} \in F(\mathcal{L}(H)) \Longleftrightarrow (u_{22} = \overline{u}_{11}\,,\ u_{12} = -\overline{u}_{21})\,,$$

and the map

$$\mathcal{L}(\ell^2(T))_{2,2} \longrightarrow \mathcal{L}(H)\,, \quad u \longmapsto \widetilde{u}$$

is an isomorphism of C^*-algebras (Proposition 5.6.4.16 d)).

b) *For every*

$$u := (u_1, u_2, u_3, u_4) \in (\mathcal{L}(\ell^2(T, \mathbb{R})))^4$$

put

$$\vec{u} := \overbrace{\begin{bmatrix} u_1 + iu_2 & iu_3 - u_4 \\ iu_3 + u_4 & u_1 - iu_2 \end{bmatrix}} \in F(\mathcal{L}(H)),$$

$$\overleftarrow{u} : \ell^2(T, \mathbb{H}) \longrightarrow \ell^2(T, \mathbb{H}),$$

$$\xi_1 1 + \xi_2 i + \xi_3 j + \xi_4 k \longmapsto (u_1 1 + u_2 i + u_3 j + u_4 k)(\xi_1 1 + \xi_2 i + \xi_3 j + \xi_4 k).$$

If we define

$$\varphi : (\mathcal{L}(\ell^2(T, \mathbb{R})))^4 \longrightarrow F(\mathcal{L}(H)), \quad u \longmapsto \vec{u},$$

$$\psi : (\mathcal{L}(\ell^2(T, \mathbb{R})))^4 \longrightarrow \mathcal{L}_{\mathbb{H}}(\ell^2(T, \mathbb{H})), \quad u \longmapsto \overleftarrow{u},$$

then $\varphi \circ \psi^{-1}$ is an isomorphism of real C^*-algebras (Example 5.6.4.2 a), Theorem 5.6.1.11 a)) and

$$\varphi \circ \psi^{-1}(\mathcal{L}_{\mathbb{H}}(\ell^2(T, \mathbb{H}) \cap \mathcal{K}(\ell^2(T, \mathbb{H})))) = F(\mathcal{K}(H)).$$

c) *The complexification of $\mathcal{L}_{\mathbb{H}}(\ell^2(T, \mathbb{H}))$ (of $\mathcal{L}_{\mathbb{H}}(\ell^2(T, \mathbb{H})) \cap \mathcal{K}(\ell^2(T, \mathbb{H}))$) is isomorphic to $\mathcal{L}(H)$ (to $\mathcal{K}(H)$).*

a) By Proposition 5.6.4.16 d),

$$\widetilde{u} \in \mathcal{L}(H)$$

for every $u \in \mathcal{L}(\ell^2(T))_{2,2}$ and the map

$$\mathcal{L}(\ell^2(T))_{2,2} \longrightarrow \mathcal{L}(H), \quad u \longmapsto \widetilde{u}$$

is an isomorphism of C^*-algebras. Take $u \in \mathcal{L}(\ell^2(T))_{2,2}$ and $\xi \in H$. Then

$$\widetilde{u}x\xi = \widetilde{u}(-\xi_2, \xi_1) = (-u_{11}\xi_2 + u_{12}\xi_1, -u_{21}\xi_2 + u_{22}\xi_1),$$

$$x\overline{\widetilde{u}}\xi = x(\overline{u}_{11}\xi_1 + \overline{u}_{12}\xi_2, \overline{u}_{21}\xi_1 + \overline{u}_{22}\xi_2) =$$

$$= (-\overline{u}_{21}\xi_1 - \overline{u}_{22}\xi_2\,,\ \overline{u}_{11}\xi_1 + \overline{u}_{12}\xi_2)\,.$$

Since $\widetilde{u} \in F(\mathcal{L}(H))$ iff $\widetilde{u}x = x\overline{\widetilde{u}}$, it follows $\widetilde{u} \in F(\mathcal{L}(H))$ iff

$$u_{22} = \overline{u}_{11}\,, \quad u_{21} = -\overline{u}_{12}\,.$$

b) By a) and Proposition 5.6.4.5,

$$\vec{u} \in F(\mathcal{L}(H))\,, \quad \vec{u}^* = \widetilde{(u_1^*, -u_2^*, -u_3^*, -u_4^*)}\,,$$

$$\overleftarrow{u} \in \mathcal{L}_{\mathbb{H}}(\ell^2(T, \mathbb{H}))\,, \quad \overleftarrow{u}^* = \overleftarrow{(u_1^*, -u_2^*, -u_3^*, -u_4^*)}$$

for every $u \in (\mathcal{L}(\ell^2(T, \mathbb{R})))^4$ and φ and ψ are bijective and linear. Hence $\varphi \circ \psi^{-1}$ is bijective, linear, and involutive.

Take $a, b \in \mathcal{L}_{\mathbb{H}}(\ell^2(T, \mathbb{H}))$ and put

$$u := \psi^{-1}a\,, \quad v := \psi^{-1}b\,, \quad w := \psi^{-1}(ab)\,,$$

$$\vec{c} := \varphi(u)\varphi(v) = \widetilde{\begin{bmatrix} u_1 + iu_2 & iu_3 - u_4 \\ iu_3 + u_4 & u_1 - iu_2 \end{bmatrix} \begin{bmatrix} v_1 + iv_2 & iv_3 - v_4 \\ iv_3 + v_4 & v_1 - iv_2 \end{bmatrix}}\,,$$

with $c \in \mathcal{L}(\ell^2(T))_{2,2}$. Then

$$\begin{cases} w_1 = u_1v_1 - u_2v_2 - u_3v_3 - u_4v_4 \\ w_2 = u_1v_2 + u_2v_1 + u_3v_4 - u_4v_3 \\ w_3 = u_1v_3 - u_2v_4 + u_3v_1 + u_4v_2 \\ w_4 = u_1v_4 + u_2v_3 - u_3v_2 + u_4v_1 \end{cases}$$

$$c_{11} = u_1v_1 - u_2v_2 - u_3v_3 - u_4v_4 + i(u_1v_2 + u_2v_1 + u_3v_4 - u_4v_3) = w_1 + iw_2\,,$$

$$c_{12} = -u_1v_4 - u_2v_3 + u_3v_2 - u_4v_1 + i(u_1v_3 - u_2v_4 + u_3v_1 + u_4v_2) = iw_3 - w_4\,,$$

$$c_{21} = u_1v_4 + u_2v_3 - u_3v_2 + u_4v_1 + i(u_1v_3 - u_2v_4 + u_3v_1 + u_4v_2) = iw_3 + w_4\,,$$

$$c_{22} = u_1v_1 - u_2v_2 - u_3v_3 - u_4v_4 + i(-u_1v_2 - u_2v_1 - u_3v_4 + u_4v_3) = w_1 - iw_2\,.$$

Hence

$$(\varphi\psi^{-1}a)(\varphi\psi^{-1}b) = \varphi(u)\varphi(v) = \vec{w} = \varphi\psi^{-1}(ab)$$

and $\varphi \circ \psi^{-1}$ is an algebra homomorphism, i.e. an isomorphism of real C^*-algebras.

The last relation is easy to see.

c) follows from b) and Proposition 5.5.7.10 a). ■

Definition 5.6.4.19 (8) *Let* $F := (F_t)_{t\in T}$, $G := (G_s)_{s\in S}$, *and* $H := (H_r)_{r\in R}$ *be families of Hilbert right* E*-modules. We endow the vector spaces of the form* $\mathcal{N}_{G,F}$ *with the norms*

$$\mathcal{N}_{G,F} \longrightarrow \mathbb{R}_+, \quad u \longmapsto \|\widetilde{u}\|$$

(Proposition 5.6.4.16 b)) and denote by $\mathcal{M}_{G,F}$ *the completion of* $\mathcal{N}_{G,F}$. *Further we define the maps*

$$\mathcal{M}_{G,F} \longrightarrow \mathcal{L}_E\left(\bigoplus_{t\in T} F_t, \bigoplus_{s\in S} G_s\right), \quad u \longmapsto \widetilde{u},$$

$$\mathcal{M}_{G,F} \longrightarrow \mathcal{M}_{F,G}, \quad u \longmapsto u^*,$$

$$\mathcal{M}_{H,G} \times \mathcal{M}_{G,F} \longrightarrow \mathcal{M}_{H,F}, \quad (v,u) \longmapsto vu$$

by extending continuously the corresponding maps of $\mathcal{N}$ *(Definition 5.6.4.15, Proposition 5.6.4.16 a)).*

The first map is linear and norm preserving, the second one is conjugate linear and norm preserving, and the third one is bilinear and continuous.

Proposition 5.6.4.20 (8) *Let* $F := (F_t)_{t\in T}$, $G := (G_s)_{s\in S}$, $H := (H_r)_{r\in R}$ *be families of Hilbert right* E*-modules.*

a) *For each* $u \in \mathcal{M}_{G,F}$ *there is a unique family* $(u_{s,t})_{(s,t)\in S\times T}$ *such that*

$$u_{s,t} \in \mathcal{L}_E(F_t, G_s)$$

for every $(s,t) \in S \times T$ *and such that*

$$\left(\sum_{t\in T} u_{s,t}\xi_t\right)_{s\in S} = \widetilde{u}\xi$$

for every $\xi \in \bigoplus_{t\in T} F_t$.

b) *For every* $u \in \mathcal{M}_{G,F}$ *and with the notation of a),*

$$\sup_{(s,t)\in S\times T} \|u_{s,t}\| \leq \sup_{s\in S} \|\widetilde{u}_{s,\cdot}\| \leq \|\widetilde{u}\| \leq$$

$$\leq \left(\sum_{s\in S}\left(\sum_{t\in T}\|u_{s,t}\|\right)^2\right)^{\frac{1}{2}} \leq \sum_{t\in T}\left(\sum_{s\in S}\|u_{s,t}\|^2\right)^{\frac{1}{2}},$$

where for every $s \in S$, $u_{s,\cdot}$ *is the element of* $\mathcal{M}_{G,F}$ *defined by*

$$\bigoplus_{t\in T} F_t \longrightarrow \bigoplus_{s'\in S} G_{s'}, \quad \xi \longmapsto ((\widetilde{u}\xi)_{s'}\delta_{s,s'})_{s'\in S}.$$

Moreover,

$$(u^*)_{t,s} = u^*_{s,t}$$

for all $(s,t) \in S \times T$ *and*

$$u^{**} = u.$$

c) *For all* $u \in \mathcal{M}_{G,F}$ *and* $v \in \mathcal{M}_{H,G}$,

$$\widetilde{v}\circ\widetilde{u} = \widetilde{vu}.$$

d) $\mathcal{M}_{F,F}$ *endowed with the multiplication*

$$\mathcal{M}_{F,F} \times \mathcal{M}_{F,F} \longrightarrow \mathcal{M}_{F,F}, \quad (u,v) \longmapsto uv$$

and with the involution

$$\mathcal{M}_{F,F} \longrightarrow \mathcal{M}_{F,F}, \quad u \longmapsto u^*$$

is a C^**-algebra. If* T *is finite then the map*

$$\mathcal{M}_{F,F} \longrightarrow \mathcal{L}_E\left(\bigoplus_{t\in T} F_t\right), \quad u \longmapsto \widetilde{u}$$

is as isomorphism of C^**-algebras.*

e) $\mathcal{M}_{G,F}$ *endowed with the bilinear map*

$$\mathcal{M}_{G,F} \times \mathcal{M}_{F,F} \longrightarrow \mathcal{M}_{G,F}, \quad (u,v) \longmapsto uv$$

and with the sesquilinear map

$$\mathcal{M}_{G,F} \times \mathcal{M}_{G,F} \longrightarrow \mathcal{M}_{F,F}, \quad (u,v) \longmapsto v^*u$$

is a Hilbert right $\mathcal{M}_{F,F}$*-module.*

f) *If we put*

$$\overset{\circ}{F} := (\overset{\circ}{F}_t)_{t\in T}, \quad \overset{\circ}{G} := (\overset{\circ}{G}_s)_{s\in S}$$

(Proposition 5.6.1.6) then the complexification of the C^**-algebra* $\mathcal{M}_{F,F}$ *is isomorphic to* $\mathcal{M}_{\overset{\circ}{F},\overset{\circ}{F}}$.

g) *For all* $u \in \mathcal{M}_{G,F}$, $A \subset S$, *and* $B \subset T$, *there is a unique* $u_{A,B} \in \mathcal{M}_{G,F}$ *such that*

$$(u_{A,B})_{s,t} = \begin{cases} u_{s,t} & \text{if } (s,t) \in A \times B \\ 0 & \text{if } (s,t) \notin A \times B \end{cases} .$$

Moreover, the map

$$\mathcal{M}_{G,F} \longrightarrow \mathcal{M}_{G,F}, \quad v \longmapsto v_{A,B}$$

is an operator (of norm 1 *if* A *and* B *are nonempty).*

h) *Put*

$$\mathfrak{M} := \{A \times B | A \in \mathfrak{P}_f(S), B \in \mathfrak{P}_f(T)\},$$

order $\mathfrak{M}$ *by inclusion, and denote by* $\mathfrak{F}$ *the upper setion filter of* $\mathfrak{M}$. *Then*

$$u = \lim_{A \times B, \mathfrak{F}} u_{A,B}$$

for every $u \in \mathcal{M}_{G,F}$.

i) *Assume* E *is a* W^**-algebra and for every* $A \in \mathfrak{P}_f(S)$ *and* $B \in \mathfrak{P}_f(T)$ *put*

$$\pi_A : \overset{W}{\bigoplus_{s \in S}} G_s \longrightarrow \overset{W}{\bigoplus_{s \in S}} G_s, \quad \eta \longmapsto \eta e_A^S,$$

$$\pi_B : \overset{W}{\bigoplus_{t \in T}} F_t \longrightarrow \overset{W}{\bigoplus_{t \in T}} F_t, \quad \xi \longmapsto \xi e_B^T,$$

Then $\pi_A \in \Pr \mathcal{L}_E \left(\overset{W}{\bigoplus_{s \in S}} G_s \right)$, $\pi_B \in \Pr \mathcal{L}_E \left(\overset{W}{\bigoplus_{t \in T}} F_t \right)$ *for every* $A \subset S$ *and* $B \subset T$,

$$u' = \lim_{A \times B, \mathfrak{F}} \pi_B u' \pi_A$$

for every $u' \in \overbrace{\overset{W}{\bigoplus_{t \in T}} F_t}^{\cdots}$ *in the topology of pointwise convergence on*

$$\mathcal{L}_E \left(\overset{W}{\bigoplus_{t \in T}} F_t, \quad \overset{W}{\bigoplus_{s \in S}} G_s \right),$$

and

$$u = \lim_{A\times B,\mathfrak{F}} \pi_A \circ u \circ \pi_B$$

for every $u \in \mathcal{L}_E\left(\overset{W}{\underset{t\in T}{\bigoplus}} F_t\,,\quad \overset{W}{\underset{s\in S}{\bigoplus}} G_s\right)$ *in the topology of pointwise convergence on* $\overset{\overset{\cdots}{W}}{\underset{t\in T}{\bigoplus}} F_t$.

j) *Let* $(S_\iota)_{\iota\in I}$ *and* $(T_\iota)_{\iota\in I}$ *be disjoint families in* $\mathfrak{P}(S)$ *and* $\mathfrak{P}(T)$, *respectively, and put*

$$\widehat{F}_\iota := (F_t)_{t\in T_\iota}\,,\quad \widehat{G}_\iota := (G_s)_{s\in S_\iota}$$

for every $\iota \in I$. *Let*

$$(u_\iota)_{\iota\in I} \in \prod_{\iota\in I} \mathcal{M}_{\widehat{G}_\iota,\widehat{F}_\iota}$$

such that $(\|\widetilde{u}_\iota\|)_{\iota\in I} \in c_0(I)$ *and define*

$$v_{st} := \begin{cases} u_{\iota,st} & \text{if } (s,t) \in S_\iota \times T_\iota \text{ for some } \iota \in I \\ 0 & \text{if } (s,t) \in S \times T \setminus \bigcup_{\iota\in I}(S_\iota \times T_\iota) \end{cases}$$

for every $(s,t) \in S \times T$, *Then there is a unique* $v \in \mathcal{M}_{G,F}$ *such that*

$$\widetilde{v}\xi = \left(\sum_{t\in T} v_{st}\xi_t\right)_{s\in S}$$

for every $\xi \in \underset{t\in T}{\bigoplus} F_t$. *Moreover,*

$$\|\widetilde{v}\| = \sup_{\iota\in I} \|\widetilde{u}_\iota\|\,.$$

a) The uniqueness is obvious. By Proposition 5.6.4.16 a), the map

$$\mathcal{N}_{G,F} \longrightarrow \mathcal{L}_E(F_t, G_s)\,,\quad v \longmapsto v_{s,t}$$

is continuous for every $(s,t) \in S \times T$. We may extend this map continuously to $\mathcal{M}_{G,F}$, getting in this way a family $(u_{s,t})_{S\times T}$ with

$$u_{s,t} \in \mathcal{L}_E(F_t, G_s)$$

for every $(s,t) \in S \times T$ and

$$\left(\sum_{t\in T} u_{s,t}\xi_t\right)_{s\in S} = \widetilde{u}\xi$$

for every $\xi \in \bigoplus_{t\in T} F_t$ with $\{t \in T \mid \xi_t \neq 0\}$ finite. By continuity, this relation holds for arbitrary $\xi \in \bigoplus_{t\in T} F_t$.

b),c), and d) follow from a) and Proposition 5.6.4.16.

e) follows from c),d), and Example 5.6.4.14.

f) is easy to see.

g) Take $u \in \mathcal{M}_{G,F}$ and assume $u_{A,B}$ exists. We want to prove the inequality

$$\|\widetilde{u}_{A,B}\| \leq \|\widetilde{u}\| .$$

Take $\xi \in \left(\bigoplus_{t\in T} F_t\right)^{\#}$ and define $\eta \in \prod_{t\in T} F_t$ by

$$\eta_t := \begin{cases} \xi_t & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{cases} .$$

By Proposition 5.6.4.1 b), $\eta \in \left(\bigoplus_{t\in T} F_t\right)^{\#\#}$

$$\|\widetilde{u}_{A,B}\xi\| = \|\widetilde{u}_{A,B}\eta\| \leq \|\widetilde{u}\eta\| \leq \|\widetilde{u}\|$$

so that

$$\|\widetilde{u}_{A,B}\| \leq \|\widetilde{u}\| .$$

The linearity of the map

$$\mathcal{M}_{G,F} \longrightarrow \mathcal{M}_{G,F}, \quad u \longmapsto u_{A,B}$$

is obvious. By the above, its norm is one if A and B are nonempty.

Now we prove the first assertion. The uniqueness and the case $u \in \mathcal{N}_{G,F}$ are trivial. Let $(u^{(n)})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{N}_{G,F}$ converging to u. By the above, $(u^{(n)}_{A,B})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{N}_{G,F}$. Put

$$v := \lim_{n\to\infty} u^{(n)}_{A,B} \in \mathcal{M}_{G,F} .$$

By b),

$$u_{s,t} = \lim_{n\to\infty} u^{(n)}_{s,t}, \quad v_{s,t} = \lim_{n\to\infty} (u^{(n)}_{A,B})_{s,t}$$

for all $(s,t) \in S \times T$. It follows

$$v_{s,t} = \begin{cases} u_{s,t} & \text{if } (s,t) \in A \times B \\ 0 & \text{if } (s,t) \notin A \times B \end{cases}$$

for all $(s,t) \in S \times T$.

h) Take $\varepsilon > 0$. There is a $v \in \mathcal{N}_{G,F}$ such that

$$\|u - v\| < \frac{\varepsilon}{2}.$$

Take $A \in \mathfrak{P}_f(S)$ and $B \in \mathfrak{P}_f(T)$ such that

$$\{(s,t) \in S \times T | v_{s,t} \neq 0\} \subset A \times B.$$

Then

$$v = v_{A,B}$$

and by g),

$$\|u_{A,B} - u\| = \|(u_{A,B} - u) - (v_{A,B} - v)\| \leq$$

$$\leq \|u_{A,B} - v_{A,B}\| + \|u - v\| \leq 2\|u - v\| < \varepsilon.$$

Hence

$$\lim_{A \times B, \mathfrak{F}} u_{A,B} = u.$$

i) It is easy to see that $\pi_A \in \Pr \mathcal{L}_E \left(\bigoplus_{s \in S}^{W} G_s \right)$ and $\pi_B \in \Pr \mathcal{L}_E \left(\bigoplus_{t \in T}^{W} F_t \right)$ for every $A \subset S$ and $B \subset T$. For the assertion concerning u', we may assume

$$u' := \widetilde{(a, \xi, \eta)},$$

where

$$(a, \xi, \eta) \in \ddot{E} \times \left(\bigoplus_{t \in T}^{W} F_t \right) \times \left(\bigoplus_{s \in S}^{W} G_s \right).$$

By Proposition 5.6.3.1 e),

$$\pi_B \widetilde{(a, \xi, \eta)} \pi_A = \widetilde{(a, \pi_B \xi, \pi_A \eta)}$$

for every $A \subset S$ and $B \subset T$, so that we have to prove

$$\lim_{A\times B,\mathfrak{F}} \widetilde{(a,\xi e_B^T,\eta e_A^S)} = \widetilde{(a,\xi,\eta)}\,.$$

Take $v \in \mathcal{L}_E\left(\bigoplus_{t\in T}^{W} F_t, \bigoplus_{s\in S}^{W} G_s\right)$ and $\varepsilon > 0$. Since the family

$$(\langle\langle \xi_t \mid (v^*\eta)_t\rangle, a\rangle)_{t\in T}$$

is summable, there is a $B_0 \in \mathfrak{P}_f(T)$ such that

$$\left|\sum_{t\in T\setminus B} \langle\langle \xi_t \mid (v^*\eta)_t\rangle, a\rangle\right| < \frac{\varepsilon}{2}$$

for every $B \in \mathfrak{P}_f(T)$, $B_0 \subset B$. For every $B \subset T$,

$$\|v\xi e_b^T\| \le \|v\|\,\|\xi e_B^T\| \le \|v\|\,\|\xi\|$$

(Corollary 4.2.1.18). By Proposition 5.6.4.6 c), there is an $A_0 \in \mathfrak{P}_f(S)$ such that

$$|\langle\langle v\xi e_B^T \mid \eta e_{S\setminus A}^S\rangle, a\rangle| < \frac{\varepsilon}{2}$$

for every $B \subset T$ and $A \in \mathfrak{P}_f(S)$, $A_0 \subset A$.

Take $A \in \mathfrak{P}_f(S)$, $A_0 \subset A$, and $B \in \mathfrak{P}_f(T)$, $B_0 \subset B$. Then

$$\left|\langle v, \widetilde{(a,\xi e_B^T,\eta e_A^S)}\rangle - \langle v, \widetilde{(a,\xi,\eta)}\rangle\right| =$$

$$= \left|\langle\langle v\xi e_B^T \mid \eta e_A^S\rangle - \langle v\xi|\eta\rangle, a\rangle\right| \le$$

$$\le \left|\langle\langle v\xi e_B^T \mid \eta e_A^S\rangle - \langle v\xi e_B^T|\eta\rangle, a\rangle\right| +$$

$$+ \left|\langle\langle v\xi e_B^T|\eta\rangle - \langle v\xi|\eta\rangle, a\rangle\right| =$$

$$= \left|\langle\langle v\xi e_B^T \mid \eta e_{S\setminus A}^S\rangle, a\rangle\right| + \left|\langle\langle \xi e_{T\setminus B}^T \mid v^*\eta\rangle, a\rangle\right| <$$

$$< \frac{\varepsilon}{2} + \left|\sum_{t\in T\setminus B} \langle\langle \xi_t|(v^*\eta)_t\rangle, a\rangle\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Hence

$$\lim_{A\times B,\mathfrak{F}} \langle v, \widetilde{(a, \xi e_B^T, \eta e_A^S)}\rangle = \langle v, \widetilde{(a,\xi,\eta)}\rangle\,,$$

so that

$$\lim_{A\times B,\mathfrak{F}} \widetilde{(a, \xi e_B^T, \eta e_A^S)} = \widetilde{(a,\xi,\eta)}$$

in the topology of pointwise convergence on $\mathcal{L}_E\left(\overset{W}{\underset{t\in T}{\bigoplus}} F_t, \overset{W}{\underset{s\in S}{\bigoplus}} G_s\right)$.

The assertion concerning u follows.

j) For every $\iota \in I$, we identify $\underset{t\in T_\iota}{\bigoplus} F_t$ and $\underset{s\in S_\iota}{\bigoplus} G_s$ in a natural way with a subspace of $\underset{t\in T}{\bigoplus} F_t$ and $\underset{s\in S}{\bigoplus} G_s$, respectively. For $\iota \in I$ and $\xi_\iota \in \underset{t\in T_\iota}{\bigoplus} F_t$,

$$\widetilde{u}_\iota \xi_\iota = \left(\sum_{t\in T_\iota} u_{\iota,st}\xi_{\iota,t}\right)_{s\in S_\iota},$$

$$\langle \widetilde{u}_\iota\xi_\iota \mid \widetilde{u}_\iota\xi_\iota\rangle \le \|\widetilde{u}_\iota\|^2\langle \xi_\iota|\xi_\iota\rangle$$

(Corollary 5.6.1.12). Moreover, for every $\lambda \in I\backslash\{\iota\}$ and $\xi_\lambda \in \underset{t\in T_\lambda}{\bigoplus} F_t$,

$$\langle \widetilde{u}_\iota\xi_\iota \mid \widetilde{u}_\lambda\xi_\lambda\rangle = 0\,.$$

Take $\xi \in \underset{t\in T}{\bigoplus} F_t$ and for every $\iota \in I$, define

$$\widehat{\xi}_\iota := (\xi_t)_{t\in T_\iota} \in \underset{t\in T}{\bigoplus} F_t\,.$$

Then for every $J \in \mathfrak{P}_f(I)$,

$$\sum_{\iota\in J}\sum_{t\in T_\iota} v_{st}\xi_t = \sum_{\iota\in J} \widetilde{u}_\iota\widehat{\xi}_\iota\,,$$

$$\left\langle \sum_{\iota\in J}\sum_{t\in T_\iota} v_{st}\xi_t \,\middle|\, \sum_{\iota\in J}\sum_{t\in T_\iota} v_{st}\xi_t\right\rangle = \left\langle \sum_{\iota\in J}\widetilde{u}_\iota\widehat{\xi}_\iota \mid \sum_{\iota\in J}\widetilde{u}_\iota\widehat{\xi}_\iota\right\rangle =$$

$$= \sum_{\iota\in J}\langle \widetilde{u}_\iota\widehat{\xi}_\iota|\widetilde{u}_\iota\widehat{\xi}_\iota\rangle \le \sum_{\iota\in J}\|u_\iota\|^2\langle\widehat{\xi}_\iota|\widehat{\xi}_\iota\rangle \le$$

$$\le \left(\sup_{\iota\in J}\|u_\iota\|^2\right)\sum_{\iota\in J}\langle\widehat{\xi}_\iota|\widehat{\xi}_\iota\rangle \le \left(\sup_{\iota\in J}\|u_\iota\|^2\right)\langle\xi|\xi\rangle\,.$$

For every $A \in \mathfrak{P}_f(S \times T)$ and $(s,t) \in S \times T$, put

$$v_{A,st} := \begin{cases} v_{st} & \text{if } (s,t) \in A \\ 0 & \text{if } (s,t) \in S \times T \backslash A . \end{cases}$$

Since $(\|\widetilde{u}_\iota\|)_{\iota \in I} \in c_0(I)$, it follows from the above that

$$v := \lim_{A,\mathfrak{F}} v_A \in \mathcal{M}_{G,F}$$

exists, where $\mathfrak{F}$ denotes the upper section filter of $\mathfrak{P}_f(S \times T)$,

$$\widetilde{v}\xi = \left(\sum_{t \in T} v_{st} \xi_t \right)_{s \in S}$$

for every $\xi \in \bigoplus_{t \in T} F_t$, and

$$\|\widetilde{v}\| \leq \sup_{\iota \in I} \|\widetilde{u}_\iota\| .$$

Since the reverse inequality is trivial, we have

$$\|\widetilde{v}\| = \sup_{\iota \in I} \|\widetilde{u}_\iota\| . \qquad \blacksquare$$

Example 5.6.4.21 *Assume that E is the C^*-direct sum of a family $(E_\iota)_{\iota \in I}$ of C^*-algebras and for every $\iota \in I$ let $\mathfrak{F}_\iota$ be an approximate unit of E_ι. Let H be a Hilbert right E-module and for each $\iota \in I$ put*

$$H_\iota := \{\xi \in H \mid \xi = \lim_{x,\mathfrak{F}_\iota} \xi x\} .$$

Define

$$G := \{\xi \in \prod_{\iota \in I} H_\iota \mid (\|\xi_\iota\|)_{\iota \in I} \in c_0(I)\} .$$

a) *G endowed with the right multiplication*

$$G \times E \longrightarrow G , \quad (\xi, x) \longmapsto (\xi_\iota x_\iota)_{\iota \in I}$$

and with the inner-product

$$G \times G \longrightarrow E , \quad (\xi, \eta) \longmapsto (\langle \xi_\iota | \eta_\iota \rangle)_{\iota \in I}$$

is a Hilbert right E-module such that

$$\|\xi\| = \sup_{\iota \in I} \|\xi_\iota\|$$

for every $\xi \in G$.

b) $\sum_{\iota\in I}\xi_\iota \in H$ *for every* $\xi \in G$ *and the map*

$$G \longrightarrow H\,, \quad \xi \longmapsto \sum_{\iota\in I}\xi_\iota$$

is an isomorphism of Hilbert right E*-modules.*

c) H *is self-dual iff* H_ι *is self-dual for every* $\iota \in I$. ■

Proposition 5.6.4.22 *Let* E *be a* C^**-algebra and* $x \in E$ *such that*

$$p := x^*x \in \Pr E\,, \quad q := xx^* \in \Pr E\,.$$

Define

$$\varphi : pE \longrightarrow qE\,, \quad y \longmapsto xy\,,$$
$$\psi : qE \longrightarrow pE\,, \quad z \longmapsto x^*z\,.$$

Then φ *and* ψ *are isomorphisms of Hilbert right* E*-modules (Example 5.6.1.5), and*

$$\varphi \in \mathcal{L}_E(pE, qE)\,, \quad \psi \in \mathcal{L}_E(qE, pE)\,, \quad \varphi^* = \psi = \varphi^{-1}\,.$$

First remark that

$$q\varphi y = xx^*xy = xy = \varphi y$$

for every $y \in pE$ (Corollary 4.1.2.22 a $\Rightarrow$ c), so that φ (and ψ) is well-defined. For $y \in pE$ and $z \in qE$,

$$\langle \varphi y|z\rangle = \langle xy|z\rangle = z^*xy = (x^*z)^*y = \langle y|x^*z\rangle = \langle y|\psi z\rangle\,,$$

so that

$$\varphi \in \mathcal{L}_E(pE, qE)\,, \quad \psi \in \mathcal{L}_E(qE, pE)\,, \quad \varphi^* = \psi\,.$$

Moreover,

$$\psi\varphi y = x^*xy = y\,, \quad \varphi\psi z = xx^*z = z\,,$$

and so

$$\psi = \varphi^{-1}\,.$$

For $y_1, y_2 \in pE$ and $y \in E$,

$$\varphi(y_1y) = x(y_1y) = (xy_1)y = (\varphi y_1)y\,,$$

$$\langle \varphi y_1|\varphi y_2\rangle = \langle xy_1|xy_2\rangle = y_2^*x^*xy_1 = y_2^*y_1 = \langle y_1|y_2\rangle\,.$$

Thus φ and ψ are isomorphisms of Hilbert right E-modules. ■

Proposition 5.6.4.23 *Let E be a W^*-algebra and $(p_\iota)_{\iota\in I}$ a family in $\Pr E$ such that*

$$p_\iota p_\lambda = 0$$

for all distinct $\iota, \lambda \in I$ and

$$\sum_{\iota\in I}^{\ddot{E}} p_\iota = 1\,.$$

Put

$$H := \bigoplus_{\iota\in I}^{W} p_\iota E$$

and

$$\widetilde{x} : H \longrightarrow H\,, \quad \xi \longmapsto \left(p_\iota x \sum_{\lambda\in I}^{E} \xi_\lambda \right)_{\iota\in I}$$

for every $x \in E$ (Proposition 5.6.3.20 b $\Rightarrow$ a).

a) *$\widetilde{x} \in \mathcal{L}_E(H)$ for every $x \in E$ and $\widetilde{x}^* = \widetilde{x^*}$. Define*

$$\varphi : E \longrightarrow \mathcal{L}_E(H)\,, \quad x \longmapsto \widetilde{x}\,.$$

b) *φ is an isomorphism of W^*-algebras (Theorem 5.6.3.5 b), Theorem 5.6.4.7 a), Proposition 5.6.2.3) such that*

$$\widetilde{\ddot{\varphi}(a,\xi,\eta)} = \sum_{\iota\in I}^{E}\sum_{\lambda\in I}^{E} \xi_\lambda a \eta_\iota^*$$

for every $(a,\xi,\eta) \in \ddot{E}\times H\times H$ (Proposition 5.6.3.17 d)), where $\ddot{\varphi}$ denotes the pretranspose of φ. Moreover, if we put

$$p := (p_\iota)_{\iota\in I} \in H$$

then

$$\varphi^{-1}u = \sum_{\iota\in I}^{\ddot{E}} (up)_\iota$$

for every $u \in \mathcal{L}_E(H)$.

c) *If there is a family* $(x_\iota)_{\iota\in I}$ *in* E *such that*

$$x_\iota^* x_\iota = p_\iota\,, \quad x_\iota x_\iota^* = 1$$

for every $\iota \in I$, *then the* W^**-algebras* E *and*

$$\mathcal{L}_E\left(\overset{W}{\bigoplus_{\iota\in I}} E\right)$$

are isomorphic.

a) For $\xi, \eta \in H$,

$$\langle \tilde{x}\xi|\eta\rangle = \sum_{\iota\in I}^{E} \eta_\iota^* p_\iota x \sum_{\lambda\in I}^{E} \xi_\lambda = \sum_{\iota\in I}^{E} \eta_\iota^* x \sum_{\lambda\in I}^{E} p_\lambda \xi_\lambda =$$

$$= \sum_{\lambda\in I}^{E} \left(\sum_{\iota\in I}^{E} \eta_\iota^*\right) x p_\lambda \xi_\lambda = \sum_{\lambda\in I}^{E} \left(p_\lambda x^* \sum_{\iota\in I}^{E} \eta_\iota\right)^* \xi_\lambda =$$

$$= \sum_{\lambda\in I}^{E} \left(\tilde{x^*}\eta\right)_\lambda \xi_\lambda = \langle \xi|\tilde{x^*}\eta\rangle$$

(Proposition 5.6.3.19 a), Proposition 5.6.3.17 c)). Hence $\tilde{x} \in \mathcal{L}_E(H)$ and $\tilde{x}^* = \tilde{x^*}$.

b) By a), φ is involutive. For $x, y \in E$ and $\xi \in H$,

$$\tilde{x}\tilde{y}\xi = \tilde{x}\left(p_\iota y \sum_{\lambda\in I}^{E} \xi_\lambda\right)_{\iota\in I} = \left(p_\mu x \sum_{\iota\in I}^{E} p_\iota y \sum_{\lambda\in I}^{E} \xi_\lambda\right)_{\mu\in I} =$$

$$= \left(p_\mu xy \sum_{\lambda\in I}^{E} \xi_\lambda\right)_{\mu\in I} = \widetilde{xy}\xi\,,$$

so that

$$\tilde{x}\tilde{y} = \widetilde{xy}\,,$$

i.e. φ is an algebra homomorphism. φ is obviously injective.

In order to prove that φ is surjective take $u \in \mathcal{L}_E(H)$ and put

$$x := \ddot{\sum_{\iota \in I}} (up)_\iota .$$

Then for $\xi \in H$ and $\iota \in I$,

$$(\tilde{x}\xi)_\iota = p_\iota x \ddot{\sum_{\lambda \in I}} \xi_\lambda = p_\iota \ddot{\sum_{\mu \in I}} (up)_\mu \ddot{\sum_{\lambda \in I}} \xi_\lambda =$$

$$= (up)_\iota \ddot{\sum_{\lambda \in I}} \xi_\lambda = \left(u \left(p \ddot{\sum_{\lambda \in I}} \xi_\lambda \right) \right)_\iota = (u\xi)_\iota ,$$

so that

$$\tilde{x} = u .$$

Hence φ is surjective and

$$\varphi^{-1} u = \ddot{\sum_{\iota \in I}} (up)_\iota .$$

For $x \in E$,

$$\langle \tilde{x}, \widetilde{(a, \xi, \eta)} \rangle = \langle \langle \tilde{x}\xi | \eta \rangle, a \rangle = \langle \ddot{\sum_{\iota \in I}} \eta_\iota^* p_\iota x \ddot{\sum_{\lambda \in I}} \xi_\lambda, a \rangle =$$

$$= \sum_{\iota \in I} \langle \eta_\iota^* x \ddot{\sum_{\lambda \in I}} \xi_\lambda, a \rangle = \sum_{\iota \in I} \sum_{\lambda \in I} \langle \eta_\iota^* x \xi_\lambda, a \rangle =$$

$$= \sum_{\iota \in I} \sum_{\lambda \in I} \langle x, \xi_\lambda a \eta_\iota^* \rangle = \langle x, \sum_{\iota \in I}^{E} \sum_{\lambda \in I}^{E} \xi_\lambda a \eta_\iota^* \rangle ,$$

so that

$$\ddot{\varphi}\widetilde{(a, \xi, \eta)} = \sum_{\iota \in I}^{E} \sum_{\lambda \in I}^{E} \xi_\lambda a \eta_\iota^* .$$

c) By Proposition 5.6.4.22, the Hilbert right E-modules E and $p_\iota E$ are isomorphic for every $\iota \in I$ and the assertion follows from b). ■

5.6.5 $\mathcal{K}_E$

Definition 5.6.5.1 (8) *Let G, H be Hilbert right E-modules. We define for every $(\xi, \eta) \in H \times G$,*

$$\xi\langle\cdot|\eta\rangle : G \longrightarrow H\,, \quad \zeta \longmapsto \xi\langle\zeta|\eta\rangle\,.$$

Proposition 5.6.5.2 (8) *Let F, G, H be Hilbert right E-modules.*

a) *For every $(\xi, \eta) \in F \times G$,*

$$\eta\langle\cdot|\xi\rangle \in \mathcal{L}_E(F,G)\,, \quad (\eta\langle\cdot|\xi\rangle)^* = \xi\langle\cdot|\eta\rangle\,,$$

$$\|\eta\langle\cdot|\xi\rangle\| = \|\langle\xi|\xi\rangle^{\frac{1}{2}}\langle\eta|\eta\rangle^{\frac{1}{2}}\| \leq \|\xi\|\,\|\eta\|\,.$$

b) *The map*

$$G \times F \longrightarrow \mathcal{L}_E(F,G)\,, \quad (\eta, \xi) \longmapsto \eta\langle\cdot|\xi\rangle$$

is sesquilinear and continuous.

c) $(\xi, \eta) \in F \times G$, $x \in E \Rightarrow (\eta x)\langle\cdot|\xi\rangle = \eta\langle\cdot|\xi x^*\rangle$.

d) $(\xi, \eta) \in F \times G$, $v \in \mathcal{L}_E(G,H) \Rightarrow v\circ(\eta\langle\cdot|\xi\rangle) = (v\eta)\langle\cdot|\xi\rangle$.

e) $(\eta, \zeta) \in G \times H$, $u \in \mathcal{L}_E(F,G) \Rightarrow (\zeta\langle\cdot|\eta\rangle))\circ u = \zeta\langle\cdot|u^*\eta\rangle$.

f) $(\xi, \eta_1) \in F \times G$, $(\eta_2, \zeta) \in G \times H \Rightarrow$
$\Rightarrow (\zeta\langle\cdot|\eta_2\rangle)\circ(\eta_1\langle\cdot|\xi\rangle) = (\zeta\langle\eta_1|\eta_2\rangle)\langle\cdot|\xi\rangle$.

g) $(\xi, \eta) \in F \times G \Rightarrow (\eta\langle|\cdot\xi\rangle)^*\circ(\eta\langle\cdot|\xi\rangle) = (\xi\langle\eta|\eta\rangle^{\frac{1}{2}})\langle\cdot|\xi\langle\eta|\eta\rangle^{\frac{1}{2}}\rangle$.

h) *If $\overset{\circ}{F}, \overset{\circ}{G}$ denote the complexifications of F and G, respectively (Proposition 5.6.1.6), then for all $(\xi_1, \xi_2) \in \overset{\circ}{F}$ and $(\eta_1, \eta_2) \in \overset{\circ}{G}$,*

$$(\eta_1, \eta_2)\langle\cdot|(\xi_1, \xi_2)\rangle = (\eta_1\langle\cdot|\xi_1\rangle + \eta_2\langle\cdot|\xi_2\rangle, \eta_2\langle\cdot|\xi_1\rangle - \eta_1\langle\cdot|\xi_2\rangle)\,,$$

where we identified $\mathcal{L}_{\overset{\circ}{E}}(\overset{\circ}{F}, \overset{\circ}{G})$ with $\overset{\circ}{\mathcal{L}}_E(F,G)$ (Proposition 5.6.1.9).

i) $\xi \in F \Rightarrow \xi\langle\cdot|\xi\rangle \in \mathcal{L}_E(F)_+$.

j) *If F, G are Hilbert E-modules then for $(\xi, \eta) \in F \times G$ and $x \in E$,*

$$x(\eta\langle\cdot|\xi\rangle) = (x\eta)\langle\cdot|\xi\rangle\,, \quad (\eta\langle\cdot|\xi\rangle)x = \eta\langle\cdot|x^*\xi\rangle\,,$$

where we used the notation of Proposition 5.6.1.8 g).

k) *If E is finite-dimensional then*

$$\eta\langle\cdot|\xi\rangle \in \mathcal{L}_f(F,G)$$

for every $(\xi,\eta) \in F \times G$.

l) *Let (ξ_1,η_1), $(\xi_2,\eta_2) \in F \times G$ such that*

$$\eta_1\langle\cdot|\xi_1\rangle = \eta_2\langle\cdot|\xi_2\rangle$$

and such that $\langle\eta_1|\eta_1\rangle$ and $\langle\xi_2|\xi_2\rangle$ are invertible in E. Then

$$\xi_1 = \xi_2 x\,, \quad \eta_2 = \eta_1 x^*\,,$$

where

$$x := \langle\xi_1|\xi_2\rangle\langle\xi_2|\xi_2\rangle^{-1}\,.$$

m) *Let $(F_t)_{t\in T}$ and $(G_s)_{s\in S}$ be families of Hilbert right E-moduls and $\xi \in \bigoplus_{t\in T} F_t$, $\eta \in \bigoplus_{s\in S} G_s$. Put*

$$\mathfrak{M} := \{A \times B \mid A \in \mathfrak{P}_f(S)\,,\ B \in \mathfrak{P}_f(T)\}\,,$$

order $\mathfrak{M}$ by inclusion, and denote by $\mathfrak{F}$ the upper section filter of $\mathfrak{M}$. Then

$$\lim_{A\times B,\mathfrak{F}} (\eta e_A^S)\langle\cdot|\xi e_B^T\rangle = \eta\langle\cdot|\xi\rangle\,.$$

a) For $(\xi_0,\eta_0) \in F \times G$,

$$\langle(\eta\langle\cdot|\xi\rangle)\xi_0|\eta_0\rangle = \langle\eta\langle\xi_0|\xi\rangle|\eta_0\rangle =$$

$$= \langle\eta|\eta_0\rangle\langle\xi_0|\xi\rangle = \langle\xi_0|\xi\langle\eta|\eta_0\rangle^*\rangle =$$

$$= \langle\xi_0|\xi\langle\eta_0|\eta\rangle\rangle = \langle\xi_0|(\xi\langle\cdot|\eta\rangle)\eta_0\rangle$$

(Proposition 5.6.1.2 a)), so that

$$\eta\langle\cdot|\xi\rangle \in \mathcal{L}_E(F,G)\,, \quad (\eta\langle\cdot|\xi\rangle)^* = \xi\langle\cdot|\eta\rangle\,.$$

Moreover, by Proposition 5.6.1.2 a),e),

$$\|(\eta\langle\cdot|\xi\rangle)\xi_0\| = \|\eta\langle\xi_0|\xi\rangle\| = \|\langle\eta\langle\xi_0|\xi\rangle|\eta\langle\xi_0|\xi\rangle\rangle\|^{\frac{1}{2}} = \|\langle\eta|\eta\rangle^{\frac{1}{2}}\langle\xi_0|\xi\rangle\| =$$

$$= \|\langle \xi_0|\xi\langle\eta|\eta\rangle^{\frac{1}{2}}\rangle\| = \|(\langle\cdot|\xi\langle\eta|\eta\rangle^{\frac{1}{2}}\rangle)\xi_0\|$$

so that, by Proposition 5.6.2.1 a) and Proposition 5.6.1.2 e),

$$\|\eta\langle\cdot|\xi\rangle\| = \sup_{\xi_0\in F^\#} \|(\eta\langle\cdot|\xi\rangle)\xi_0\| =$$

$$= \sup_{\xi_0\in F^\#} \|(\langle\cdot|\xi\langle\eta|\eta\rangle^{\frac{1}{2}}\rangle)\xi_0\| = \|\langle\cdot|\xi\langle\eta|\eta\rangle^{\frac{1}{2}}\rangle\| =$$

$$= \|\xi\langle\eta|\eta\rangle^{\frac{1}{2}}\| = \|\langle\xi|\xi\rangle^{\frac{1}{2}}\langle\eta|\eta\rangle^{\frac{1}{2}}\| \leq$$

$$\leq \|\langle\xi|\xi\rangle^{\frac{1}{2}}\|\,\|\langle\eta|\eta\rangle^{\frac{1}{2}}\| = \|\xi\|\,\|\eta\|\,.$$

b) follows from a).

c) For every $\xi_0 \in F$,

$$((\eta x)\langle\cdot|\xi\rangle)\xi_0 = (\eta x)\langle\xi_0|\xi\rangle = \eta(x\langle\xi_0|\xi\rangle) =$$

$$= \eta\langle\xi_0|\xi x^*\rangle = (\eta\langle\cdot|\xi x^*\rangle)\xi_0$$

(Proposition 5.6.1.2 a)), so that

$$(\eta x)\langle\cdot|\xi\rangle = \eta\langle\cdot|\xi x^*\rangle\,.$$

d) For every $\xi_0 \in F$,

$$(v\circ(\eta\langle\cdot|\xi\rangle))\xi_0 = v(\eta\langle\xi_0|\xi\rangle) = (v\eta)\langle\xi_0|\xi\rangle = ((v\eta)\langle\cdot|\xi\rangle)\xi_0$$

(Proposition 5.6.1.8 d)), so that

$$v\circ(\eta\langle\cdot|\xi\rangle) = (v\eta)\langle\cdot|\xi\rangle\,.$$

e) For every $\xi_0 \in F$,

$$((\zeta\langle\cdot|\eta\rangle)\circ u)\xi_0 = (\zeta\langle\cdot|\eta\rangle)u\xi_0 = \zeta\langle u\xi_0|\eta\rangle =$$

$$= \zeta\langle\xi_0|u^*\eta\rangle = (\zeta\langle\cdot|u^*\eta\rangle)\xi_0$$

so that

$$(\zeta\langle\cdot|\eta\rangle)\circ u = \zeta\langle\cdot|u^*\eta\rangle\,.$$

f) follows from d).

g) By a), f), and c),

$$(\eta\langle\cdot|\xi\rangle)^*\circ(\eta\langle\cdot|\xi\rangle) = (\xi\langle\cdot|\eta\rangle)\circ(\eta\langle\cdot|\xi\rangle) =$$

$$= (\xi\langle\eta|\eta\rangle)\langle\cdot|\xi\rangle = \xi\langle\eta|\eta\rangle^{\frac{1}{2}}\langle\cdot|\xi\langle\eta|\eta\rangle^{\frac{1}{2}}\rangle .$$

h) For all $(\zeta_1,\zeta_2) \in \overset{\circ}{F}$,

$$((\eta_1,\eta_2)\langle\cdot|(\xi_1,\xi_2))(\zeta_1,\zeta_2) =$$

$$= (\eta_1,\eta_2)\langle(\zeta_1,\zeta_2)|(\xi_1,\xi_2)\rangle =$$

$$= (\eta_1,\eta_2)(\langle\zeta_1|\xi_1\rangle + \langle\zeta_2|\xi_2\rangle\,,\ \langle\zeta_2|\xi_1\rangle - \langle\zeta_1|\xi_2\rangle) =$$

$$= \begin{pmatrix} \eta_1\langle\zeta_1|\xi_1\rangle + \eta_1\langle\zeta_2|\xi_2\rangle - \eta_2\langle\zeta_2|\xi_1\rangle + \eta_2\langle\zeta_1|\xi_2\rangle\,, \\ \eta_1\langle\zeta_2|\xi_1\rangle - \eta_1\langle\zeta_1|\xi_2\rangle + \eta_2\langle\zeta_1|\xi_1\rangle + \eta_2\langle\zeta_2|\xi_2\rangle \end{pmatrix} =$$

$$= (\eta_1\langle\cdot|\xi_1\rangle + \eta_2\langle\cdot|\xi_2\rangle\,,\ \eta_2\langle\cdot|\xi_1\rangle - \eta_1\langle\cdot|\xi_2\rangle)(\zeta_1,\zeta_2)$$

(Proposition 5.6.1.9), so that

$$(\eta_1,\eta_2)\langle\cdot|(\xi_1,\xi_2)\rangle =$$

$$= (\eta_1\langle\cdot|\xi_1\rangle + \eta_2\langle\cdot|\xi_2\rangle\,,\ \eta_2\langle\cdot|\xi_1\rangle - \eta_1\langle\cdot|\xi_2\rangle) .$$

i) For every $\eta \in F$,

$$\langle(\xi\langle\cdot|\xi\rangle)\eta|\eta\rangle = \langle\xi\langle\eta|\xi\rangle|\eta\rangle = \langle\xi|\eta\rangle\langle\eta|\xi\rangle \in E_+ ,$$

so that

$$\xi\langle\cdot|\xi\rangle \in \mathcal{L}_E(F)_+ ,$$

by a) and Theorem 5.6.1.11 $c_2 \Rightarrow c_1$.

j) For $\xi' \in F$,

$$((\eta\langle\cdot|\xi\rangle)x)\xi' = (\eta\langle\cdot|\xi\rangle)(x\xi') = \eta\langle x\xi'|\xi\rangle =$$

$$= \eta\langle\xi'|x^*\xi\rangle = (\eta\langle\cdot|x^*\xi\rangle)\xi'$$

so that

$$(\eta\langle\cdot|\xi\rangle)x = \eta\langle\cdot|x^*\xi\rangle\,.$$

The other relation is easy to see.

k) We have

$$\operatorname{Im}\eta\langle\cdot|\xi\rangle \subset \eta E\,,$$

$$\operatorname{Dim}\operatorname{Im}\eta\langle\cdot|\xi\rangle \leq \operatorname{Dim} E < \infty\,.$$

l) We have

$$\eta_1\langle\xi_2|\xi_1\rangle = \eta_2\langle\xi_2|\xi_2\rangle\,,$$

$$\eta_2 = \eta_1\langle\xi_2|\xi_1\rangle\langle\xi_2|\xi_2\rangle^{-1} = \eta_1 x^*\,.$$

It follows by c),

$$\eta_1\langle\cdot|\xi_1\rangle = \eta_2\langle\cdot|\xi_2\rangle = \eta_1 x^*\langle\cdot|\xi_2\rangle = \eta_1\langle\cdot|\xi_2 x\rangle\,,$$

$$\eta_1\langle\cdot|\xi_1 - \xi_2 x\rangle = 0\,.$$

By a),

$$\|\langle\xi_1 - \xi_2 x|\xi_1 - \xi_2 x\rangle\langle\eta_1|\eta_1\rangle\| = 0\,,$$

$$\langle\xi_1 - \xi_2 x|\xi_1 - \xi_2 x\rangle\langle\eta_1|\eta_1\rangle = 0\,,$$

$$\langle\xi_1 - \xi_2 x|\xi_1 - \xi_2 x\rangle = 0\,,$$

$$\|\xi_1 - \xi_2 x\| = 0\,,$$

$$\xi_1 = \xi_2 x\,.$$

m) By Proposition 5.6.4.1 e), there are $A_0 \in \mathfrak{P}_f(S)$ and $B_0 \in \mathfrak{P}_f(T)$ such that

$$\|\xi - \xi e_B^T\| < \frac{\varepsilon}{2(1 + |\eta\|)}\,, \quad \|\eta - \eta e_A^S\| < \frac{\varepsilon}{2(1 + \|\xi\|)}$$

for every $A \in \mathfrak{P}_f(S)$, $A_0 \subset A$, and $B \in \mathfrak{P}_f(T)$, $B_0 \subset B$. By a) (and Corollary 4.2.1.18) for such A and B,

$$\|(\eta e_A^S)\langle\cdot|(\xi e_B^T)\rangle - \eta\langle\cdot|\xi\rangle\| \leq$$

$$\leq \|(\eta e_A^S)\langle\cdot|(\xi e_B^T)\rangle - (\eta e_A^S)\langle\cdot|\xi\rangle\| +$$

$$+\|(\eta e_A^S)\langle\cdot|\xi\rangle - \eta\langle\cdot|\xi\rangle\| \leq$$

$$\leq \|(\eta e_A^S)\langle\cdot|(\xi e_B^T) - \xi\rangle\| + \|(\eta e_A^S - \eta)\langle\cdot|\xi\rangle\| <$$

$$< \|\eta\|\frac{\varepsilon}{2(1+\|\eta\|)} + \frac{\varepsilon}{2(1+\|\xi\|)}\|\xi\| < \varepsilon\,. \qquad \blacksquare$$

Definition 5.6.5.3 (8) *Let G, H be Hilbert right E-modules. We denote by $\mathcal{K}_E(G,H)$ the closed vector subspace of $\mathcal{L}_E(G,H)$ generated by*

$$\{\eta\langle\cdot|\xi\rangle|(\xi,\eta) \in G \times H\}$$

(Proposition 5.6.5.2 a)) and put

$$\mathcal{K}_E(H) := \mathcal{K}_E(H,H)\,.$$

Corollary 5.6.5.4 (8) *Let F, G, H, I be Hilbert right E modules.*

a) *If $u \in \mathcal{L}_E(F,G)$, $v \in \mathcal{K}_E(G,H)$, $w \in \mathcal{L}_E(H,I)$ then*

$$v^* \in \mathcal{K}_E(H,G)\,, \quad w \circ v \circ u \in \mathcal{K}_E(F,I)\,.$$

b) *$\mathcal{K}_E(H)$ is a closed ideal of $\mathcal{L}_E(H)$ and so a hereditary C^*-subalgebra of $\mathcal{L}_E(H)$.*

c) *If G, H are Hilbert E-modules then $\mathcal{K}_E(G,H)$ is an E-submodule of $\mathcal{L}_E(G,H)$ (Proposition 5.6.1.8 g)).*

d) *If E is finite-dimensional then*

$$\mathcal{K}_E(G,H) \subset \mathcal{K}(G,H)\,.$$

e) $u \in \mathcal{L}_E(G,H)$, $u \circ u^* \in \mathcal{K}_E(H) \Longrightarrow u \in \mathcal{K}_E(G,H)$.

f) *If we identify* $\overset{\circ}{\mathcal{L}}_E(G,H)$ *with* $\mathcal{L}_{\overset{\circ}{E}}(\overset{\circ}{G},\overset{\circ}{H})$ *using the map*

$$\varphi : \overset{\circ}{\mathcal{L}}_E(G,H) \longrightarrow \mathcal{L}_{\overset{\circ}{E}}(\overset{\circ}{G},\overset{\circ}{H}), \quad (u,v) \longmapsto \widetilde{(u,v)}$$

of Proposition 5.6.1.9, then

$$\varphi(\overset{\circ}{\mathcal{K}}_E(G,H)) = \mathcal{K}_{\overset{\circ}{E}}(\overset{\circ}{G},\overset{\circ}{H}),$$

i.e. $\overset{\circ}{\mathcal{K}}_E(G,H)$ *and* $\mathcal{K}_{\overset{\circ}{E}}(\overset{\circ}{G},\overset{\circ}{H})$ *may be identified.*

a) follows from Proposition 5.6.5.2 a),d),e).

b) follows from a), Theorem 5.6.1.11 a), and Proposition 4.3.4.5 e).

c) follows from Proposition 5.6.5.2 j).

d) follows from Proposition 5.6.5.2 k).

e) For every $v \in \mathcal{K}_E(H)$,

$$\|v\circ u - u\|^2 = \|(v\circ u - u)\circ(v\circ u - u)^*\| =$$

$$= \|(v\circ u - u)\circ(u^*\circ v^* - u^*)\| =$$

$$= \|v\circ u\circ u^*\circ v^* - u\circ u^*\circ v^* - v\circ u\circ u^* + u\circ u^*\|$$

(Proposition 5.6.1.8 c),e),f)). By b), $\mathcal{K}_E(H)$ has an approximate unit $\mathfrak{F}$ so that

$$\lim_{v,\mathfrak{F}} \|v\circ u - u\| = 0$$

(Proposition 2.3.4.9). By a),

$$u = \lim_{v,\mathfrak{F}} v\circ u \in \mathcal{K}_E(G,H).$$

f) For $(\xi,\eta) \in G \times H$,

$$\varphi(\eta\langle\cdot|\xi\rangle, 0) = ((\eta,0)\langle\cdot \mid (\xi,0)\rangle) \in \mathcal{K}_{\overset{\circ}{E}}(\overset{\circ}{G},\overset{\circ}{H})$$

so that

$$\varphi(\overset{\circ}{\mathcal{K}}_E(G,H)) \subset \mathcal{K}_{\overset{\circ}{E}}(\overset{\circ}{G},\overset{\circ}{H}).$$

Take $(\xi_1,\xi_2) \in \overset{\circ}{G}$ and $(\eta_1,\eta_2) \in \overset{\circ}{H}$. Then

$$(\eta_1\langle\cdot|\xi_1\rangle + \eta_2\langle\cdot|\xi_2\rangle, \eta_2\langle\cdot|\xi_1\rangle - \eta_1\langle\cdot|\xi_2\rangle) \in \overset{\circ}{\mathcal{K}}_E(G,H)$$

and by a simple calculation,

$$\varphi(\eta_1\langle\cdot|\xi_1\rangle + \eta_2\langle\cdot|\xi_2\rangle, \eta_2\langle\cdot|\xi_1\rangle - \eta_1\langle\cdot|\xi_2\rangle) = (\eta_1,\eta_2)\langle\cdot|(\xi_1,\xi_2)\rangle.$$

Hence

$$\varphi(\overset{\circ}{\mathcal{K}}_E(G,H) = \mathcal{K}_{\overset{\circ}{E}}(\overset{\circ}{G},\overset{\circ}{H}).$$

■

Proposition 5.6.5.5 (8)

a) *For every finite family* $((x_\iota, y_\iota))_{\iota\in I}$ *in* $E \times E$,

$$\left\|\sum_{\iota\in I} x_\iota\langle\cdot|y_\iota\rangle\right\| = \left\|\sum_{\iota\in I} x_\iota y_\iota^*\right\| \quad \textit{(Example 5.6.1.5)}.$$

b) $\mathcal{K}_E(E) = \{x\langle\cdot|y\rangle | x, y \in E\}$ *(Example 5.6.1.5).*

c) *The map*

$$\mathcal{K}_E(E) \longrightarrow E\,, \quad x\langle\cdot|y\rangle \longmapsto xy^*$$

is an isometry of C^**-algebras (Corollary 5.6.5.4 b)).*

a) $$\left\|\sum_{\iota\in I} x_\iota\langle\cdot|y_\iota\rangle\right\| = \sup_{z\in E^\#}\left\|\left(\sum_{\iota\in I} x_\iota\langle\cdot|y_\iota\rangle\right)z\right\| =$$

$$= \sup_{z\in E^\#}\left\|\sum_{\iota\in I} x_\iota\langle z|y_\iota\rangle\right\| = \sup_{z\in E^\#}\left\|\sum_{\iota\in I} x_\iota y_\iota^* z\right\| =$$

$$= \sup_{z\in E^\#}\left\|\left(\sum_{\iota\in I} x_\iota y_\iota^*\right)z\right\| = \left\|\sum_{\iota\in I} x_\iota y_\iota^*\right\|$$

(Proposition 4.1.1.12).

b) and c). Let $\mathcal{F}$ be the vector subspace of $\mathcal{L}_E(E)$ generated by

$$\{x\langle\cdot|y\rangle | x, y \in E\}\,.$$

By a), the linear map

$$\varphi : \mathcal{F} \longrightarrow E\,, \quad \sum_{\iota\in I} x_\iota\langle\cdot|y_\iota\rangle \longmapsto \sum_{\iota\in I} x_\iota y_\iota^*$$

is well-defined and preserves the norms.

Take $x \in E$. By Corollary 4.2.4.14, there is a $y \in E$ such that

$$x = y|x|^{\frac{1}{2}}\,.$$

Then

$$x = \varphi(y\langle\cdot||x|^{\frac{1}{2}}\rangle)$$

so that φ is surjective. In particular, $\mathcal{F}$ is complete so that

$$\mathcal{K}_E(E) = \mathcal{F}$$

and

$$\mathcal{K}_E(E) = \{x\langle\cdot|y\rangle | x, y \in E\}\,.$$

For $a, b, c, d \in E$,

$$\varphi((a\langle\cdot|b\rangle \circ (c\langle\cdot|d\rangle)) = \varphi(a\langle c|b\rangle\langle\cdot|d\rangle) =$$

$$= \varphi(ab^*c\langle\cdot|d\rangle) = ab^*cd^* = \varphi(a\langle\cdot|b\rangle)\varphi(c\langle\cdot|d\rangle)$$

(Proposition 5.6.5.2 f)),

$$\varphi((a\langle\cdot|b\rangle)^*) = \varphi(b\langle\cdot|a\rangle) = ba^* = (ab^*)^* = (\varphi(a\langle\cdot|b\rangle))^*$$

(Proposition 5.6.5.2 a)), so that φ is an isometry of C^*-algebras. ■

Proposition 5.6.5.6 (8) *Let H be a Hilbert right E-module.*

a) $(\xi, x) \in H \times E \Rightarrow x\langle\cdot|\xi\rangle = \langle\cdot|\xi x^*\rangle$.

b) *For every $\xi \in H$,*

$$\langle\cdot|\xi\rangle = \lim_{x,\mathfrak{F}} x\langle\cdot|\xi\rangle \in \mathcal{K}_E(H, E)\,,$$

where $\mathfrak{F}$ is an approximate unit of E.

c) *The map*

$$H \longrightarrow \mathcal{K}_E(H, E)\,, \quad \xi \longmapsto \langle\cdot|\xi\rangle$$

is a conjugate linear isometry of real Banach spaces with inverse

$$\mathcal{K}_E(H, E) \longrightarrow H\,, \quad u \longmapsto \lim_{x,\mathfrak{F}} u^*x\,,$$

where $\mathfrak{F}$ is an approximate unit of E.

d) *Put*

$$\widetilde{\xi} : E \longrightarrow H\,, \quad x \longmapsto \xi x$$

for every $\xi \in H$. Then

$$\widetilde{\xi} = \langle\cdot|\xi\rangle^*$$

for every $\xi \in H$ *and the map*

$$H \longrightarrow \mathcal{K}_E(E,H)\,, \quad \xi \longmapsto \widetilde{\xi}$$

is an isometry of Banach spaces with inverse

$$\mathcal{K}_E(E,H) \longrightarrow H\,, \quad u \longmapsto \lim_{x,\mathfrak{F}} ux\,,$$

where $\mathfrak{F}$ *is an approximate unit of* E.

e) *If* E *is unital then*

$$\mathcal{K}_E(E,H) = \mathcal{L}_E(E,H) = \widehat{\mathcal{L}}_E(E,H)\,,$$

$$\mathcal{K}_E(H,E) = \mathcal{L}_E(H,E)\,.$$

a) For every $\eta \in H$,

$$(x\langle\cdot|\xi\rangle)\eta = x\langle\eta|\xi\rangle = \langle\eta|\xi x^*\rangle$$

(Proposition 5.6.1.2 a)), so that

$$x\langle\cdot|\xi\rangle = \langle\cdot|\xi x^*\rangle\,.$$

b) By a) and Proposition 5.6.2.1 a),

$$\|x\langle\cdot|\xi\rangle - \langle\cdot|\xi\rangle\| = \|\langle\cdot|\xi x^*\rangle - \langle\cdot|\xi\rangle\| =$$

$$= \|\langle\cdot|\xi x^* - \xi\rangle\| = \|\xi x^* - \xi\|$$

for every $x \in E$ so that

$$\langle\cdot|\xi\rangle = \lim_{x,\mathfrak{F}} x\langle\cdot|\xi\rangle \in \mathcal{K}_E(H,E)$$

(Proposition 2.3.4.9, Proposition 5.6.1.2 f)).

c) By b) (and Theorem 4.2.8.2), the map is well-defined. It is obviously conjugate linear and by Proposition 5.6.2.1 a), it preserves the norms. By a), the map is surjective.

Take $u \in \mathcal{K}_E(H;E)$ and let $\xi \in H$ such that

$$u = \langle\cdot|\xi\rangle\,.$$

For $x \in E$ and $\eta \in H$,

$$\langle u^* x|\eta\rangle = \langle x|u\eta\rangle = \langle x|\langle \eta|\xi\rangle\rangle = \langle \xi|\eta\rangle x = \langle \xi x|\eta\rangle\,,$$

so that

$$\|\langle u^* x - \xi|\eta\rangle\| = \|\langle \xi x - \xi|\eta\rangle\| \leq \|\xi x - \xi\|\,\|\eta\|\,,$$

$$\|u^* x - \xi\| \leq \|\xi x - \xi\|\,.$$

It follows

$$\lim_{x,\mathfrak{F}} u^* x = \xi\,.$$

d) By Proposition 5.6.2.1 a),

$$\widetilde{\xi} = \langle \cdot|\xi\rangle^*$$

for every $\xi \in H$. By Proposition 5.6.1.8 a),b),d) and Corollary 5.6.5.4 a), the map

$$\mathcal{K}_E(E,H) \longrightarrow \mathcal{K}_E(H,E)\,, \quad u \longmapsto u^*$$

is a conjugate linear isometry of real Banach spaces, and so the assertion follows from c).

e) The equality

$$\mathcal{K}_E(H,E) = \mathcal{L}_E(H,E)$$

follows from c) and Proposition 5.6.2.1 b). This equality implies

$$\mathcal{K}_E(E,H) = \mathcal{L}_E(E,H)\,.$$

The equality

$$\mathcal{L}_E(E,H) = \widehat{\mathcal{L}}_E(E,H)$$

follows from the Proposition 5.6.2.3 and 5.6.2.4. ■

Remark. The equality

$$\mathcal{L}_E(H,E) = \widehat{\mathcal{L}}_E(H,E)$$

does not hold in general as it was shown in the remark of Proposition 5.6.2.4.

Proposition 5.6.5.7 *Let H be a Hilbert right E module, $\xi, \eta \in H$, and $\alpha \in \mathbb{K}\backslash\{0\}$. We denote by F the C^*-algebra $\mathcal{L}_E(H)$ (Theorem 5.6.1.11 a)). The following are equivalent:*

a) $\alpha \notin \sigma_E(\langle\xi|\eta\rangle)$.

b) $\alpha \notin \sigma_F(\xi\langle\cdot|\eta\rangle)$ *(Proposition 5.6.5.2 a)).*

If these conditions are fulfilled then

$$(\alpha 1 - \xi\langle\cdot|\eta\rangle)^{-1} = \frac{1}{\alpha}(1 + \xi(\alpha 1 - \langle\xi|\eta\rangle)^{-1}\langle\cdot|\eta\rangle),$$

$$(\alpha 1 - \langle\xi|\eta\rangle)^{-1} = \frac{1}{\alpha}(1 + \langle\zeta|\eta\rangle),$$

where

$$\zeta := (\alpha 1 - \xi\langle\cdot|\eta\rangle)^{-1}\xi.$$

First assume that a) holds. By Proposition 5.6.5.2 f),

$$(\alpha 1 - \xi\langle\cdot|\eta\rangle)\frac{1}{\alpha}(1 + \xi(\alpha 1 - \langle\xi|\eta\rangle)^{-1}\langle\cdot|\eta\rangle) =$$

$$= 1 - \frac{1}{\alpha}\xi\langle\cdot|\eta\rangle + \xi(\alpha 1 - \xi|\eta\rangle)^{-1}\langle\cdot|\eta\rangle - \frac{1}{\alpha}\xi\langle\xi|\eta\rangle(\alpha 1 - \langle\xi|\eta\rangle)^{-1}\langle\cdot|\eta\rangle =$$

$$= 1 - \frac{1}{\alpha}\xi(1 - (\alpha 1 - \langle\xi|\eta\rangle)(\alpha 1 - \langle\xi|\eta\rangle)^{-1})\langle\cdot|\eta\rangle) = 1,$$

$$\frac{1}{\alpha}(1 + \xi(\alpha 1 - \langle\xi|\eta\rangle)^{-1}\langle\cdot|\eta\rangle)(\alpha 1 - \xi\langle\cdot|\eta\rangle) =$$

$$= 1 - \frac{1}{\alpha}\xi\langle\cdot|\eta\rangle + \xi(\alpha 1 - \langle\xi|\eta\rangle)^{-1}\langle\cdot|\eta\rangle - \frac{1}{\alpha}\xi(\alpha 1 - \langle\xi|\eta\rangle)^{-1}\langle\xi|\eta\rangle\langle\cdot|\eta\rangle =$$

$$= 1 - \frac{1}{\alpha}\xi(1 - (\alpha 1 - \langle\xi|\eta\rangle)^{-1}(\alpha 1 - \langle\xi|\eta\rangle))\langle\cdot|\eta\rangle = 1.$$

Hence $\alpha \notin \sigma_F(\xi\langle\cdot|\eta\rangle)$ and

$$(\alpha 1 - \xi\langle\cdot|\eta\rangle)^{-1} = \frac{1}{\alpha}(1 + \xi(\alpha 1 - \langle\xi|\eta\rangle)^{-1}\langle\cdot|\eta\rangle).$$

Now assume that b) holds. Then

$$\xi = (\alpha 1 - \xi\langle\cdot|\eta\rangle)\zeta = \alpha\zeta - \xi\langle\zeta|\eta\rangle$$

so that

$$\langle\xi|\eta\rangle = \alpha\langle\zeta|\eta\rangle - \langle\xi|\eta\rangle\langle\zeta|\eta\rangle\,,$$

$$(\alpha 1 - \langle\xi|\eta\rangle)\frac{1}{\alpha}(1 + \langle\zeta|\eta\rangle) = 1 - \frac{1}{\alpha}\langle\xi|\eta\rangle + \langle\zeta|\eta\rangle - \frac{1}{\alpha}\langle\xi|\eta\rangle\langle\zeta|\eta\rangle = 1\,.$$

By Proposition 2.3.1.14 and Proposition 5.6.5.2 a),

$$\overline{\alpha} \notin \sigma_F(\eta\langle\cdot|\xi\rangle)\,.$$

Put

$$\zeta_0 := (\overline{\alpha}1 - \eta\langle\cdot|\xi\rangle)^{-1}\eta\,.$$

Then

$$\eta = (\overline{\alpha}1 - \eta\langle\cdot|\xi\rangle)\zeta_0 = \overline{\alpha}\zeta_0 - \eta\langle\zeta_0|\xi\rangle$$

so that

$$\langle\xi|\eta\rangle = \alpha\langle\xi|\zeta_0\rangle - \langle\xi|\zeta_0\rangle\langle\xi|\eta\rangle$$

(Proposition 5.6.1.2 a)),

$$\frac{1}{\alpha}(1 + \langle\xi|\zeta_0\rangle)(\alpha 1 - \langle\xi|\eta\rangle) = 1 - \frac{1}{\alpha}\langle\xi|\eta\rangle + \langle\xi|\zeta_0\rangle - \frac{1}{\alpha}\langle\xi|\zeta_0\rangle\langle\xi|\eta\rangle = 1\,.$$

Hence $\alpha \notin \sigma_E(\langle\xi|\eta\rangle)$ (Proposition 2.1.2.3) and

$$(\alpha 1 - \langle\xi|\eta\rangle)^{-1} = \frac{1}{\alpha}(1 + \langle\zeta|\eta\rangle)\,. \qquad \blacksquare$$

Proposition 5.6.5.8 *Let H be a Hilbert right E-module. Put*

$$u \uparrow v := v \circ u$$

for all $u, v \in \mathcal{L}_E(H)$ and denote by F the C^*-algebra obtained from the C^*-algebra $\mathcal{L}_E(H)$ (Theorem 5.6.1.11 a)) by reversing the multiplication i.e. by endowing F with the multiplication

$$F \times F \longrightarrow F\,, \quad (u, v) \longmapsto u \uparrow v\,.$$

a) *H endowed with the right multiplication*

$$H \times F \longrightarrow H\,, \quad (\xi, u) \longmapsto \xi \uparrow u :\Longrightarrow u\xi$$

and with the inner-product

$$H \times H \longrightarrow F\,, \quad (\xi, \eta) \longmapsto \langle \xi \uparrow \eta \rangle := \xi\langle \cdot | \eta \rangle$$

is a Hilbert right F-module, denoted by $\vec{H}$, such that its norm coincides with the initial norm of H.

b) *For every $x \in E$, the map*

$$\widetilde{x} : \vec{H} \longrightarrow \vec{H}\,, \quad \xi \longmapsto \xi x$$

belongs to $\mathcal{L}_F(\vec{H})$ and the map

$$E \longrightarrow \mathcal{L}_F(\vec{H})\,, \quad x \longmapsto \widetilde{x}$$

is an involutive algebra homomorphism with kernel

$$\{x \in E | \xi \in H \Longrightarrow \xi x = 0\}\,.$$

If there are $\xi, \eta \in H$ such that $\langle \xi | \eta \rangle$ is invertible in E then this map is surjective.

c) *Let G be the C^*-algebra obtained from the C^*-algebra $\mathcal{L}_F(\vec{H})$ by reversing the multiplication, i.e. by endowing G with the multiplication*

$$G \times G \longrightarrow G\,, \quad (u, v) \longmapsto u \downarrow v := v \circ u\,.$$

Then H endowed with the right multiplication

$$H \times G \longrightarrow H\,, \quad (\xi, u) \longmapsto \xi \downarrow u := u\xi$$

and with the inner-product

$$H \times H \longrightarrow G\,, \quad (\xi, \eta) \longmapsto \langle \xi \downarrow \eta \rangle := \xi\langle \cdot \uparrow \eta \rangle$$

is a Hilbert right G-module, denoted by $\overleftarrow{H}$, such that its norm coincides with the initial norm of H,

$$\zeta\langle \xi | \eta \rangle = \zeta \downarrow \langle \xi \downarrow \eta \rangle$$

for all $\xi, \eta, \zeta \in H$, and $\mathcal{L}_E(H)$ is a unital C^*-subalgebra of $\mathcal{L}_G(\overleftarrow{H})$. Moreover, for every $u \in \mathcal{L}_G(\overleftarrow{H})$ and $\xi, \eta, \zeta \in H$,

$$\zeta(\langle u\xi|\eta\rangle - \langle \xi|u^*\eta\rangle) = 0\,.$$

In particular, if

$$\{x \in E | \xi \in H \Longrightarrow \xi x = 0\} = \{0\}$$

then

$$\mathcal{L}_E(H) = \mathcal{L}_G(\overleftarrow{H})\,.$$

a) We have

$$(\xi \uparrow u) \uparrow v = v(u\xi) = (v \circ u)\xi = (u \uparrow v)\xi = \xi \uparrow (u \uparrow v)$$

for all $\xi \in H$ and $u, v \in F$.

By Proposition 5.6.5.2 a), the map

$$H \times H \longrightarrow F\,, \quad (\xi, \eta) \longmapsto \xi\langle\cdot|\eta\rangle$$

is sesquilinear and Hermititan such that

$$\|\xi\langle\cdot|\xi\rangle\| = \|\langle\xi|\xi\rangle\| = \|\xi\|^2$$

for every $\xi \in H$ and by Proposition 5.6.5.2 i),

$$\xi \in H \Longrightarrow \xi\langle\cdot|\xi\rangle \in F_+\,.$$

By Proposition 5.6.5.2 d),

$$(\xi \uparrow u)\langle\cdot|\eta\rangle = (u\xi)\langle\cdot|\eta\rangle) = u \circ (\xi\langle\cdot|\eta\rangle) = (\xi\langle\cdot|\eta\rangle) \uparrow u$$

for all $\xi, \eta \in H$ and $u \in F$.

Take $\xi, \eta \in H$. By the above and Proposition 5.6.5.2 h),

$$(\xi\langle\cdot|\xi\rangle + \eta\langle\cdot|\eta\rangle, \eta\langle\cdot|\xi\rangle - \xi\langle\cdot|\eta\rangle) =$$

$$= (\xi, \eta)\langle\cdot|(\xi, \eta)\rangle \in \overset{\circ}{\mathcal{L}}_E(H)_+\,.$$

b) For $\xi, \eta \in H$,

$$\langle\widetilde{x}\xi \uparrow \eta\rangle = \langle \xi x \uparrow \eta\rangle = (\xi x)\langle\cdot|\eta\rangle =$$

$$= \xi\langle\cdot|\eta x^*\rangle = \xi\langle\cdot|\widetilde{x^*}\eta\rangle = \langle \xi \uparrow \widetilde{x^*}\eta\rangle$$

(Proposition 5.6.5.2 c)), so that

$$\widetilde{x} \in \mathcal{L}_F(\overrightarrow{H}), \quad \widetilde{x}^* = \widetilde{x^*}.$$

Take $x, y \in E$. For every $\xi \in H$,

$$\widetilde{xy}\xi = \xi(xy) = (\xi x)y = \widetilde{y}\widetilde{x}\xi = (\widetilde{x} \uparrow \widetilde{y})\xi$$

so that

$$\widetilde{xy} = \widetilde{x} \uparrow \widetilde{y}.$$

Hence the map

$$\varphi : E \longrightarrow \mathcal{L}_F(\overrightarrow{H}), \quad x \longmapsto \widetilde{x}$$

is an involutive algebra homomorphism. The relation

$$\operatorname{Ker}\varphi = \{x \in E | \xi \in H \Longrightarrow \xi x = 0\}$$

is easy to see.

We prove now the last assertion. Take $u \in \mathcal{L}_F(\overrightarrow{H})$ and put

$$x := \langle \xi | u^*\eta \rangle \langle \xi | \eta \rangle^{-1}.$$

Then for every $\zeta \in H$,

$$(u\zeta)\langle \cdot | \eta \rangle = \langle u\zeta \uparrow \eta \rangle = \langle \zeta \uparrow u^*\eta \rangle = \zeta \langle \cdot | u^*\eta \rangle,$$

$$(u\zeta)\langle \xi | \eta \rangle = \zeta \langle \xi | u^*\eta \rangle,$$

$$u\zeta = \zeta \langle \xi | u^*\eta \rangle \langle \xi | \eta \rangle^{-1} = \zeta x.$$

Hence $u = \widetilde{x}$ and φ is surjective.

c) By a), $\overleftarrow{H}$ is an Hilbert right G-module such that its norm coincides with the initial norm of H. Take $u \in \mathcal{L}_E(H)$. Then $u \in F$, so that by b), $\widetilde{u} \in \mathcal{L}_G(\overleftarrow{H})$. But

$$\widetilde{u}\xi = \xi \uparrow u = u\xi$$

for every $\xi \in H$. Hence $\widetilde{u} = u$ and $\mathcal{L}_E(H)$ is a subset of $\mathcal{L}_G(\overleftarrow{H})$. By b), $\mathcal{L}_E(H)$ is a unital C^*-subalgebra of $\mathcal{L}_G(\overleftarrow{H})$.

We have

$$\zeta \uparrow \langle \xi \downarrow \eta \rangle = \zeta \downarrow (\langle \xi \langle \cdot \uparrow \eta \rangle) = (\xi \langle \cdot \uparrow \eta \rangle) \zeta =$$

$$= \xi \uparrow \langle \zeta \uparrow \eta \rangle = \xi \uparrow (\zeta \langle \cdot | \eta \rangle) = (\zeta \langle \cdot | \eta \rangle) \xi = \zeta \langle \xi | \eta \rangle .$$

Hence for every $u \in \mathcal{L}_G(\overleftarrow{H})$,

$$\zeta(\langle u\xi|\eta\rangle - \langle \xi | u^* \eta \rangle) = \zeta \langle u\xi|\eta\rangle - \zeta \langle \xi | u^* \eta \rangle =$$

$$= \zeta \downarrow \langle u\xi \downarrow \eta \rangle - \zeta \downarrow \langle \xi \downarrow u^* \eta \rangle = 0 . \qquad \blacksquare$$

Proposition 5.6.5.9 (8) *Let* $F := (F_t)_{t \in T}$ *and* $G := (G_s)_{s \in S}$ *be families of Hilbert right* E *modules and*

$$\mathcal{P}_{G,F} := \{ u \in \mathcal{N}_{G,F} \mid (s,t) \in S \times T \Longrightarrow u_{st} \in \mathcal{K}_E(F_t, G_s) \} .$$

Put

$$H := \bigoplus_{t \in T} F_t , \quad \bigoplus_{s \in S} G_s ,$$

and

$$\widetilde{u} : H \longrightarrow K , \quad \xi \longmapsto \left(\sum_{t \in T} u_{st} \xi_t \right)_{s \in S}$$

for every $u \in \mathcal{P}_{G,F}$. *Define*

$$\mathfrak{M} := \{ A \times B \mid A \in \mathfrak{P}_f(S) , \quad B \in \mathcal{P}_f(T) \} ,$$

order $\mathfrak{M}$ *by inclusion, and denote by* $\mathfrak{F}$ *the upper section filter of* $\mathfrak{M}$.

a) *Let* $\xi \in H$ *and* $\eta \in K$ *and for every* $A \times B \in \mathfrak{M}$ *define* $u_{A,B} \in \mathcal{P}_{G,F}$ *by*

$$(u_{A,B})_{st} := \begin{cases} \eta_s \langle \cdot | \xi_t \rangle & \text{if } (s,t) \in A \times B \\ 0 & \text{if } (s,t) \in S \times T \backslash A \times B . \end{cases}$$

Then

$$\widetilde{u}_{A,B} = (\eta e_A^S) \langle \cdot | \xi e_B^T \rangle$$

for any $A \times B \in \mathfrak{M}$ *and*

$$\eta \langle \cdot | \xi \rangle = \lim_{A \times B, \mathfrak{F}} \widetilde{u}_{A,B} .$$

b) *$\mathcal{K}_E(H,K)$ is the closed vector subspace of $\mathcal{L}_E(H,K)$ generated by $\{\widetilde{u}|u \in \mathcal{P}_{G,F}\}$.*

a) For $A \times B \in \mathfrak{M}$ and $\zeta \in H$,

$$\widetilde{u}_{A,B}\zeta = \left(\sum_{t\in T}(u_{A,B})_{st}\zeta_t\right)_{s\in S} = \left(e_A^S(s)\sum_{t\in B}\eta_s\langle\zeta_t|\xi_t\rangle\right)_{s\in S},$$

so that

$$\widetilde{u}_{A,B} = \sum_{(s,t)\in A\times B}\eta_s\langle\cdot|\xi_t\rangle = \eta e_A^S\langle\cdot|\xi e_B^T\rangle.$$

By Proposition 5.6.5.2 m),

$$\eta\langle\cdot|\xi\rangle = \lim_{A\times B,\mathfrak{F}}\widetilde{u}_{A,B}.$$

b) Let $\mathcal{F}$ be the closed vector subspace of $\mathcal{L}_E(H,K)$ generated by $\{\widetilde{u}|u \in \mathcal{P}_{G,F}\}$ (Proposition 5.6.4.16 a)). By a),

$$\mathcal{K}_E(H,K) \subset \mathcal{F}.$$

Take $(a,b) \in S \times T$ and $(\xi,\eta) \in F_a \times G_b$. Define $\widetilde{\xi} \in \bigoplus_{t\in T} F_t$, $\widetilde{\eta} \in \bigoplus_{s\in S} G_s$, and $u \in \mathcal{P}_{G,F}$ by

$$\widetilde{\xi}_t := \begin{cases} \xi & \text{if } t = b \\ 0 & \text{if } t \neq b, \end{cases} \qquad \widetilde{\eta}_s := \begin{cases} \eta & \text{if } s = a \\ 0 & \text{if } s \neq a, \end{cases}$$

$$u_{s,t} := \begin{cases} \eta\langle\cdot|\xi\rangle & \text{if } (s,t) = (a,b) \\ 0 & \text{if } (s,t) \neq (a,b) \end{cases}$$

for all $(s,t) \in S \times T$. Then for any $\zeta \in \bigoplus_{t\in T} F_t$,

$$(\widetilde{u}\zeta)_s = \sum_{t\in T}u_{s,t}\zeta_t = \begin{cases} \eta\langle\zeta_b|\xi\rangle & \text{if } s = a \\ 0 & \text{if } s \neq a \end{cases}$$

so that

$$\widetilde{u}\zeta = \widetilde{\eta}\langle\zeta|\widetilde{\xi}\rangle,$$

$$\widetilde{u} = \widetilde{\eta}\langle\cdot|\widetilde{\xi}\rangle \in \mathcal{K}_E\left(\bigoplus_{t\in T}F_t, \bigoplus_{s\in S}G_s\right).$$

It follows

$$\widetilde{v} \in \mathcal{K}_E\left(\bigoplus_{t\in T} F_t\,,\ \bigoplus_{s\in S} G_s\right)$$

for every $v \in \mathcal{P}_{G,F}$ such that

$$v_{s,t} = v_{s,t}\delta_{s,a}\delta_{t,b}$$

for all $(s,t) \in S \times T$. We deduce further

$$\{\widetilde{v} | v \in \mathcal{P}_{G,F}\} \subset \mathcal{K}_E\left(\bigoplus_{t\in T} F_t\,,\ \bigoplus_{s\in S} G_s\right),$$

$$\mathcal{F} \subset \mathcal{K}_E\left(\bigoplus_{t\in T} F_t\,,\ \bigoplus_{s\in S} G_s\right),$$

$$\mathcal{F} = \mathcal{K}_E\left(\bigoplus_{t\in T} F_t\,,\ \bigoplus_{s\in S} G_s\right). \qquad \blacksquare$$

Proposition 5.6.5.10 (8) *We use the notation of Proposition 5.6.4.13.*

a) $\widetilde{u} \in \mathcal{K}_E(G,H)$ *for every*

$$u \in \prod_{\iota\in I} \mathcal{K}_{E_\iota}(Ge_\iota, He_\iota)$$

and the map

$$\prod_{\iota\in I} \mathcal{K}_{E_\iota}(Ge_\iota, He_\iota) \longrightarrow \mathcal{K}_E(G,H)$$

is an isomorphism of Banach spaces.

b) *If*

$$\mathcal{K}_{E_\iota}(Ge_\iota, He_\iota) = \mathcal{L}_{E_\iota}(Ge_\iota, He_\iota) \cap \mathcal{K}(Ge_\iota, He_\iota)$$

for every $\iota \in I$ *then*

$$\mathcal{K}_E(G,H) = \mathcal{L}_E(G,H) \cap \mathcal{K}(G,H).$$

a) Fix $\iota \in I$, let $((\xi_\lambda, \eta_\lambda))_{\lambda\in L}$ be a finite family in $(Ge_\iota) \times (He_\iota)$ and put

$$u_\iota := \sum_{\lambda\in L} \eta_\lambda \langle \cdot | \xi_\lambda \rangle,$$

$u_\kappa = 0$ for every $\kappa \in I\backslash\{\iota\}$, and

$$u := (u_\kappa)_{\kappa \in I} .$$

Then

$$\widetilde{u} := \sum_{\lambda \in L} \eta_\lambda \langle \cdot | \xi_\lambda \rangle \in \mathcal{K}_E(G, H) .$$

By Proposition 5.6.4.13 d), $\widetilde{u} \in \mathcal{K}_E(G, H)$ for every

$$u \in \prod_{\iota \in I} \mathcal{K}_{E_\iota}(Ge_\iota, He_\iota) .$$

Now let $((\xi_\lambda, \eta_\lambda))_{\lambda \in L}$ be a finite family in $G \times H$ and put

$$v := \sum_{\lambda \in L} \eta_\lambda \langle \cdot | \xi_\lambda \rangle ,$$

$$u_\iota := \sum_{\lambda \in L} \eta_\lambda e_\iota \langle \cdot | \xi_\lambda e_\iota \rangle$$

for every $\iota \in I$ and

$$u := (u_\iota)_{\iota \in I} \in \prod_{\iota \in I} \mathcal{K}_{E_\iota}(Ge_\iota, He_\iota) .$$

Then for every $\xi \in G$,

$$\widetilde{u}\xi = \sum_{\iota \in I} u_\iota(\xi e_\iota) = \sum_{\iota \in I} \sum_{\lambda \in L} \eta_\lambda e_\iota \langle \xi e_\iota | \xi_\lambda e_\iota \rangle =$$

$$= \sum_{\lambda \in L} \eta_\lambda \langle \xi | \xi_\lambda \rangle = v\xi$$

so that

$$\widetilde{u} = v$$

and the map is surjective.

The other assertions are easy to see.

b) follows from a) and Proposition 5.6.4.13 d). ■

Proposition 5.6.5.11 (8) *Let H be a Hilbert right E-module, $p \in \Pr \mathcal{L}_E(H)$, and A a finite Fourier set of* $\operatorname{Im} p$ *(which obviously is a Hilbert right E-module).*

a) $\sum_{\xi \in A} \xi \langle \cdot | \xi \rangle \in \Pr \mathcal{L}_E(H)$, $\sum_{\xi \in A} \xi \langle \cdot | \xi \rangle \leq p$.

b) *If E is $\mathbb{C}$-order σ-complete and p is a minimal element of* $\Pr \mathcal{L}_E(H)\backslash\{0\}$, *then there is a* $\xi \in \operatorname{Im} p$ *with*

$$p = \xi\langle\cdot|\xi\rangle\,, \quad \langle\xi|\xi\rangle \in \Pr E\backslash\{0\}\,.$$

c) *If E is $\mathbb{C}$-order σ-complete and A is a Fourier basis of* $\operatorname{Im} p$ *then*

$$p = \sum_{\xi\in A} \xi\langle\cdot|\xi\rangle\,.$$

a) By Proposition 5.6.5.2 a),f) and Proposition 5.6.3.9 a $\Rightarrow$ b, $\sum_{\xi\in A} \xi\langle\cdot|\xi\rangle$ is selfadjoint and

$$\left(\sum_{\xi\in A} \xi\langle\cdot|\xi\rangle\right)^2 = \left(\sum_{\xi\in A} \xi\langle\cdot|\xi\rangle\right) \circ \left(\sum_{\eta\in A} \eta\langle\cdot|\eta\rangle\right) =$$

$$= \sum_{\xi,\eta\in A} \xi\langle\eta|\xi\rangle\langle\cdot|\eta\rangle = \sum_{\xi\in A} \xi\langle\cdot|\xi\rangle$$

so that

$$\sum_{\xi\in A} \xi\langle\cdot|\xi\rangle \in \Pr \mathcal{L}_E(H)\,.$$

Moreover,

$$p\circ \left(\sum_{\xi\in A} \xi\langle\cdot|\xi\rangle\right) = \sum_{\xi\in A} p\xi\langle\cdot|\xi\rangle = \sum_{\xi\in A} \xi\langle\cdot|\xi\rangle$$

(Proposition 5.6.5.2 d)), so that by Corollary 4.2.7.6 d $\Rightarrow$ a,

$$\sum_{\xi\in A} \xi\langle\cdot|\xi\rangle \leq p\,.$$

b) follows from a) and Proposition 5.6.3.10 a),d).

c) Take $\eta \in H$. For every $\zeta \in A$,

$$\left\langle \left(p - \sum_{\xi\in A} \xi\langle\cdot|\xi\rangle\right)\eta \middle| \zeta \right\rangle = \left\langle p\eta - \sum_{\xi\in A} \xi\langle\eta|\xi\rangle \middle| \zeta \right\rangle =$$

$$= \langle p\eta|\zeta\rangle - \sum_{\xi\in A} \langle\xi\langle\eta|\xi\rangle|\zeta\rangle = \langle\eta|p\zeta\rangle - \sum_{\xi\in A} \langle\xi|\zeta\rangle\langle\eta|\xi\rangle =$$

$$= \langle \eta|\zeta\rangle - \langle \eta|\zeta\rangle = 0$$

so that by Theorem 5.6.3.13 b),

$$\left(p - \sum_{\xi\in A} \xi\langle\cdot|\xi\rangle\right)\eta = 0\,.$$

It follows

$$p = \sum_{\xi\in A} \xi\langle\cdot|\xi\rangle\,. \qquad \blacksquare$$

Proposition 5.6.5.12 (8) *Assume* $\mathbb{K} = \mathbb{R}$ *and take* $E := \mathbb{C}$ $(E := \mathbb{H})$. *Let* G *and* H *be Hilbert right* E*-modules,* $\xi \in G$, $\eta \in H$, *and put*

$$\xi_1 := \xi\,, \quad \xi_2 := \xi i\,, \quad \eta_1 := \eta\,, \quad \eta_2 := \eta i$$

$$\left(\begin{array}{cccc} \xi_1 := \xi\,, & \xi_2 := \xi i\,, & \xi_3 := \xi j\,, & \xi_4 := \xi k \\ \eta_1 := \eta\,, & \eta_2 := \eta i\,, & \eta_3 := \eta j\,, & \eta_4 := \eta k \end{array}\right).$$

a) *For* $s,t \in \mathbb{N}_2$ $(s,t \in \mathbb{N}_4)$,

$$\mathrm{re}\langle\xi_s|\xi_t\rangle = \delta_{st}\langle\xi_t|\xi_t\rangle\,, \quad \mathrm{re}\langle\eta_s|\eta_t\rangle = \delta_{st}\langle\eta_t|\eta_t\rangle\,.$$

b) $\eta\langle\cdot|\xi\rangle = \sum_{t=1}^{2} \eta_t\,\mathrm{re}\langle\cdot|\xi_t\rangle$

$$\left(\eta\langle\cdot|\xi\rangle = \sum_{t=1}^{4} \eta_t\,\mathrm{re}\langle\cdot|\xi_t\rangle\right).$$

a) is easy to see.

b) We do the proof only for $E = \mathbb{H}$. For $\zeta \in G$,

$$\eta_1\,\mathrm{re}\langle\zeta|\xi_1\rangle = \eta\,\mathrm{re}\langle\zeta|\xi\rangle\,,$$

$$\eta_2\,\mathrm{re}\langle\zeta|\xi_2\rangle = \eta i\,\mathrm{re}(-i\langle\zeta|\xi\rangle)\,,$$

$$\eta_3\,\mathrm{re}\langle\zeta|\xi_3\rangle = \eta j\,\mathrm{re}(-j\langle\zeta|\xi\rangle)\,,$$

$$\eta_4\,\mathrm{re}\langle\zeta|\xi_4\rangle = \eta k\,\mathrm{re}(-k\langle\zeta|\xi\rangle)$$

so that

$$\sum_{t=4}^{4} \eta_t\,\mathrm{re}\langle\eta|\xi_t\rangle = \eta\langle\zeta|\xi\rangle\,.$$

Hence

$$\sum_{t=1}^{4} \eta_t\,\mathrm{re}\langle\cdot|\xi_t\rangle = \eta\langle\cdot|\xi\rangle\,. \qquad \blacksquare$$

Proposition 5.6.5.13 *Let E be a W^*-algebra and H a von Neumann right E-module.*

a) *$\mathcal{K}_E(H)$ is $\mathbb{C}$-order complete.*

b) *For every $u \in \mathcal{K}_E(H)$, the space*

$$\{v \in \mathcal{K}_E(H) \mid v^*v \leq u^*u\}_{\ddot{H}}$$

(Definition 5.6.3.2) is compact.

c) *$\mathrm{Re}\,\mathcal{K}_E(H)$ and $\mathcal{K}_E(H)_+$ are closed in $(\mathcal{K}_E(H))_{\ddot{H}}$.*

d) *A family $(u_\iota)_{\iota \in I}$ in $\mathcal{K}_E(H)_+$ is summable in $(\mathcal{K}_E(H))_{\ddot{H}}$ iff*

$$\left\{ \sum_{\iota \in J} u_\iota \;\middle|\; J \in \mathfrak{P}_f(I) \right\}$$

is upper bounded in $\mathcal{K}_E(H)$ and in this case

$$\sum_{\iota \in I}^{\ddot{H}} u_\iota = \bigvee_{J \in \mathfrak{P}_f(I)} \sum_{\iota \in J} u_\iota .$$

e) *If A is a Fourier basis of H then*

$$u = \lim_{B, \mathfrak{F}_A} u \circ \left(\sum_{\xi \in B} \xi \langle \cdot | \xi \rangle \right) = \lim_{B, \mathfrak{F}_A} \left(\sum_{\xi \in B} \xi \langle \cdot | \xi \rangle \right) \circ u$$

for every $u \in \mathcal{L}_E(H)$, where the limits are considered in $(\mathcal{L}_E(H))_{\ddot{H}}$.

a) $\mathcal{L}_E(H)$ is a W^*-algebra (Theorem 5.6.3.5 b)) and so it is $\mathbb{C}$-order complete (Theorem 4.4.1.8 c)). Since $\mathcal{K}_E(H)$ is a hereditary C^*-subalgebra of $\mathcal{L}_E(H)$ (Corollary 5.6.5.4 b)), it is also $\mathbb{C}$-order complete (Theorem 4.4.1.2, Corollary 5.6.5.4 f)).

b) Let $v \in \mathcal{L}_E(H)$ with $v^*v \leq u^*u$. Since $\mathcal{K}_E(H)$ is a hereditary C^*-subalgebra of $\mathcal{L}_E(H)$ (Corollary 5.6.5.4 b)), we have $v^*v \in \mathcal{K}_E(H)$. By Corollary 5.6.5.4 e), $v \in \mathcal{K}_E(H)$. Hence

$$\{v \in \mathcal{K}_E(H) \mid v^*v \leq u^*u\} = \{v \in \mathcal{L}_E(H) \mid v^*v \leq u^*u\} .$$

By Proposition 4.4.2.23 and Theorem 5.6.3.5 b),

$$\{v \in \mathcal{L}_E(H) \mid v^*v \leq u^*u\}_{\ddot{H}}$$

is compact so that

$$\{u \in \mathcal{K}_E(H) \mid v^*v \leq u^*u\}_{\ddot{H}}$$

is also compact.

c) By Proposition 4.4.1.3 (and Theorem 5.6.3.5 b)), $\mathrm{Re}\,\mathcal{L}_E(H)$ and $\mathcal{L}_E(H)_+$ are closed in $(\mathcal{L}_E(H))_{\ddot{H}}$. Hence $\mathrm{Re}\,\mathcal{K}_E(H)$ and $\mathcal{K}_E(H)_+$ are closed in $(\mathcal{K}_E(H))_{\ddot{H}}$.

d) If $(u_\iota)_{\iota\in I}$ is summable in $(\mathcal{K}_E(H))_{\ddot{H}}$ then by c),

$$\sum_{\iota\in J} u_\iota \leq \overset{\ddot{H}}{\sum_{\iota\in I}} u_\iota$$

for every $J \in \mathfrak{P}_f(I)$. Hence

$$\left\{\sum_{\iota\in J} u_\iota \;\middle|\; J \in \mathfrak{P}_f(I)\right\}$$

is upper bounded.

Now assume

$$\left\{\sum_{\iota\in J} u_\iota \;\middle|\; J \in \mathfrak{P}_f(I)\right\}$$

is upper bounded. Let $\mathfrak{F}, \mathfrak{G}$ be ultrafilters on $\mathfrak{P}_f(I)$ finer than $\mathfrak{F}_I$. By b), $\mathfrak{F}$ and $\mathfrak{G}$ converge in $(\mathcal{K}_E(H))_{\ddot{H}}$ to u and v, respectively. By c),

$$u \geq \sum_{\iota\in J} u_\iota$$

for every $J \in \mathfrak{P}_f(I)$ so that $u \geq v$. By symmetry $u = v$. It follows that $\mathfrak{F}_I$ converges in $(\mathcal{K}_E(H))_{\ddot{H}}$, i.e. the family $(u_\iota)_{\iota\in I}$ is summable in $(\mathcal{K}_E(H))_{\ddot{H}}$ and

$$\overset{\ddot{H}}{\sum_{\iota\in I}} u_\iota = \bigvee_{J\in\mathfrak{P}_f(I)} \sum_{\iota\in J} u_\iota .$$

e) Take $(a, \eta, \zeta) \in \ddot{E} \times H \times H$. Then for every $B \in \mathfrak{P}_f(A)$,

$$\left\langle u\circ\left(\sum_{\xi\in B} \xi\langle\cdot|\xi\rangle\right), \widetilde{(a,\eta,\zeta)}\right\rangle = \left\langle \sum_{\xi\in B} (u\xi)\langle\cdot|\xi\rangle, \widetilde{(a,\eta,\zeta)}\right\rangle =$$

$$= \left\langle \sum_{\xi\in B} \langle(u\xi)\langle\eta|\xi\rangle|\zeta\rangle, a\right\rangle = \left\langle \sum_{\xi\in B} \langle u\xi|\zeta\rangle\langle\eta|\xi\rangle, a\right\rangle =$$

$$= \left\langle \sum_{\xi\in B} \langle \xi | u^*\zeta\rangle\langle\eta|\xi\rangle, a \right\rangle ,$$

$$\left\langle \left(\sum_{\xi\in B} \xi\langle\cdot|\xi\rangle\right) \circ u, \widetilde{(a,\eta,\zeta)} \right\rangle = \left\langle \sum_{\xi\in B} \xi\langle\cdot|u^*\xi\rangle, \widetilde{(a,\eta,\zeta)} \right\rangle =$$

$$= \left\langle \sum_{\xi\in B} \langle \xi\langle\eta|u^*\xi\rangle|\zeta\rangle, a \right\rangle = \left\langle \sum_{\xi\in B} \langle \xi|\zeta\rangle\langle\eta|u^*\xi\rangle, a \right\rangle =$$

$$= \left\langle \sum_{\xi\in B} \langle \xi|\zeta\rangle\langle u\eta|\xi\rangle, a \right\rangle$$

(Proposition 5.6.5.2 d),e)). By Theorem 5.6.3.13 f),

$$\lim_{B,\mathfrak{F}_A} \left\langle u\circ \left(\sum_{\xi\in B} \xi\langle\cdot|\xi\rangle\right), \widetilde{(a,\eta,\zeta)} \right\rangle = \langle\langle\eta|u^*\zeta\rangle, a\rangle =$$

$$= \langle\langle u\eta|\zeta\rangle, a\rangle = \langle u, \widetilde{(a,\eta,\zeta)}\rangle ,$$

$$\lim_{B,\mathfrak{F}_A} \left\langle \left(\sum_{\xi\in B} \xi\langle\cdot|\xi\rangle\right) \circ u, \widetilde{(a,\eta,\zeta)} \right\rangle = \langle\langle u\eta\rangle|\zeta\rangle, a\rangle = \langle u, \widetilde{(a,\eta,\zeta)}\rangle$$

so that

$$u = \lim_{B,\mathfrak{F}_A} u\circ \left(\sum_{\xi\in B} \xi\langle\cdot|\xi\rangle\right) = \lim_{B,\mathfrak{F}_A} \left(\sum_{\xi\in B} \xi\langle\cdot|\xi\rangle\right) \circ u$$

in $(\mathcal{L}_E(H))_{\ddot{H}}$. ■

Proposition 5.6.5.14 *Let E be a W^*-algebra, H a von Neumann right E-module, and $(G_\iota)_{\iota\in I}$ a family of Hilbert right $\mathcal{K}_E(H)$-modules (Corollary 5.6.5.4 b)). Put*

$$G := \overset{W'}{\bigoplus_{\iota\in I}} G_\iota := \left\{ (\xi_\iota)_{\iota\in I} \in \prod_{\iota\in I} G_\iota \;\middle|\; ((\langle\xi_\iota|\xi_\iota\rangle))_{\iota\in I} \text{ is summable in } (\mathcal{K}_E(H))_{\ddot{H}} \right\} .$$

a) *If $\xi,\eta\in G$ then $\xi+\eta\in G$ and the family $((\langle\xi_\iota|\eta_\iota\rangle))_{\iota\in I}$ is summable in $(\mathcal{K}_E(H))_{\ddot{H}}$. Put*

$$\langle\xi|\eta\rangle := \overset{\ddot{H}}{\sum_{\iota\in I}} \langle\xi_\iota|\eta_\iota\rangle .$$

b) *G endowed with the right multiplication*

$$G \times \mathcal{K}_E(H) \longrightarrow G\,, \quad (\xi, u) \longmapsto (\xi_\iota u)_{\iota \in I}$$

and with the inner-product

$$G \times G \longrightarrow \mathcal{K}_E(H)\,, \quad (\xi, \eta) \longmapsto \langle \xi | \eta \rangle$$

is an inner-product right $\mathcal{K}_E(H)$-module. If G_ι is self-dual for every $\iota \in I$, the it is even a Hilbert right $\mathcal{K}_E(H)$-module.

a) By Proposition 5.6.1.2 b),

$$\langle \xi_\iota + \eta_\iota | \xi_\iota + \eta_\iota \rangle \leq 2(\langle \xi_\iota | \xi_\iota \rangle + \langle \eta_\iota | \eta_\iota \rangle)$$

for every $\iota \in I$. By Proposition 5.6.5.13 d), $\xi + \eta \in G$. If $\mathbb{K} = \mathbb{C}$ then by the Polarization Identity (Proposition 2.3.3.2 d)), the family $(\langle \xi_\iota | \eta_\iota \rangle)_{\iota \in I}$ is summable in $(\mathcal{K}_E(H))_{\ddot{H}}$. Using the complexification (Theorem 4.4.1.2, Proposition 5.6.1.6, Proposition 5.6.1.9, Corollary 5.6.5.4 f)), it follows that this assertion holds also for $\mathbb{K} = \mathbb{R}$.

b) Take $\xi, \eta \in G$ and $u \in \mathcal{K}_E(H)$. Then

$$\sum_{\iota \in J} \langle \xi_\iota u | \xi_\iota u \rangle = \sum_{\iota \in J} u^* \langle \xi_\iota | \xi_\iota \rangle u \leq u^* \left(\sum_{\iota \in I}^{\ddot{H}} \langle \xi_\iota | \xi_\iota \rangle \right) u$$

for every $J \in \mathfrak{P}_f(I)$ (Proposition 5.1.5.13 d), Corollary 4.2.2.3) so that $\xi u \in G$. Moreover,

$$\langle \xi u | \eta \rangle = \lim_{J, \mathfrak{F}_I} \sum_{\iota \in J} \langle \xi_\iota u | \eta_\iota \rangle =$$

$$= \lim_{J, \mathfrak{F}_I} \sum_{\iota \in J} \langle \xi_\iota | \eta_\iota \rangle u = \langle \xi | \eta \rangle u\,,$$

$$\langle \xi | \eta \rangle = \lim_{J, \mathfrak{F}_I} \sum_{\iota \in J} \langle \xi_\iota | \eta_\iota \rangle = \lim_{J, \mathfrak{F}_I} \sum_{\iota \in J} \langle \eta_\iota | \xi_\iota \rangle^* =$$

$$= \lim_{J, \mathfrak{F}_I} \left(\sum_{\iota \in J} \langle \eta_\iota | \xi_\iota \rangle \right)^* = \langle \eta | \xi \rangle^*\,,$$

$$\langle\xi|\xi\rangle = \lim_{J,\mathfrak{F}_I} \sum_{\iota\in J} \langle\xi_\iota|\xi_\iota\rangle \in \mathcal{K}_E(H)_+$$

(Proposition 5.6.5.13 c)),

$$(\langle\xi|\xi\rangle + \langle\eta|\eta\rangle\,,\ \langle\eta|\xi\rangle - \langle\xi|\eta\rangle) =$$

$$= \lim_{J,\mathfrak{F}_I} \sum_{\iota\in J} (\langle\xi_\iota|\xi_\iota\rangle + \langle\eta_\iota|\eta_\iota\rangle\,,\ \langle\eta_\iota|\xi_\iota\rangle - \langle\xi_\iota|\eta_\iota\rangle) \in \mathring{\mathcal{K}}_E(H)_+$$

(Proposition 4.4.1.3, Theorem 5.6.3.5 b), Corollary 5.6.5.4 f)),

$$\xi 1 = (\xi_\iota 1)_{\iota\in I} = (\xi_\iota)_{\iota\in I} = \xi\,,$$

$$\langle\xi|\xi\rangle = 0 \Longrightarrow (\iota \in I \Longrightarrow \langle\xi_\iota|\xi_\iota\rangle = 0) \Longrightarrow (\iota \in I \Longrightarrow \xi_\iota = 0) \Longrightarrow \xi = 0\,.$$

Now we want to show that G is complete. By Proposition 5.6.2.12 (and Corollary 5.6.5.4 b)), for every $\iota \in I$ it is possible to define a right multiplication

$$G_\iota \times \mathcal{L}_E(H) \longrightarrow G_\iota$$

such that G_ι becomes a Hilbert right $\mathcal{L}_E(H)$–module. Put

$$F := \overset{w}{\bigoplus_{\iota\in I}} G_\iota := \left\{ (\xi_\iota)_{\iota\in I} \in \prod_{\iota\in I} G_\iota \,\middle|\, (\langle\xi_\iota|\xi_\iota\rangle)_{\iota\in I} \text{ is summable in } (\mathcal{L}_E(H))_{\dot{H}} \right\}.$$

By Proposition 5.6.4.6 b), F is a Hilbert right $\mathcal{L}_E(H)$–module so that it is sufficient to show that G is closed in F. Let $\xi \in F$ and let $(\xi^{(n)})_{n\in\mathbb{N}}$ be a sequence in G converging to ξ. Since $\mathcal{K}_E(H)$ is closed in $\mathcal{L}_E(H)$ it is sufficent to show that $(\langle\xi^{(n)}|\xi^{(n)}\rangle)_{n\in\mathbb{N}}$ converges to $\langle\xi|\xi\rangle$. For every $n \in \mathbb{N}$,

$$\|\langle\xi^{(n)}|\xi^{(n)}\rangle - \langle\xi|\xi\rangle\| \leq$$

$$\leq \|\langle\xi^{(n)}|\xi^{(n)}\rangle - \langle\xi^{(n)}|\xi\rangle\| + \|\langle\xi^{(n)}|\xi\rangle - \langle\xi|\xi\rangle\| =$$

$$= \|\langle\xi^{(n)}|\xi^{(n)} - \xi\rangle\| + \|\langle\xi^{(n)} - \xi|\xi\rangle\| \leq \|\xi^{(n)}\|\,\|\xi^{(n)} - \xi\| + \|\xi^{(n)} - \xi\|\,\|\xi\|$$

(Proposition 5.6.1.2 d)) so that

$$\langle\xi|\xi\rangle = \lim_{n\to\infty} \langle\xi^{(n)}|\xi^{(n)}\rangle \in \mathcal{K}_E(H)\,. \qquad \blacksquare$$

Proposition 5.6.5.15 *Let E be a W^*-algebra, H a von Neumann right E-module, G a self-dual Hilbert right $\mathcal{K}_E(H)$-module (Corollary 5.6.5.4 b)), and A a Fourier basis of G. For every*

$$(a,\alpha,\beta,\xi) \in \ddot{E} \times H \times H \times G$$

put

$$\widetilde{(a,\alpha,\beta,\xi)} : G \longrightarrow \mathbb{K}, \quad \eta \longmapsto \langle\langle\eta|\xi\rangle\alpha|\beta\rangle, a\rangle$$

and denote by $\dot{G}$ the closed vector subspace of G' generated by

$$\left\{ \widetilde{(a,\alpha,\beta,\xi)} \mid (a,\alpha,\beta,\xi) \in \ddot{E} \times H \times H \times G \right\}.$$

Define (Proposition 5.6.5.14, Proposition 5.6.2.3)

$$\bigoplus_{\xi\in A}^{W'} \langle\xi|\xi\rangle\mathcal{K}_E(H) :=$$

$$:= \left\{ x \in \prod_{\xi\in A} \langle\xi|\xi\rangle\mathcal{K}_E(H) \mid (x_\xi^* x_\xi)_{\xi\in A} \text{ is summable in } (\mathcal{K}_E(H))_{\ddot{H}} \right\}.$$

a) $G^{\#}_{G}$ *is compact.*

b) *For every*

$$x \in \bigoplus_{\xi\in A}^{W'} \langle\xi|\xi\rangle\mathcal{K}_E(H)$$

the family $(\xi x_\xi)_{\xi\in A}$ is summable in $G_{\dot{G}}$,

$$\left\langle \sum_{\xi\in A}^{\dot{G}} \xi x_\xi \,\middle|\, \eta \right\rangle = x_\eta$$

for every $\eta \in A$, the map

$$\bigoplus_{\xi\in A}^{W'} \langle\xi|\xi\rangle\mathcal{K}_E(H) \longrightarrow G, \quad x \longmapsto \sum_{\xi\in A}^{\dot{G}} \xi x_\xi$$

is an isomorphism of Hilbert right $\mathcal{K}_E(H)$-modules, and

$$G \longrightarrow \overset{W'}{\underset{\xi\in A}{\textcircled{\oplus}}} \langle\xi|\xi\rangle\mathcal{K}_E(H)\,, \quad \eta \longmapsto (\langle\eta|\xi\rangle)_{\xi\in A}$$

is its inverse. In particular, for $\eta, \zeta \in G$,

$$\eta = \sum_{\xi\in A}^{\dot{G}} \xi\langle\eta|\xi\rangle\,,$$

$$\langle\eta|\zeta\rangle = \sum_{\xi\in A}^{\dot{G}} \langle\xi|\zeta\rangle\langle\eta|\xi\rangle\,.$$

c) *If* F *is a self-dual Hilbert right* $\mathcal{K}_E(H)$*-module and*

$$u \in \mathcal{L}_{\mathcal{K}_E(H)}(G, F)$$

then the map

$$G_{\dot{G}} \longrightarrow F_{\dot{F}}\,, \quad \xi \longmapsto u\xi$$

is continuous. In particular,

$$u\eta = \sum_{\xi\in A}^{\dot{G}} (u\xi)\langle\eta|\xi\rangle$$

for every $\eta \in G$.

a) Let $\mathfrak{F}$ be an ultrafilter on $G^{\#}$. Take $\xi \in G$. Then

$$\langle\eta|\xi\rangle\langle\xi|\eta\rangle \leq \|\langle\eta|\eta\rangle\|\langle\xi|\xi\rangle \leq \langle\xi|\xi\rangle$$

for every $\eta \in G^{\#}$ (Proposition 5.6.1.2 c)). By Proposition 5.6.5.13 b),

$$\widetilde{\xi} := \lim_{\eta,\mathfrak{F}} \langle\xi|\eta\rangle$$

exists in $(\mathcal{K}_E(H))_H$ and $\|\widetilde{\xi}\| \leq \|\xi\|$. Hence for

$$(a, \alpha, \beta) \in \ddot{E} \times H \times H\,,$$

$$\left\langle \langle\widetilde{\xi}\alpha|\beta\rangle, a\right\rangle = \left\langle \widetilde{\xi}, \widetilde{(a,\alpha,\beta)}\right\rangle = \lim_{\eta,\mathfrak{F}} \langle\langle\xi|\eta\rangle, \widetilde{(a,\alpha,\beta)}\rangle =$$

$$= \lim_{\eta,\mathfrak{F}} \langle\langle\langle\xi|\eta\rangle\alpha|\beta\rangle, a\rangle .$$

It follows for $u \in \mathcal{K}_E(H)$,

$$\langle\langle\widetilde{\xi u}\alpha|\beta\rangle, a\rangle = \lim_{\eta,\mathfrak{F}} \langle\langle\langle\xi u|\eta\rangle\alpha|\beta\rangle, a\rangle =$$

$$= \lim_{\eta,\mathfrak{F}} \langle\langle\langle\xi|\eta\rangle u\alpha|\beta\rangle, a\rangle = \langle\langle\widetilde{\xi}(u\alpha)|\beta\rangle, a\rangle ,$$

$$\widetilde{\xi u} = \widetilde{\xi} u .$$

Since G is self-dual, there is a $\zeta \in G^{\#}$ with

$$\langle\xi|\zeta\rangle = \widetilde{\xi}$$

for every $\xi \in G$. For every

$$(a, \alpha, \beta, \xi) \in \ddot{E} \times H \times H \times G ,$$

$$\left\langle \zeta, \widetilde{(a, \alpha, \beta, \xi)} \right\rangle = \langle\langle\langle\zeta|\xi\rangle\alpha|\beta\rangle, a\rangle = \overline{\langle\langle\langle\xi|\zeta\rangle\beta|\alpha\rangle, a^*\rangle} =$$

$$= \overline{\langle\langle\widetilde{\xi}\beta|\alpha\rangle, a^*\rangle} = \lim_{\eta,\mathfrak{F}} \overline{\langle\langle\langle\xi|\eta\rangle\beta|\alpha\rangle, a^*\rangle} =$$

$$= \lim_{\eta,\mathfrak{F}} \langle\langle\langle\eta|\xi\rangle\alpha|\beta\rangle, a\rangle = \lim_{\eta,\mathfrak{F}} \left\langle \eta, \widetilde{(a, \alpha, \beta, \xi)} \right\rangle ,$$

so that $\mathfrak{F}$ converges to ζ in $G_{\ddot{G}}$. Thus $G^{\#}_{\ddot{G}}$ is compact.

b) For every $B \in \mathfrak{P}_f(A)$,

$$\left\langle \sum_{\xi\in B} \xi x_\xi \middle| \sum_{\xi\in B} \xi x_\xi \right\rangle = \sum_{\xi,\eta\in B} x_\eta \langle\xi|\eta\rangle x_\xi =$$

$$= \sum_{\xi\in B} x_\xi^* \langle\xi|\xi\rangle x_\xi = \sum_{\xi\in B} x_\xi^* x_\xi$$

so that (Proposition 5.6.5.13 d))

$$\sup_{B\in\mathfrak{P}_f(A)} \left\| \sum_{\xi\in B} \xi x_\xi \right\| < \infty .$$

Let $\mathfrak{F}$ be an ultrafilter on $\mathfrak{P}_f(A)$ finer than $\mathfrak{F}_A$. By a), there is a $\zeta \in G$ such that

$$\zeta = \lim_{B,\mathfrak{F}} \sum_{\xi \in B} \xi x_\xi$$

in $G_{\overset{\cdot}{G}}$. For $\eta \in A$ and $(a, \alpha, \beta) \in \ddot{E} \times H \times H$,

$$\langle\langle\langle \zeta | \eta \rangle \alpha | \beta \rangle, a \rangle = \left\langle \zeta, \overbrace{\widetilde{(a, \alpha, \beta, \eta)}} \right\rangle =$$

$$= \lim_{B,\mathfrak{F}} \left\langle \sum_{\xi \in B} \xi x_\xi, \overbrace{\widetilde{(a, \alpha, \beta, \eta)}} \right\rangle =$$

$$= \lim_{B,\mathfrak{F}} \left\langle \left\langle \left\langle \sum_{\xi \in B} \xi x_\xi \,\middle|\, \eta \right\rangle \alpha \,\middle|\, \beta \right\rangle, a \right\rangle = \langle \langle x_\eta \alpha | \beta \rangle, a \rangle$$

so that

$$\langle \zeta | \eta \rangle = x_\eta .$$

Since η is arbitrary and $\mathcal{K}_E(H)$ is $\mathbb{C}$-order complete (Proposition 5.6.5.13 a)), ζ does not depend on $\mathfrak{F}$ (Theorem 5.6.3.13 b)). Hence $(\xi x_\xi)_{\xi \in A}$ is summable in $G_{\overset{\cdot}{G}}$ and

$$\left\langle \sum_{\xi \in A}^{\overset{\cdot}{G}} \xi x_\xi \,\middle|\, \eta \right\rangle = x_\eta$$

for every $\eta \in A$.

Now take $\eta \in G$. Then

$$\langle \eta | \xi \rangle = \langle \xi | \xi \rangle \langle \eta | \xi \rangle \in \langle \xi | \xi \rangle \mathcal{K}_E(H)$$

for every $\xi \in A$ (Proposition 5.6.3.9 a $\Rightarrow$ c) and

$$\sum_{\xi \in B} \langle \xi | \eta \rangle \langle \eta | \xi \rangle \leq \langle \eta | \eta \rangle$$

for every $B \in \mathfrak{P}_f(A)$ (Proposition 5.6.3.12 b)). By Proposition 5.6.5.13 d),

$$(\langle \eta | \xi \rangle)_{\xi \in A} \in \overset{W'}{\bigoplus_{\xi \in A}} \langle \xi | \xi \rangle \mathcal{K}_E(H) .$$

By the above,

$$\eta = \overset{\ddot{G}}{\underset{\xi\in A}{\oplus}} \xi\langle\eta|\xi\rangle .$$

Hence the map

$$\overset{W'}{\underset{\xi\in A}{\oplus}} \langle\xi|\xi\rangle\mathcal{K}_E(H) \longrightarrow G , \quad x \longmapsto \overset{\ddot{G}}{\underset{\xi\in A}{\oplus}} \xi x_\xi$$

is bijective and

$$G \longrightarrow \overset{W'}{\underset{\xi\in A}{\oplus}} \langle\xi|\xi\rangle\mathcal{K}_E(H) , \quad \eta \longmapsto (\langle\eta|\xi\rangle)_{\xi\in A}$$

is its inverse.

Now take $\eta, \zeta \in G$. For $(a, \alpha, \beta) \in \ddot{E} \times H \times H$,

$$\left\langle \langle\eta|\zeta\rangle, \widetilde{(a,\alpha,\beta)} \right\rangle = \langle\langle\langle\eta|\zeta\rangle\alpha|\beta\rangle, a\rangle = \left\langle \eta, \widetilde{\widetilde{(a,\alpha,\beta,\zeta)}} \right\rangle =$$

$$= \sum_{\xi\in A} \left\langle \xi\langle\eta|\xi\rangle, \widetilde{(a,\alpha,\beta,\zeta)} \right\rangle = \sum_{\xi\in A} \langle\langle\langle\xi\langle\eta|\xi\rangle|\zeta\rangle\alpha|\beta\rangle, a\rangle =$$

$$= \sum_{\xi\in A} \langle\langle\langle\xi|\zeta\rangle\langle\eta|\xi\rangle\alpha|\beta\rangle, a\rangle = \sum_{\xi\in A} \left\langle \langle\xi|\zeta\rangle\langle\eta|\xi\rangle, \widetilde{(a,\alpha,\beta)} \right\rangle .$$

Hence

$$\langle\eta|\zeta\rangle = \sum_{\xi\in A}^{\ddot{H}} \langle\xi|\zeta\rangle\langle\eta|\xi\rangle .$$

In particular the map

$$\overset{W'}{\underset{\xi\in A}{\oplus}} \langle\xi|\xi\rangle\mathcal{K}_E(H) \longrightarrow G , \quad x \longmapsto \overset{\ddot{G}}{\underset{\xi\in A}{\oplus}} \xi x_\xi$$

is an isomorphism of Hilbert right $\mathcal{K}_E(H)$-modules.

c) Take

$$(a, \alpha, \beta, \zeta) \in \ddot{E} \times H \times H \times F .$$

Then for $\xi \in G$,

$$\left\langle u\xi, \widetilde{(a,\alpha,\beta,\zeta)} \right\rangle = \langle\langle\langle u\xi|\zeta\rangle\alpha|\beta\rangle, a\rangle =$$

$$= \langle\langle\langle\xi|u^*\zeta\rangle\alpha|\beta\rangle, a\rangle = \left\langle \xi, \widetilde{(a,\alpha,\beta,u^*\zeta)} \right\rangle$$

and this proves the continuity.

The final assertion follows from b). ■

Proposition 5.6.5.16 *Let $(F)_{F\in\mathfrak{F}}$ be a family in $\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, $(T_F)_{F\in\mathfrak{F}}$ a family of sets, and E the $\mathbb{C}^*$-direct sum of the family*

$$(\mathcal{K}_F(\ell^2(T_F,F)))_{F\in\mathfrak{F}}.$$

For every $F\in\mathfrak{F}$, we identify $\mathcal{K}_F(\ell^2(T_F,F))$ with the corresponding closed ideal of E. For $F\in\mathfrak{F}$ and $s,t\in T_F$, put

$$e_t^F : T_F \longrightarrow F, \quad r \longmapsto \delta_{rt},$$

$$e_{st}^F := e_s^F\langle\cdot|e_t^F\rangle \in \mathcal{K}_F(\ell^2(T_F,F)).$$

a) *E is $\mathbb{C}$-order complete.*

b) *If $F,G\in\mathfrak{F}$, $q,r\in T_F$, and $s,t\in T_G$ then*

$$e_{qr}^F e_{st}^G = \delta_{FG}\delta_{rs}e_{qt}^F,$$

$$(e_{qr}^F)^* = e_{rq}^F.$$

c) *If $\mathbb{K}=\mathbb{C}$ ($\mathbb{K}=\mathbb{R}$), $F\in\mathfrak{F}$, and $t\in T_F$ then e_{tt}^F is a minimal element of $\Pr E\backslash\{0\}$ and $e_{tt}^F E\, e_{tt}^F$ ($e_{tt}^F(\operatorname{Re} E)e_{tt}^F$) is one-dimensional.*

d) *Denote by T the disjoint union of the family of sets $(T_F)_{F\in\mathfrak{F}}$, for every $t\in T$ denote by F_t the element F of $\mathfrak{F}$ with $t\in T_F$ and put*

$$e_{tt} := e_{tt}^{F_t}.$$

If $\mathfrak{G}$ is the upper section filter of $\mathfrak{P}_f(T)$ and if φ denotes the map

$$\mathfrak{P}_f(T) \longrightarrow \Pr E, \quad S \longmapsto \sum_{s\in S} e_{tt}$$

then $\varphi(\mathfrak{G})$ is an approximate unit of E.

a) By Proposition 5.6.5.13 a) (and Proposition 5.6.2.5 f)), $\mathcal{K}_F(\ell^2(T_F, F))$ is $\mathbb{C}$-order complete for every $F \in \mathfrak{F}$ so that E is $\mathbb{C}$-order complete.

b) is easy to see (Proposition 5.6.5.2 a),f).

c) By b), $e^F_{tt} \in \Pr E\backslash\{0\}$. Take $q \in \Pr E$ with $q \leq e^F_{tt}$. Then

$$q = e^F_{tt} q e^F_{tt} = (e^F_t\langle\cdot|e^F_t\rangle)\circ q \circ (e^F_t\langle\cdot|e^F_t\rangle) =$$

$$= (e^F_t\langle qe^F_t|e^F_t\rangle)\langle\cdot|e^F_t\rangle = (e^F_t\langle qe^F_t|qe^F_t\rangle)\langle\cdot|e^F_t\rangle$$

(Corollary 4.2.7.6 a $\Rightarrow$ f, Proposition 5.6.5.2 d),f)). Since

$$\langle qe^F_t|qe^F_t\rangle \in F_+ = \mathbb{R}_+$$

there is an $\alpha \in \mathbb{R}_+$ such that

$$q = \alpha e^F_{tt}.$$

Then

$$\alpha = \|\alpha e^F_{tt}\| = \|q\| \in \{0, 1\}$$

(Proposition 4.1.2.21). Hence e^F_{tt} is a minimal element of $\Pr E\backslash\{0\}$. By a) and Corollary 4.3.2.12 a $\Rightarrow$ d, $e^F_{tt} E e^F_{tt}$ $(e^F_{tt}(\operatorname{Re} E)e^F_{tt})$ is one-dimensional.

d) Take $F \in \mathfrak{F}$, $\xi \in \ell^2(T_F, F)$, and denote by $\mathfrak{F}_{T_F}$ the upper section filter of $\mathfrak{P}_f(T_F)$. For every $t \in T_F$,

$$e^F_{tt}\xi = e^F_t\langle\xi|e^F_t\rangle = \xi_t e^F_t$$

so that

$$\left(\sum_{t\in S} e^F_{tt}\right)\xi = \xi e^{T_F}_S$$

for every $S \in \mathfrak{P}_f(T_F)$. It follows

$$\lim_{S,\mathfrak{F}_{T_F}} \left(\sum_{t\in S} e^F_{tt}\right)\xi = \xi$$

(Example 5.6.4.2 e)). In particular

$$\lim_{S,\mathfrak{F}_{T_F}} (\xi\langle\cdot|\eta\rangle)\circ\left(\sum_{t\in S} e^F_{tt}\right) = \lim_{S,\mathfrak{F}_{T_F}} \left(\sum_{t\in S} e^F_{tt}\right)\circ(\xi\langle\cdot|\eta\rangle) = \xi\langle\cdot|\eta\rangle$$

for all $\xi, \eta \in \ell^2(T_F, F)$. Hence

$$\lim_{S,\mathfrak{F}_{T_F}} u\circ\left(\sum_{t\in S} e_{tt}^F\right) = \lim_{S,\mathfrak{F}_{T_F}} \left(\sum_{t\in S} e_{tt}^F\right)\circ u = u$$

for every $u \in \mathcal{K}_F(\ell^2(T_F, F))$. The assertion follows. ■

Remark. A C^*-algebra E is of the above form iff there is a faithful representation (H,φ) of E such that $\varphi(E) \subset \mathcal{K}(H)$ (Theorem 6.3.6.14).

Proposition 5.6.5.17 *We use the notation of Proposition 5.6.5.16 and denote by H a Hilbert right E-module. Take*

$$(t_F)_{F\in\mathfrak{F}} \in \prod_{F\in\mathfrak{F}} T_F$$

and denote by $\mathfrak{A}$ the set of Fourier sets A of H such that

$$\{\langle\xi|\xi\rangle \mid \xi \in A\} \subset \left\{e_{t_F,t_F}^F \mid F \in \mathfrak{F}\right\} .$$

Let A be a maximal element of $\mathfrak{A}$ and put

$$A_F := \{\xi \in A \mid \langle\xi|\xi\rangle = e_{t_F,t_F}^F\} ,$$

$$\ell^{W'}\left(A_F, e_{t_F,t_F}^F E\right) := \bigoplus_{\xi\in A_F}^{W'} \langle\xi|\xi\rangle E$$

for every $F \in \mathfrak{F}$, where $\bigoplus_{\xi\in A_F}^{W'} \langle\xi|\xi\rangle E$ was defined in Proposition 5.6.5.14 (Proposition 5.6.2.3).

a) *A is a Fourier basis of H and H is isomorphic to*

$$\left\{\xi \in \prod_{F\in\mathfrak{F}} \ell^{W'}\left(A_F, e_{t_F,t_F}^F E\right) \;\middle|\; (\|\xi_F\|)_{F\in\mathfrak{F}} \in c_0(\mathfrak{F})\right\}$$

(Example 5.6.4.21).

b) *For every*

$$u \in \mathcal{L}_E(H) \cap (\mathcal{K}_E(H))^c$$

there is a $\lambda \in \ell^\infty(\mathfrak{F})$ with $\|u\| = \|\lambda\|_\infty$ and

$$u\eta = \sum_{F\in\mathfrak{F}} \lambda_F \eta_F$$

for every $\eta \in H$, *where*

$$\eta_F := \sum_{\xi \in A_F}^{H} \xi\langle\eta|\xi\rangle$$

for any $F \in \mathfrak{F}$. *In particular, if* $\operatorname{Card}\mathfrak{F} = 1$ *then*

$$\mathcal{L}_E(H) \cap (\mathcal{K}_E(H))^c = \mathbb{K}\,1\,,$$

where 1 *is the identity map of* H.

a) Assume A is not a Fourier basis of H. Then there is an $\eta \in H$ with

$$\langle\eta|\eta\rangle \in \Pr E\backslash\{0\}\,,$$

$$\xi \in A \Longrightarrow \langle\eta|\xi\rangle = 0\,.$$

By Proposition 5.6.5.16 d) and Proposition 5.6.1.2 f), there is an $F \in \mathfrak{F}$ and an $s \in T_F$ with

$$\eta e^F_{ss} \neq 0\,.$$

Then

$$e^F_{ss}\langle\eta|\eta\rangle e^F_{ss} = \langle\eta e^F_{ss} \mid \eta e^F_{ss}\rangle \neq 0\,.$$

By Proposition 5.6.5.16 c), there is an $\alpha > 0$ such that

$$e^F_{ss}\langle\eta|\eta\rangle e^F_{ss} = \alpha e^F_{ss}\,.$$

Put

$$\zeta := \frac{1}{\sqrt{\alpha}}\eta e^F_{s,t_F}\,.$$

By Proposition 5.6.5.16 b),c),

$$\langle\zeta|\zeta\rangle = \frac{1}{\alpha}\langle\eta e^F_{s,t_F} \mid \eta e^F_{s,t_F}\rangle = \frac{1}{\alpha}e^F_{t_F,s}\langle\eta|\eta\rangle e^F_{s,t_F} =$$

$$= e^F_{t_F,s}e^F_{ss}e^F_{s,t_F} = e^F_{t_F,t_F}\,.$$

For $\xi \in A$,

$$\langle\zeta|\xi\rangle = \frac{1}{\sqrt{\alpha}}\langle\eta|\xi\rangle e^F_{s,t_F} = 0\,.$$

Hence $A \cup \{\zeta\} \in \mathfrak{A}$ and this contradicts the maximality of A.

By Proposition 5.6.5.15 b) and Example 5.6.4.21, H is isomorphic to

$$\left\{ \xi \in \prod_{F \in \mathfrak{F}} \ell^{W'}\left(A_F, e^F_{t_F, t_F} E\right) \;\middle|\; (\|\xi_F\|)_{F \in \mathfrak{F}} \in c_0(F) \right\}.$$

b) Put

$$B := \{\xi \in H \mid \langle \xi | \xi \rangle \in \Pr E\}$$

and

$$x_\xi := \langle u\xi | \xi \rangle$$

for every $\xi \in B$. Then

$$x_\xi = x_\xi \langle \xi | \xi \rangle = \langle \xi | \xi \rangle x_\xi$$

for every $\xi \in B$ (Proposition 5.6.3.9 a $\Rightarrow$ b). For $\xi, \eta \in B$,

$$(u\xi)\langle \cdot | \eta \rangle = u \circ (\xi \langle \cdot | \eta \rangle) = (\xi \langle \cdot | \eta \rangle) \circ u = \xi \langle \cdot | u^* \eta \rangle$$

(Proposition 5.6.5.2 d),e)) so that

$$(u\xi)\langle \eta | \eta \rangle = \xi \langle \eta | u^* \eta \rangle = \xi \langle u\eta | \eta \rangle = \xi x_\eta .$$

If $\langle \xi | \xi \rangle = \langle \eta | \eta \rangle$, then we obtain successively

$$u\xi = (u\xi)\langle \xi | \xi \rangle = (u\xi)\langle \eta | \eta \rangle = \xi x_\eta ,$$

$$\xi(x_\xi - x_\eta) = \xi x_\xi - \xi x_\eta = u\xi - u\xi = 0 ,$$

$$0 = \langle \xi(x_\xi - x_\eta) \mid \xi(x_\xi - x_\eta) \rangle =$$

$$= (x_\xi - x_\eta)^* \langle \xi | \xi \rangle (x_\xi - x_\eta) = (x_\xi - x_\eta)^* (x_\xi - x_\eta) ,$$

$$x_\xi = x_\eta .$$

Let $F \in \mathfrak{F}$ and $\xi \in A_F$. By the above, we may define

$$x_F := x_\xi , \quad \lambda_F := \langle x_F e^F_{t_F} \mid e^F_{t_F} \rangle \in F .$$

Then

$$x_F = e^F_{t_F,t_F} x_F e^F_{t_F,t_F} =$$

$$= \left(e^F_{t_F}\langle\cdot|e^F_{t_F}\rangle\right) \circ x_F \circ \left(e^F_{t_F}\langle\cdot|e^F_{t_F}\rangle\right) =$$

$$= \left(e^F_{t_F}\langle x_F e^F_{t_F} \mid e^F_{t_F}\rangle\right)\langle\cdot|e^F_{t_F}\rangle = \left(e^F_{t_F}\lambda_F\right)\langle\cdot|e^F_{t_F}\rangle .$$

Take $\mu \in F$ with $\|\mu\| = 1$ and put

$$\eta := \xi\left(\left(e^F_{t_F}\mu\right)\langle\cdot|e^F_{t_F}\rangle\right) .$$

Then

$$\langle\eta|\eta\rangle = \left(\left(e^F_{t_F}\mu\right)\langle\cdot|e^F_{t_F}\rangle\right)^* \langle\xi|\xi\rangle \left(\left(e^F_{t_F}\mu\right)\langle\cdot|e^F_{t_F}\rangle\right) =$$

$$= \left(e^F_{t_F}\left\langle\cdot \mid e^F_{t_F}\mu\right\rangle\right) \circ \left(\left(e^F_{t_F}\mu\right)\langle\cdot \mid e^F_{t_F}\rangle\right) =$$

$$= \left(e^F_{t_F}\left\langle e^F_{t_F}\mu \mid e^F_{t_F}\mu\right\rangle\right)\left\langle\cdot \mid e^F_{t_F}\right\rangle =$$

$$= \left(e^F_{t_F}\|\mu\|^2\right)\left\langle\cdot \mid e^F_{t_F}\right\rangle = e^F_{t_F,t_F} .$$

It follows

$$u\eta = \eta x_F = \eta\left(\left(e^F_{t_F}\lambda_F\right)\left\langle\cdot \mid e^F_{t_F}\right\rangle\right) =$$

$$= \xi\left(\left(e^F_{t_F}\mu\right)\left\langle\cdot \mid e^F_{t_F}\right\rangle\right)\left(\left(e^F_{t_F}\lambda_F\right)\langle\cdot|e^F_{t_F}\rangle\right) =$$

$$= \xi\left(\left(e^F_{t_F}\mu\right)\left\langle e^F_{t_F}\lambda_F \mid e^F_{t_F}\right\rangle\right)\left\langle\cdot \mid e^F_{t_F}\right\rangle =$$

$$= \xi\left(e^F_{t_F}\mu\lambda_F\right)\left\langle\cdot \mid e^F_{t_F}\right\rangle .$$

On the other side,

$$u\eta = (u\xi)\left(\left(e^F_{t_F}\mu\right)\left\langle\cdot \mid e^F_{t_F}\right\rangle\right) =$$

$$= \xi\left(\left(e^F_{t_F}\lambda_F\right)\left\langle\cdot \mid e^F_{t_F}\right\rangle\right)\left(\left(e^F_{t_F}\mu\right)\langle\cdot|e^F_{t_F}\rangle\right) =$$

$$= \xi\left(\left(e^F_{t_F}\lambda_F\right)\left\langle e^F_{t_F}\mu|e^F_{t_F}\right\rangle\right)\left\langle\cdot \mid e^F_{t_F}\right\rangle =$$

$$= \xi \left(e^F_{t_F} \lambda_F \mu \right) \langle \cdot \mid e^F_{t_F} \rangle$$

so that, successivley,

$$\xi \left(e^F_{t_F} \mu \lambda_F \right) \langle \cdot \mid e^F_{t_F} \rangle = \xi \left(e^F_{t_F} \lambda_F \mu \right) \langle \cdot \mid e^F_{t_F} \rangle \,,$$

$$\xi \left(e^F_{t_F} (\mu \lambda_F - \lambda_F \mu) \right) \langle \cdot \mid e^F_{t_F} \rangle = 0 \,,$$

$$0 = \left(\left(e^F_{t_F} (\mu \lambda_F - \lambda_F \mu) \right) \langle \cdot \mid e^F_{t_F} \right)^* \langle \xi | \xi \rangle \left(e^F_{t_F} (\mu \lambda_F - \lambda_F \mu) \langle \cdot \mid e^F_{t_F} \rangle \right) =$$

$$= \left(e^F_{t_F} \left\langle e^F_{t_F} (\mu \lambda_F - \lambda_F \mu) \,\middle|\, e^F_{t_F} (\mu \lambda_F - \lambda_F \mu) \right\rangle \right) \langle \cdot \mid e^F_{t_F} \rangle =$$

$$= \left(e^F_{t_F} \| \mu \lambda_F - \lambda_F \mu \|^2 \right) \langle \cdot | e^F_{t_F} \rangle = \| \mu \lambda_F - \lambda_F \mu \|^2 e^F_{t_F, t_F} \,,$$

$$\| \mu \lambda_F - \lambda_F \mu \| = 0 \,,$$

$$\mu \lambda_F = \lambda_F \mu \,,$$

$$\lambda_F \in \mathbb{K} \,,$$

$$x_F = \lambda_F e^F_{t_F, t_F} \,,$$

$$u\xi = \xi x_F = \xi \lambda_F \,.$$

By Proposition 5.6.5.15 b),c), for $\eta \in H$,

$$u\eta_F = u \left(\sum_{\xi \in A_F}^{\dot{H}} \xi \langle \eta | \xi \rangle \right) = \sum_{\xi \in A_F}^{\dot{H}} (u\xi) \langle \eta | \xi \rangle =$$

$$= \sum_{\xi \in A_F}^{\dot{H}} \xi \lambda_F \langle \eta | \xi \rangle = \lambda_F \eta_F \,.$$

In particular, $|\lambda_F| \leq \|u\|$. Since

$$\eta = \sum_{F \in \mathfrak{F}} \eta_F \,,$$

$$u\eta = \sum_{F\in\mathfrak{F}} u\eta_F = \sum_{F\in\mathfrak{F}} \lambda_F\eta_F\,.$$

The relation

$$\|\lambda\|_\infty = \|u\|\,,$$

where

$$\lambda := (\lambda_F)_{F\in\mathfrak{F}}$$

is easy to see. ■

Proposition 5.6.5.18 (8) *We use the notation of Proposition 5.6.5.9, denote by* $\mathcal{F}_{F,G}$ *the set of*

$$u' := (u'_{ts})_{(t,s)\in T\times S} \in \prod_{(t,s)\in T\times S} \mathcal{K}_E(F_t, G_s)'$$

such that the map

$$\{\widetilde{u}|u \in \mathcal{P}_{G,F}\} \longrightarrow \mathbb{K}\,, \quad \widetilde{u} \longmapsto \sum_{(s,t)\in S\times T} \langle u_{st}, u'_{ts}\rangle$$

is continuous (Proposition 5.6.4.16 b)), and for every $u' \in \mathcal{F}_{F,G}$ *denote by* $\widetilde{u'}$ *the extension of this map to*

$$\mathcal{K}_E\left(\bigoplus_{t\in T} F_t\,,\ \bigoplus_{s\in S} G_s\right)$$

(Proposition 5.6.5.9 b)) and set

$$u'^* := ((u'_{st})^*)_{(s,t)\in S\times T}\,.$$

a) *The map*

$$\mathcal{F}_{F,G} \longrightarrow \mathcal{K}_E\left(\bigoplus_{t\in T} F_t\,,\ \bigoplus_{s\in S} G_s\right)'\,, \quad u' \longmapsto \widetilde{u'}$$

is linear and bijective.

b) *For every* $u' \in \mathcal{F}_{F,G}$, $u'^* \in \mathcal{F}_{G,F}$ *and* $\widetilde{u'^*} = \widetilde{u'}^*$.

c) *Let* $u' \in \mathcal{F}_{G,F}$ *and* $A \subset T \times S$ *and define*

$$v' \in \prod_{(t,s)\in T\times S} \mathcal{K}_E(F_t, G_s)'$$

by

$$v'_{ts} := \begin{cases} u'_{ts} & \text{if } (t,s) \in A \\ 0 & \text{if } (t,s) \in T\times S\backslash A\,. \end{cases}$$

Then $v' \in \mathcal{F}_{G,F}$ *and* $\|\widetilde{v'}\| \leq \|\widetilde{u'}\|$

a) Only the surjectivity needs a proof. For every $(a,b) \in S \times T$ and $u_{ab} \in \mathcal{K}_E(F_b, G_a)$ put

$$\widehat{u}_{ab} : \bigoplus_{t\in T} F_t \longrightarrow \bigoplus_{s\in S} G_s\,, \quad \xi \longmapsto (\delta_{sa} u_{ab} \xi_b)_{s\in S}\,.$$

Take

$$\theta \in \mathcal{K}_E\left(\bigoplus_{t\in T} F_t\,,\ \bigoplus_{s\in S} G_s\right)'$$

and define

$$u'_{ts} : \mathcal{K}_E(F_t, G_s) \longrightarrow \mathbb{K}\,, \quad u_{st} \longmapsto \langle \widehat{u}_{st}, \theta\rangle$$

for every $(t,s) \in T \times S$ and

$$u' := (u'_{ts})_{(t,s)\in T\times S}\,.$$

For $u \in \mathcal{P}_{G,F}$ and $\xi \in \bigoplus_{t\in T} F_t$,

$$\sum_{(s,t)\in S\times T} \widehat{u}_{st}\xi = \sum_{(s,t)\in S\times T} (\delta_{s's} u_{st}\xi_t)_{s'\in S} =$$

$$= \sum_{s\in S} (\delta_{s's}(\widetilde{u}\xi)_s)_{s'\in S} = \widetilde{u}\xi\,,$$

so that

$$\sum_{(s,t)\in S\times T} \widehat{u}_{st} = \widetilde{u}\,.$$

It follows

$$\sum_{(s,t)\in S\times T} \langle u_{st}, u'_{ts}\rangle = \sum_{(s,t)\in S\times T} \langle \widehat{u}_{st}, \theta\rangle =$$

$$= \left\langle \sum_{(s,t)\in S\times T} \widehat{u}_{st}\,,\ \theta \right\rangle = \langle \widetilde{u}, \theta\rangle$$

for every $u \in \mathcal{P}_{G,F}$. Hence

$$\theta = \widetilde{u'}$$

and the map of a) is surjective.

b) For $u \in \mathcal{P}_{F,G}$,

$$\sum_{(t,s)\in T\times S} \langle u_{ts}\,,\,(u'^*)_{st}\rangle = \sum_{(t,s)\in T\times S} \langle u_{ts}, (u'_{ts})^*\rangle =$$

$$= \sum_{(t,s)\in T\times S} \overline{\langle u^*_{ts}, u'_{ts}\rangle} = \overline{\sum_{(t,s)\in T\times S} \langle (u^*)_{st}, u'_{ts}\rangle} =$$

$$= \overline{\langle \widetilde{u^*}, \widetilde{u'}\rangle} = \overline{\langle \widetilde{u}^*, \widetilde{u'}\rangle} = \langle \widetilde{u}, \widetilde{u'}^*\rangle\,.$$

Hence $u'^* \in \mathcal{F}_{G,F}$ and $\widetilde{u'^*} = \widetilde{u'}^*$.

c) In fact, $\widetilde{v'}$ is the restriction of $\widetilde{u'}$ to

$$\{\widetilde{u} | u \in \mathcal{P}_{G,F}\,,\ (s,t) \in S \times T \backslash A \Longrightarrow u_{st} = 0\}\,. \qquad \blacksquare$$

Proposition 5.6.5.19 (8) *Let S, T be sets. For every $x \in E^{(S\times T)}$ put*

$$\widetilde{x} : \ell^2(T,E) \longrightarrow \ell^2(S,E)\,, \quad \xi \longmapsto \left(\sum_{t\in T} x_{st}\xi_t\right)_{s\in S}$$

and denote by $\mathcal{F}$ the set of $x' \in (E')^{T\times S}$ such that the map

$$\{\widetilde{x} | x \in E^{(S\times T)}\} \longrightarrow \mathbb{K}\,, \quad \widetilde{x} \longmapsto \sum_{(s,t)\in S\times T} \langle x_{st}, x'_{ts}\rangle$$

is continuous (Proposition 5.6.1.8 h), Proposition 5.6.4.16 b)) and for every $x' \in \mathcal{F}$ denote by $\widetilde{x'}$ the extension of this map to

$$\mathcal{K}_E(\ell^2(T,E)\,,\,\ell^2(S,E))$$

(Proposition 5.6.5.6 d), Proposition 5.6.5.9 b)). Then $(E')^{(T\times S)} \subset \mathcal{F}$ and

$$\{\widetilde{x'} \mid x' \in (E')^{T\times S}\}$$

is dense in

$$\mathcal{K}_E(\ell^2(T,E)\,,\,\ell^2(S,E))'\,.$$

The inclusion $(E')^{(T\times S)} \subset \mathcal{F}$ is easy to see. Take

$$\theta \in \mathcal{K}_E(\ell^2(T,E)\,,\quad \ell^2(S,E))'$$

and $\varepsilon > 0$. By Proposition 5.6.5.18 a) (and Proposition 5.6.5.6 d)), there is an $x' \in \mathcal{F}$ with $\widetilde{x'} = \theta$. For $A \subset T \times S$ put

$$x'_A : T \times S \longrightarrow E', \quad (t,s) \longmapsto \begin{cases} x'_{ts} & \text{if } (t,s) \in A \\ 0 & \text{if } (t,s) \in T \times S \backslash A . \end{cases}$$

Since $\{\widetilde{x} | x \in E^{(S \times T)}\}$ is dense in

$$\mathcal{K}_E(\ell^2(T,E), \ell^2(S,E))$$

(Proposition 5.6.5.9 b)) it is sufficient to show that there is an $A \in \mathfrak{P}_f(T \times S)$ such that $\|\widetilde{x'_B}\| \leq \varepsilon$ for every $B \in \mathfrak{P}_f(T \times S \backslash A)$. Indeed, it follows from this that the limit

$$\theta_0 := \lim_{A,\mathfrak{F}} \widetilde{x'_A}$$

exists in

$$\mathcal{K}_E(\ell^2(T,E), \ell^2(S,E))' ,$$

where $\mathfrak{F}$ denotes the upper section filter of $\mathfrak{P}_f(T \times S)$, and $\theta_0 = \theta$ on $\{\widetilde{x} | x \in E^{(S \times T)}\}$.

Case 1 T is a one-point set (e.g. $T = \mathbb{N}_1$)

Assume the contrary. By induction, we may construct a disjoint sequence $(S_n)_{n \in \mathbb{N}}$ in $\mathfrak{P}_f(S)$ such that

$$\|\widetilde{x}'_{T \times S_n}\| > \varepsilon$$

for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, there is an $x_n \in E^{(S \times T)}$ such that

$$\|\widetilde{x}_n\| \leq 1 , \quad \left\langle \widetilde{x}_n, \widetilde{x'_{T \times S_n}} \right\rangle > \varepsilon$$

$$s \in S \backslash S_n \Longrightarrow x_{n,s1} = 0 .$$

By Proposition 5.6.5.6 d),

$$\|\widetilde{x}_n\| = \|(x_{n,s1})_{s \in S}\| ,$$

so that

$$\sum_{s \in S_n} x^*_{n,s1} x_{ns1} \leq 1 .$$

Take $p \in \mathbb{N}$ and put

$$y : S \times T \longrightarrow E , \quad (s,1) \longmapsto \begin{cases} \frac{1}{n} x_{n,s1} & \text{if } s \in S_n \text{ for some } n \in \mathbb{N}_p \\ 0 & \text{otherwise} . \end{cases}$$

Then $y \in E^{(S\times T)}$,

$$\langle \widetilde{y}, \widetilde{x'} \rangle = \sum_{n=1}^{p} \frac{1}{n} \langle \widetilde{x}_n, \widetilde{x}'_{T\times S} \rangle > \varepsilon \sum_{n=1}^{p} \frac{1}{n},$$

$$\sum_{s\in S} y_{s1}^* y_{s1} = \sum_{n=1}^{p} \frac{1}{n^2} \sum_{s\in S_n} x_{n,s1}^* x_{n,s1} \leq \sum_{n=1}^{p} \frac{1}{n^2} 1 \leq \frac{\pi^2}{6} 1.$$

Hence

$$(y_{s1})_{s\in S} \in \ell^2(S, E), \quad \|(y_{s1})_{s\in S}\| \leq \pi$$

and by Proposition 5.6.5.6 d),

$$\|\widetilde{y}\| = \|(y_{s1})_{s\in S}\| \leq \pi.$$

This contradicts $x' \in \mathcal{F}$.

Case 2 S is a one-point set

The assertion follows from Case 1 and Proposition 5.6.5.18 b).

Case 3 S and T are arbitrary

Step 1 There are $S_0 \in \mathfrak{P}_f(S)$ and $T_0 \in \mathfrak{P}_f(T)$ such that $\|\widetilde{x'_A}\| \leq \frac{\varepsilon}{3}$ for every $A \in \mathfrak{P}_f((T\backslash T_0) \times (S\backslash S_0))$.

Assume the contrary. By Case 1 and Case 2, we may assume S and T infinite. By induction, we may construct disjoint sequences $(S_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}}$ in $\mathfrak{P}_f(S)$ and $\mathfrak{P}_f(T)$, respectively, such that

$$\|\widetilde{x'_{T_n\times S_n}}\| > \frac{\varepsilon}{3}$$

for every $n \in \mathbb{N}$ (Proposition 5.6.5.18 c)). For every $n \in \mathbb{N}$, there is an $x_n \in E^{(S\times T)}$ with

$$\|\widetilde{x}_n\| \leq 1, \quad \left\langle \widetilde{x}_n, \widetilde{x'_{T_n\times S_n}} \right\rangle > \frac{\varepsilon}{3},$$

$$(s,t) \in S \times T\backslash S_n \times T_n \Longrightarrow x_{n,st} = 0.$$

Take $p \in \mathbb{N}$ and put

$$y := \sum_{n=1}^{p} x_n.$$

Then $y \in E^{(S\times T)}$,

$$\langle \widetilde{y}, \widetilde{x'} \rangle = \sum_{n=1}^{p} \left\langle \widetilde{x}_n, \widetilde{x'_{T_n \times S_n}} \right\rangle > \frac{\varepsilon p}{3},$$

and by Proposition 5.6.4.20 j),

$$\|\widetilde{y}\| \leq \sup_{n \in \mathbb{N}} \|\widetilde{x_n}\| \leq 1.$$

This contradicts $x' \in \mathcal{F}$.

Step 2 The assertion

By Case 1, for every $t \in T_0$, there is an $S_t \in \mathfrak{P}_f(S)$ such that

$$\left\| \widetilde{x'_{\{t\} \times A}} \right\| < \frac{\varepsilon}{3(1 + \operatorname{Card} T_0)}$$

for every $A \in \mathfrak{P}_f(S \backslash S_t)$. By Case 2, for every $s \in S_0$, there is a $T_s \in \mathfrak{P}_f(T)$ such that

$$\left\| \widetilde{x'_{B \times \{s\}}} \right\| < \frac{\varepsilon}{3(1 + \operatorname{Card} S_0)}$$

for every $B \in \mathfrak{P}_f(T \backslash T_0)$. Put

$$C := \left(T_0 \cup \left(\bigcup_{s \in S_0} T_s \right) \right) \times \left(S_0 \cup \left(\bigcup_{t \in T_0} S_t \right) \right) \in \mathfrak{P}_f(T \times S)$$

and take $D \in \mathfrak{P}_f(T \times S \backslash C)$. Then

$$D = (D \cap ((T \backslash T_0) \times (S \backslash S_0))) \cup$$

$$\cup \left(\bigcup_{t \in T_0} D \cap (\{t\} \times (S \backslash S_t)) \right) \cup \left(\bigcup_{s \in S_0} D \cap ((T \backslash T_s) \times \{s\}) \right)$$

and

$$\left\| \widetilde{x'_{D \cap ((T \backslash T_0) \times (S \backslash S_0))}} \right\| < \frac{\varepsilon}{3},$$

$$t \in T_0 \Longrightarrow \left\| \widetilde{x'_{D \cap (\{t\} \times (S \backslash S_t))}} \right\| < \frac{\varepsilon}{3(1 + \operatorname{Card} T_0)},$$

$$s \in S_0 \Longrightarrow \left\| x'_{D \cap ((T \backslash T_s) \times \{s\})} \right\| < \frac{\varepsilon}{3(1 + \operatorname{Card} S_0)}.$$

It follows

$$\left\| \widetilde{x'_D} \right\| < \varepsilon. \qquad \blacksquare$$

Proposition 5.6.5.20 *Let E be a W^*-algebra, T a set, and*

$$H := \bigoplus_{t\in T}^{W} E .$$

For every $x \in E^c$, put

$$\widetilde{x} : H \longrightarrow H , \quad \xi \longmapsto \xi x .$$

Then

$$\mathcal{K}_E(H)^c = \mathcal{L}_E(H)^c ,$$

$\widetilde{x} \in \mathcal{L}_E(H)^c$ for every $x \in E^c$, and the map

$$E^c \longrightarrow \mathcal{L}_E(H)^c , \quad x \longmapsto \widetilde{x}$$

is an isomorphism of W^*-algebras (Theorem 5.6.4.7 a), Theorem 5.6.3.5 b), Corollary 4.4.4.12 e)).

By Proposition 5.6.4.8 c),d), $\widetilde{x} \in \mathcal{L}_E(H)$ for every $x \in E^c$ and the map

$$E^c \longrightarrow \mathcal{L}_E(H) , \quad x \longmapsto \widetilde{x}$$

is a W^*-homomorphism.

Let $x \in E^c$ and $u \in \mathcal{L}_E(H)$. For every $\xi \in H$,

$$u\widetilde{x}\xi = u(\xi x) = (u\xi)x = \widetilde{x}u\xi ,$$

so that

$$u\widetilde{x} = \widetilde{x}u ,$$

$$\widetilde{x} \in \mathcal{L}_E(H)^c .$$

Since

$$\mathcal{L}_E(H)^c \subset \mathcal{K}_E(H)^c ,$$

it is sufficient to show that for every $u \in \mathcal{K}_E(H)^c$ there is an $x \in E^c$ with

$$\widetilde{x} = u .$$

For every $a, b \in T$, put

$$e_a := (\delta_{at}1)_{t\in T}\,,$$

$$u_{ab} := \langle ue_a|e_b\rangle \in E\,.$$

Fix $a,b\in T$ and $y\in E$ and define

$$v := e_a y\langle\cdot|e_b\rangle \in \mathcal{K}_E(H)\,.$$

Then, for every $c\in T$,

$$u_{aa}y\delta_{bc} = u(e_a y\delta_{bc})_a = (uve_c)_a =$$

$$= (vue_c)_a = (e_a yu_{cb})_a = yu_{cb}\,.$$

We deduce

$$u_{cb} = 0$$

for $b\neq c$. Putting $b=c$, we get

$$u_{aa}y = yu_{bb}\,,$$

so that

$$u_{aa} = u_{bb}\in E^c\,.$$

Moreover,

$$\langle ue_a|e_b\rangle = \delta_{ab}u_{aa} = \langle e_a u_{aa}|e_b\rangle = \langle \widetilde{u}_{aa}e_a|e_b\rangle\,,$$

$$ue_a = \widetilde{u}_{aa}e_a\,.$$

For $\xi\in H$,

$$\xi = \sum_{t\in T}^{H} e_t\xi_t$$

(Proposition 5.6.4.6 c)), so that by Proposition 5.6.3.4 c),

$$\widetilde{u}_{aa}\xi = \sum_{t\in T}^{H}\widetilde{u}_{aa}e_t\xi_t = \sum_{t\in T}^{H} ue_t\xi_t = u\left(\sum_{t\in T}^{H} e_t\xi_t\right) = u\xi\,.$$

Thus

$$\widetilde{u}_{aa} = u\,.$$

■

5.6.6 Matrices over C^*-algebras

Theorem 5.6.6.1 (0) *Let F be a closed ideal of E such that*

$$\|x\| = \sup_{y \in F^{\#}} \|y^* x y\|$$

for every $x \in E$ (since E is quasiunital (Theorem 4.2.8.2) this condition is automatically fulfilled if $F = E$ (Proposition 2.3.4.9)). Given $n \in \mathbb{N}$, let $F^n := \ell^2(\mathbb{N}_n, F)$ be the Hilbert (unital) E-module defined in Example 5.6.4.2 c). Take $m, n, p \in \mathbb{N}$ and for $a \in E_{m,n}$ define

$$\widetilde{a} : F^n \longrightarrow F^m, \quad \xi \longmapsto \left(\sum_{j=1}^{n} a_{ij} \xi_j \right)_{i \in \mathbb{N}_m} .$$

a) *For all $a \in E_{m,n}$,*

$$\widetilde{a} \in \mathcal{L}_E(F^n, F^m), \quad \widetilde{a}^* = \widetilde{a^*},$$

$$\sup_{\substack{i \in \mathbb{N}_m \\ j \in \mathbb{N}_n}} \|a_{ij}\| \leq \sup_{j \in \mathbb{N}_n} \|a_{\cdot j}\| \leq \|\widetilde{a}\| \leq$$

$$\leq \inf \left\{ \sum_{j=1}^{n} \|a_{\cdot j}\|, \quad \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} \|a_{ij}\| \right)^2 \right)^{\frac{1}{2}} \right\} .$$

If $F = E$ then $\widetilde{a} \in \mathcal{K}_E(E^n, E^m)$.

b) *The map*

$$E_{m,n} \longrightarrow \mathcal{L}_E(F^n, F^m), \quad a \longmapsto \widetilde{a}$$

is linear and injective and its image is a closed vector subspace of $\mathcal{L}_E(F^n, F^m)$.

c) *If the map of b) is surjective for $m = n = 1$ (this is the case if $F = E$ and E is unital), then it is bijective for all $m, n \in \mathbb{N}$. In particular,*

$$\mathcal{L}_E(E^n, E^m) = \mathcal{K}_E(E^n, E^m).$$

d) *$E_{m,n}$ endowed with the norm*

$$E_{m,n} \longrightarrow \mathbb{R}_+, \quad a \longmapsto \|\widetilde{a}\|$$

and with the maps

$$E \times E_{m,n} \longrightarrow E_{m,n}\,, \quad (x,a) \longmapsto xa\,,$$

$$E_{m,n} \times E \longrightarrow E_{m,n}\,, \quad (a,x) \longmapsto ax$$

(Definition 2.2.7.23)) is a (unital) E-module and the map of b) is a homomorphism of E-modules (Proposition 5.6.1.8 g)).

e) *If $a \in E_{m,n}$ and $b \in E_{n,p}$ then*

$$\widetilde{ab} = \widetilde{a} \circ \widetilde{b}\,.$$

f) *$E_{n,n}$ is an E-C^*-algebra. If E is unital then the map*

$$E_{n,n} \longrightarrow \mathcal{L}_E(E^n)\,, \quad a \longmapsto \widetilde{a}$$

is an isomorphism of E-C^*-algebras (Theorem 5.6.1.11 d)).

g) *Let $a \in (E_{n,n})_+$. Then $a_{ii} \in E_+$ for every $i \in \mathbb{N}_n$ and*

$$\|\widetilde{a}\| \leq n^{\frac{5}{2}} \left\| \sum_{i=1}^{n} a_{ii} \right\|\,.$$

h) *$E_{m,n}$ endowed with the bilinear map*

$$E_{m,n} \times E_{n,n} \longrightarrow E_{m,n}\,, \quad (a,b) \longmapsto ab$$

and with the sesquilinear map

$$E_{m,n} \times E_{m,n} \longrightarrow E_{n,n}\,, \quad (a,b) \longmapsto b^*a$$

is a Hilbert right $E_{n,n}$-module.

i) *The norm of $E_{m,n}$ (defined in d)) does not depend on F.*

j) *Let $p \in \mathbb{N}_m$ and $q \in \mathbb{N}_n$. Take $a \in E_{p,q}$ and define $b \in E_{m,n}$ by*

$$b_{ij} := \begin{cases} a_{ij} & \text{if } (i,j) \in \mathbb{N}_p \times \mathbb{N}_q \\ 0 & \text{otherwise} \end{cases}\,.$$

Then

$$\|a\| = \|b\|\,.$$

k) *If E is unital then for all $(\xi, \eta) \in E^n \times E^m$,*

$$\eta\langle\cdot|\xi\rangle = \widetilde{a}\,,$$

where

$$a = \left[\eta_i \xi_j^*\right]_{(i,j)\in \mathbb{N}_m \times \mathbb{N}_n}\,.$$

a) Take $\xi \in F^n$ and $\eta \in F^m$. Then

$$\langle \widetilde{a}\xi|\eta\rangle = \sum_{i=1}^m \eta_i^* (\widetilde{a}\xi)_i = \sum_{i=1}^m \eta_i^* \left(\sum_{j=1}^n a_{ij}\xi_j\right) =$$

$$= \sum_{j=1}^n \left(\sum_{i=1}^m \eta_i^* a_{ij}\right) \xi_j = \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij}^* \eta_i\right)^* \xi_j =$$

$$= \sum_{j=1}^n \left(\widetilde{a}^*\eta\right)_j^* \xi_j = \langle \xi | \widetilde{a}^*\eta\rangle\,.$$

Hence $\widetilde{a} \in \mathcal{L}_E(F^n, F^m)$ and $\widetilde{a}^* = \widetilde{a^*}$. By Proposition 5.6.1.2 d),e),f) and Example 5.6.4.2 b),

$$\|\widetilde{a}\xi\|^2 = \left\|\sum_{i=1}^m (\widetilde{a}\xi)_i^* (\widetilde{a}\xi)_i\right\| = \left\|\sum_{i=1}^m \left(\sum_{j=1}^n a_{ij}\xi_j\right)^* \left(\sum_{k=1}^n a_{ik}\xi_k\right)\right\| =$$

$$= \left\|\sum_{i=1}^m \sum_{j,k=1}^n (a_{ij}\xi_j)^* (a_{ik}\xi_k)\right\| = \left\|\sum_{j,k=1}^n \sum_{i=1}^m (a_{ij}\xi_j)^* (a_{ik}\xi_k)\right\| =$$

$$= \left\|\sum_{j,k=1}^n \langle a_{\cdot,j}\xi_j | a_{\cdot,k}\xi_k\rangle\right\| \leq \sum_{j,k=1}^n \|\langle a_{\cdot,j}\xi_j | a_{\cdot,k}\xi_k\rangle\| \leq$$

$$\leq \sum_{j,k=1}^n \|a_{\cdot,j}\xi_j\|\,\|a_{\cdot,k}\xi_k\| = \left(\sum_{j=1}^n \|a_{\cdot,j}\xi_j\|\right)^2 \leq$$

$$\leq \left(\sum_{j=1}^n \|a_{\cdot,j}\|\,\|\xi_j\|\right)^2 \leq \left(\sum_{j=1}^n \|a_{\cdot,j}\|\right)^2 \sup_{j\in\mathbb{N}_n} \|\xi_j\|^2 \leq \|\xi\|^2 \left(\sum_{j=1}^n \|a_{\cdot,j}\|\right)^2.$$

Hence

$$\|\widetilde{a}\| \leq \sum_{j=1}^{n} \|a_{\cdot,j}\| .$$

By Proposition 5.6.4.16 a),

$$\|\widetilde{a}\| \leq \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} \|a_{ij}\| \right)^2 \right)^{\frac{1}{2}} .$$

Let $j \in \mathbb{N}_n$. Then for every $x \in F^{\#}$,

$$\|(\delta_{jk}x)_{k\in\mathbb{N}_n}\| \leq 1$$

so that

$$\|\widetilde{a}\| \geq \|\widetilde{a}(\delta_{jk}x)_{k\in\mathbb{N}_n}\| = \left\| \left(\sum_{k=1}^{n} a_{ik}\delta_{jk}x \right)_{i\in\mathbb{N}_n} \right\| =$$

$$= \|(a_{ij}x)_{i\in\mathbb{N}}\| = \left\| \sum_{i=1}^{m} x^* a_{ij}^* a_{ij} x \right\|^{\frac{1}{2}} =$$

$$= \left\| x^* \left(\sum_{i=1}^{m} a_{ij}^* a_{ij} \right) x \right\|^{\frac{1}{2}} .$$

By our hypothesis on F,

$$\|\widetilde{a}\| \geq \left\| \sum_{i=1}^{m} a_{ij}^* a_{ij} \right\|^{\frac{1}{2}} = \|a_{\cdot,j}\| .$$

It follows

$$\|\widetilde{a}\| \geq \sup_{j\in\mathbb{N}_n} \|a_{\cdot,j}\| \geq \sup_{\substack{i\in\mathbb{N}_m \\ j\in\mathbb{N}_n}} \|a_{ij}\|$$

(Example 5.6.4.2 b)).

Now assume $F = E$ and take $x \in E$. Then for every $\xi \in E^n$,

$$\left(\sum_{j=1}^{n} a_{\cdot,j} \langle \cdot | x^* \delta_{j,\cdot} \rangle \right) \xi = \sum_{j=1}^{n} a_{\cdot,j} \langle \xi | x^* \delta_{j,\cdot} \rangle = \sum_{j=1}^{n} a_{\cdot,j} x \xi_j = \widetilde{ax}\xi$$

so that

$$\widetilde{ax} = \sum_{j=1}^{n} a_{\cdot j} \langle \cdot | x^* \delta_{j,\cdot} \rangle \in \mathcal{K}_E(E^n, E^m) .$$

By the above

$$\|\widetilde{ax} - \tilde{a}\| \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \|a_{ij}x - a_{ij}\|^2 \right)^{\frac{1}{2}}$$

and therefore

$$\tilde{a} = \lim_{x,\mathfrak{F}} \widetilde{ax} \in \mathcal{K}_E(E^n, E^m) ,$$

where $\mathfrak{F}$ denotes an approximate unit of E (Theorem 4.2.8.2).

b) follows from a).

c) Let $u \in \mathcal{L}_E(F^n, F^m)$. Take $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$ and put

$$u_{ij} : F \longrightarrow F , \quad x \longmapsto (u(\delta_{jk}x)_{k \in \mathbb{N}_n})_i ,$$

$$v_{ij} : F \longrightarrow F , \quad y \longmapsto (u^*(\delta_{\ell i}y)_{\ell \in \mathbb{N}_m})_j .$$

For $x, y \in F$,

$$\langle u_{ij}x | y \rangle = y^*(u(\delta_{jk}x)_{k \in \mathbb{N}_n})_i =$$

$$= \sum_{\ell=1}^{m} \delta_{\ell i} y^* (u(\delta_{jk}x)_{k \in \mathbb{N}_n})_\ell =$$

$$= \langle u(\delta_{jk}x)_{k \in \mathbb{N}_n} | (\delta_{\ell i}y)_{\ell \in \mathbb{N}_m} \rangle =$$

$$= \langle (\delta_{jk}x)_{k \in \mathbb{N}_n} | u^*(\delta_{\ell i}y)_{\ell \in \mathbb{N}_m} \rangle =$$

$$= \sum_{k=1}^{n} (u^*(\delta_{\ell i}y)_{\ell \in \mathbb{N}_m})_k^* \, \delta_{jk}x =$$

$$= (u^*(\delta_{\ell i}y)_{\ell \in \mathbb{N}_m})_j^* \, x = (v_{ij}y)^* x = \langle x | v_{ij}y \rangle .$$

Hence $u_{ij} \in \mathcal{L}_E(F)$ and so there is an $a_{ij} \in E$ with

$$u_{ij}x = a_{ij}x$$

for every $x \in F$.

Take $\xi \in F^n$. Then

$$\xi = \sum_{j=1}^{n} (\delta_{jk}\xi_j)_{k\in\mathbb{N}_n},$$

so that for $i \in \mathbb{N}_m$,

$$(u\xi)_i = \left(\sum_{j=1}^{n} u(\delta_{jk}\xi_j)_{k\in\mathbb{N}_n}\right)_i = \sum_{j=1}^{n} u_{ij}\xi_j = \sum_{j=1}^{n} a_{ij}\xi_j = (\tilde{a}\xi)_i.$$

Hence

$$u\xi = \tilde{a}\xi$$

and

$$u = \tilde{a}.$$

Thus the map

$$E_{m,n} \longrightarrow \mathcal{L}_E(F^n, F^m), \quad a \longmapsto \tilde{a}$$

is surjective.

Now assume $F = E$ and E unital. Take $u \in \mathcal{L}_E(E)$ and put

$$a := u1.$$

Then

$$\tilde{a}x = ax = (u1)x = ux$$

for every $x \in E$ (Proposition 5.6.1.8 d)), so that

$$\tilde{a} = u$$

and the map

$$E \longrightarrow \mathcal{L}_E(E), \quad a \longmapsto \tilde{a}$$

is surjective.

The relation

$$\mathcal{L}_E(E^n, E^m) = \mathcal{K}_E(E^n, E^m)$$

follows from the final assertion of a).

d) By b) and Proposition 5.6.1.8 b), $E_{m,n}$ is a Banach space. Take $x \in E$ and $a \in E_{m,n}$. Then for all $\xi \in F^n$,

$$\|\widetilde{xa}\xi\| = \|x\tilde{a}\xi\| \leq \|x\| \, \|\tilde{a}\xi\| \leq \|x\| \, \|\tilde{a}\| \, \|\xi\| ,$$

$$\|\widetilde{ax}\xi\| = \|\tilde{a}(x\xi)\| \leq \|\tilde{a}\| \, \|x\xi\| \leq \|\tilde{a}\| \, \|x\| \, \|\xi\|$$

(Example 5.6.4.2 c)), so that

$$\|\widetilde{xa}\| \leq \|x\| \, \|\tilde{a}\| , \quad \|\widetilde{ax}\| \leq \|\tilde{a}\| \, \|x\| .$$

The final assertion is easy to see.

e) Put $c := ab$. For $\xi \in E^p$,

$$\tilde{a}\tilde{b}\xi = \tilde{c}\xi = \left(\sum_{k=1}^{p} c_{ik}\xi_k \right)_{i \in \mathbb{N}_n} = \left(\sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij}b_{jk} \right) \xi_k \right)_{i \in \mathbb{N}_n} =$$

$$= \left(\sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{p} b_{jk}\xi_k \right) \right)_{i \in \mathbb{N}_n} = \left(\sum_{j=1}^{n} a_{ij}(\tilde{b}\xi)_j \right)_{i \in \mathbb{N}_n} = \tilde{a}\tilde{b}\xi$$

so that

$$\widetilde{ab} = \tilde{a} \circ \tilde{b} .$$

f) For $x, y \in E$ and $a, b \in E_{n,n}$,

$$(xy)a = x(ya) , \quad (xa)y = x(ay) , \quad (ax)y = a(xy) ,$$

$$(ab)x = a(bx) , \quad (ax)b = a(xb) , \quad (xa)b = x(ab) ,$$

$$(xa)^* = a^*x^* , \quad (ax)^* = x^*a^* , \quad \widetilde{xa} = x\tilde{a} , \quad \widetilde{ax} = \tilde{a}x ,$$

$$\|a^*\| = \|\tilde{a}^*\| = \|\tilde{a}^*\| = \|\tilde{a}\| = \|a\|$$

by a),b), and Proposition 5.6.1.8 b). Hence $E_{n,n}$ is an E-C^*-algebra and by c) and d), the mentioned map is an isomorphism of E-C^*-algebras.

g) Put

$$x := \sum_{i=1}^{n} a_{ii} .$$

Take $b \in E_{n,n}$ with

$$a = b^* b .$$

We obtain successively for all $i, j \in \mathbb{N}_n$,

$$a_{ij} = \sum_{k=1}^{n} (b^*)_{ik} b_{kj} = \sum_{k=1}^{n} b_{ki}^* b_{kj} ,$$

$$a_{ii} = \sum_{k=1}^{n} b_{ki}^* b_{ki} \in E_+ ,$$

$$b_{ji}^* b_{ji} \leq \sum_{k=1}^{n} b_{ki}^* b_{ki} = a_{ii} \leq x ,$$

$$\|b_{ji}\|^2 = \|b_{ji}^* b_{ji}\| \leq \|x\|$$

(Corollary 4.2.1.18),

$$\|a_{ij}\| \leq \sum_{k=1}^{n} \|b_{ki}\| \, \|b_{kj}\| \leq \left(\sum_{k=1}^{n} \|b_{ki}\|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \|b_{kj}\|^2 \right)^{\frac{1}{2}} \leq n\|x\| .$$

Hence by a),

$$\|\widetilde{a}\| \leq \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \|a_{ij}\| \right)^2 \right)^{\frac{1}{2}} \leq n^{\frac{5}{2}} \|x\| .$$

h) We may assume E unital. In this case the assertion follows from b),c),e),f), and Example 5.6.4.14.

i) Take $a \in E_{m,n}$ and denote by $\|\widetilde{a}\|_E$ the norm of a in the case $E = F$. By f), Proposition 5.6.1.8 f), and Corollary 4.1.1.21,

$$\|\widetilde{a}\|^2 = \|\widetilde{a}^* \circ \widetilde{a}\| = \|\widetilde{a}^* \circ \widetilde{a}\|_E = \|\widetilde{a}\|_E^2 ,$$

$$\|\widetilde{a}\| = \|\widetilde{a}\|_E .$$

j) is easy to see.

k) By c), there is an $a \in E_{m,n}$ with

$$\widetilde{a} = \eta \langle \cdot | \xi \rangle .$$

Put

$$e_j := [\delta_{jk}]_{k\in\mathbb{N}_n} \in E^n$$

for every $j \in \mathbb{N}_n$. Then for $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$,

$$(\widetilde{a}e_j)_i = \sum_{k=1}^{n} a_{ik}\delta_{jk} = a_{ij},$$

$$(\eta\langle e_j|\xi\rangle)_i = \eta_i\xi_j^*$$

so that

$$a = \left[\eta_i\xi_j^*\right]_{(i,j)\in\mathbb{N}_m\times\mathbb{N}_n}. \qquad \blacksquare$$

Corollary 5.6.6.2 (8) *If E is unital then*

$$\mathcal{L}_E(E^n, E^m) = \widehat{\mathcal{L}}_E(E^n, E^m)$$

for all $m, n \in \mathbb{N}$.

By Proposition 5.6.2.3, E as Hilbert right E-module is self-dual and so E^m and E^n are also self-dual (Theorem 5.6.4.7 a)). By Proposition 5.6.2.4,

$$\mathcal{L}_E(E^n, E^m) = \widehat{\mathcal{L}}_E(E^n, E^m). \qquad \blacksquare$$

Corollary 5.6.6.3 (0) *Let $n \in \mathbb{N}$.*

a) $\mathbb{H}_{n,n}$ *is isomorphic to* $F(\mathbb{C}_{2n,2n})$ *(Proposition 5.5.7.10).*

b) *The complexification of* $\mathbb{H}_{n,n}$ *is isomorphic to* $\mathbb{C}_{2n,2n}$.

The assertions follow from Proposition 5.6.4.18 b),c) and Theorem 5.6.6.1 f). $\blacksquare$

Remark. b) was already proved in Example 2.3.1.46.

Proposition 5.6.6.4 (0) *We use the notation of Theorem 5.6.6.1 with $F = E$ and define*

$$\widetilde{\xi'} : E^m \longrightarrow \mathbb{K}, \quad \xi \longmapsto \sum_{i=1}^{m}\langle\xi_i, \xi_i'\rangle$$

for every $\xi' \in E'^m$ and

$$\widetilde{a'} : E_{m,n} \longrightarrow \mathbb{K}, \quad a \longmapsto \sum_{i=1}^{m}\sum_{j=1}^{n}\langle a_{ij}, a'_{ji}\rangle$$

for every $a' \in E'_{n,m}$. Then for all $m, n \in \mathbb{N}$:

a) *If* $a' \in E'_{n,m}$, *then* $\widetilde{a'} \in (E_{m,n})'$, $\widetilde{a'}^* = \widetilde{a'^*}$, *and*

$$\sup_{j\in\mathbb{N}_n}\left(\sum_{i=1}^{m}\|a'_{ji}\|^2\right)^{\frac{1}{2}} \leq \sup_{j\in\mathbb{N}_n}\|\widetilde{a'_{j,\cdot}}\| \leq \|\widetilde{a'}\| \leq \sum_{j=1}^{n}\|\widetilde{a'_{j,\cdot}}\| \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\|a'_{ji}\|$$

(Proposition 5.6.4.3).

b) *The map*

$$E'_{n,m} \longrightarrow (E_{m,n})', \quad a' \longmapsto \widetilde{a'}$$

is linear and bijective.

c) $E'_{n,m}$ *endowed with the norm*

$$E'_{n,m} \longrightarrow \mathbb{R}_+, \quad a' \longmapsto \|\widetilde{a'}\|$$

and with the maps

$$E \times E'_{n,m} \longrightarrow E'_{n,m}, \quad (x, a') \longmapsto xa',$$

$$E'_{n,m} \times E \longrightarrow E'_{n,m}, \quad (a', x) \longmapsto a'x$$

(Definition 2.2.7.23) is an E*-module.*

d) *The map*

$$E'_{n,m} \longrightarrow (E_{m,n})', \quad a' \longmapsto \widetilde{a'}$$

is an isometry of E*-modules.*

a) Take $a \in E_{m,n}$. Then by Theorem 5.6.6.1 a),

$$|\widetilde{a'}(a)| = \left|\sum_{i=1}^{m}\sum_{j=1}^{n}\langle a_{ij}, a'_{ji}\rangle\right| \leq \sum_{j=1}^{n}\left|\sum_{i=1}^{m}\langle a_{ij}, a'_{ji}\rangle\right| =$$

$$= \sum_{j=1}^{n}|\langle a_{\cdot,j}, \widetilde{a'_{j,\cdot}}\rangle| \leq \sum_{j=1}^{n}\|a_{\cdot,j}\|\,\|\widetilde{a'_{j,\cdot}}\| \leq \|\widetilde{a}\|\sum_{j=1}^{n}\|\widetilde{a'_{j,\cdot}}\|,$$

so that $\widetilde{a'} \in (E_{m,n})'$ and

$$\|\widetilde{a'}\| \leq \sum_{j=1}^{n}\|\widetilde{a'_{j,\cdot}}\| \leq \sum_{j=1}^{n}\sum_{i=1}^{m}\|a'_{ji}\|$$

(Proposition 5.6.4.3). Let $k \in \mathbb{N}_n$ with

$$\|\widetilde{a'_{k,\cdot}}\| = \sup_{j \in \mathbb{N}_n} \|\widetilde{a'_{j,\cdot}}\| .$$

Take $\theta \in]0,1[$ and $x \in (E^m)^\#$ with

$$\widetilde{a'_{k,\cdot}}(x) \geq \theta \|\widetilde{a'_{k,\cdot}}\|$$

and define

$$a_{ij} = \delta_{jk} x_i$$

for all $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$. Then $\widetilde{a} \in E^\#_{m,n}$ (Theorem 5.6.6.1 a)) and

$$\|\widetilde{a'}\| \geq \widetilde{a'}(a) = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle a_{ij}, a'_{ji} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle \delta_{jk} x_i, a'_{ji} \rangle = \sum_{i=1}^{m} \langle x_i, a'_{ki} \rangle =$$

$$= \widetilde{a'_{k,\cdot}}(x) \geq \theta \|\widetilde{a'_{k,\cdot}}\| = \theta \sup_{j \in \mathbb{N}_n} \|\widetilde{a}_{j,\cdot}\| .$$

Since θ is arbitrary, we get

$$\|\widetilde{a'}\| \geq \sup_{j \in \mathbb{N}_n} \|\widetilde{a'_{j,\cdot}}\| \geq \sup_{j \in \mathbb{N}_n} \left(\sum_{i=1}^{m} \|a'_{ji}\|^2 \right)^{\frac{1}{2}}$$

(Proposition 5.6.4.3).

Given $a \in E_{n,m}$,

$$\widetilde{a'^*}(a) = \overline{\widetilde{a'}(a^*)} = \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{\langle (a^*)_{ij}, a'_{ji} \rangle} =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{\langle a^*_{ji}, a'_{ji} \rangle} = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle a_{ji}, a'^*_{ji} \rangle =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \langle a_{ji}, (a'^*)_{ij} \rangle = \widetilde{a'^*}(a) ,$$

so that

$$\widetilde{a'}^* = \widetilde{a'^*} .$$

b) follows from Theorem 5.6.6.1 a) and Proposition 2.1.4.25.

c) and d). By b),

$$E'_{n,m} \longrightarrow (E_{m,n})', \quad a' \longmapsto \widetilde{a'}$$

is an isomorphism of vector spaces. Take $x \in E$ and $a' \in E'_{n,m}$. Then for every $a \in E_{m,n}$,

$$\langle a, \widetilde{xa'} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle a_{ij}, xa'_{ji} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle a_{ij}x, a'_{ji} \rangle = \langle ax, \widetilde{a'} \rangle = \langle a, x\widetilde{a'} \rangle ,$$

$$\langle a, \widetilde{a'x} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle a_{ij}, a'_{ji}x \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle xa_{ij}, a'_{ji} \rangle = \langle xa, \widetilde{a'} \rangle = \langle a, \widetilde{a'}x \rangle ,$$

so that

$$\widetilde{xa'} = x\widetilde{a'}, \quad \widetilde{a'x} = \widetilde{a'}x ,$$

$$\|\widetilde{xa'}\| = \|x\widetilde{a'}\| \leq \|x\| \, \|\widetilde{a'}\| ,$$

$$\|\widetilde{a'x}\| = \|\widetilde{a'}x\| \leq \|\widetilde{a'}\| \, \|x\| . \qquad \blacksquare$$

Theorem 5.6.6.5 (0) *Let E be a W^*-algebra. Take $n \in \mathbb{N}$, endow $E_{n,n}$ with its canonical structure of a C^*-algebra (Theorem 5.6.6.1 f)) and put*

$$\widetilde{a} : E_{n,n} \longrightarrow \mathbb{K}, \quad x \longmapsto \sum_{i,j=1}^{n} \langle x_{ij}, a_{ji} \rangle$$

for every $a \in \ddot{E}_{n,n}$.

a) *$\widetilde{a} \in (E_{n,n})'$ for every $a \in \ddot{E}_{n,n}$, the map*

$$u : \ddot{E}_{n,n} \longrightarrow (E_{n,n})', \quad a \longmapsto \widetilde{a}$$

is injective and involutive, and $\operatorname{Im} u$ is an involutive $E_{n,n}$-submodule of $(E_{n,n})'$. We identify $\ddot{E}_{n,n}$ with $\operatorname{Im} u$.

b) *$(E_{n,n})^{\#}_{\operatorname{Im} u}$ is compact.*

c) *The map*

$$E_{n,n} \longrightarrow (\ddot{E}_{n,n})', \quad x \longmapsto \langle x, \cdot \rangle | \ddot{E}_{n,n}$$

is a isometry of Banach spaces. In particular, $E_{n,n}$ is a W^*-algebra with $\ddot{E}_{n,n}$ as predual.

d) *If we identify* $\ddot{E}_{n,n}$ *with the predual of* $E_{n,n}$ *(like in c)) and* $E_{n,n}$ *with* $\mathcal{L}_E(E^n)$ *(Theorem 5.6.6.1 c)) then for every* $a \in \ddot{E}$ *and* $\xi, \eta \in E^n$,

$$\widetilde{(a, \xi, \eta)} = [\xi_j a \eta_i^*]_{i,j \in \mathbb{N}_n}.$$

a) By Proposition 5.6.6.4 a), $\widetilde{a} \in (E_{n,n})'$ and u is injective. For every $x \in E_{n,n}$,

$$\widetilde{a}^*(x) = \overline{\widetilde{a}(x^*)} = \sum_{i,j=1}^{n} \overline{\langle (x^*)_{ij}, a_{ji} \rangle} =$$

$$= \sum_{i,j=1}^{n} \overline{\langle x_{ji}^*, a_{ji} \rangle} = \sum_{i,j=1}^{n} \langle x_{ji} a_{ji}^* \rangle = \sum_{i,j=1}^{n} \langle x_{ji}, (a)_{ij}^* \rangle = \widetilde{a^*}(x)$$

so that

$$\widetilde{a}^* = \widetilde{a^*}$$

and u is involutive.

Let $(a_m)_{m \in \mathbb{N}}$ be a sequence in $\ddot{E}_{n,n}$ such that $(\widetilde{a}_m)_{m \in \mathbb{N}}$ converges to an $x' \in (E_{n,n})'$. By Proposition 5.6.6.4 a),

$$\lim_{m \to \infty} (\widetilde{a}_m)_{ij} = x'_{ij}$$

so that $x'_{ij} \in \ddot{E}$ for all $i, j \in \mathbb{N}_n$. Hence $x' \in \operatorname{Im} u$ and $\operatorname{Im} u$ is closed in $(E_{n,n})'$. Now it can be easily seen that $\operatorname{Im} u$ is an involutive $E_{n,n}$-submodule of $(E_{n,n})'$.

b) Let $\mathfrak{F}$ be an ultrafilter on $(E_{n,n})^{\#}$. Put

$$\varphi_{ij} : E_{n,n} \longrightarrow E, \quad x \longmapsto x_{ij}$$

for all $i, j \in \mathbb{N}_n$. By Proposition 5.6.6.4 a), $\varphi_{ij}(\mathfrak{F})$ is an ultrafilter on $E^{\#}$. By the Alaoglu–Bourbaki Theorem, it converges in $E^{\#}_{\ddot{E}}$ to an element y_{ij}. Put

$$y := [y_{ij}]_{i,j \in \mathbb{N}_n} \in E_{n,n}.$$

By Proposition 5.6.6.4 a),

$$\widetilde{a}(y) = \sum_{i,j=1}^{n} \langle y_{ij}, a_{ji} \rangle = \lim_{x, \mathfrak{F}} \sum_{i,j=1}^{n} \langle \varphi_{ij}(x), a_{ji} \rangle = \lim_{x, \mathfrak{F}} \widetilde{a}(x)$$

for every $a \in \ddot{E}_{n,n}$. Hence $\mathfrak{F}$ converges to y in $(E_{n,n})_{\operatorname{Im} u}$.

We have to show that $y \in (E_{n,n})^{\#}$. Take $\xi \in \ell^2(\mathbb{N}, E)^{\#}$, $a \in \ddot{E}^{\#}$, and $u, v \in (E_{n,n})^{\#}$. Then $u\xi, v\xi \in \ell^2(\mathbb{N}_n, E)^{\#}$ so that

$$\langle u\xi | v\xi \rangle \in E^{\#}$$

(Proposition 5.6.1.2 d)). Thus

$$1 \geq |\langle \langle u\xi | v\xi \rangle, a \rangle| = \left| \left\langle \sum_{i=1}^{n} (v\xi)_i^* (u\xi)_i, a \right\rangle \right| =$$

$$= \left| \left\langle \sum_{i,j,k=1}^{n} \xi_j^* v_{ij}^* u_{ik} \xi_k, a \right\rangle \right| = \left| \sum_{i,j,k=1}^{n} \langle \xi_j^* v_{ij}^* u_{ik} \xi_k, a \rangle \right| =$$

$$= \left| \sum_{i,j,k=1}^{n} \langle v_{ij}^*, u_{ik} \xi_k a \xi_j^* \rangle \right| .$$

It follows

$$1 \geq \lim_{v,\mathfrak{F}} \left| \sum_{i,j,k=1}^{n} \langle v_{ij}^*, u_{ik} \xi_k a \xi_j^* \rangle \right| =$$

$$= \left| \sum_{i,j,k=1}^{n} \langle y_{ij}^*, u_{ik} \xi_k a \xi_j^* \rangle \right| = \left| \sum_{i,j,k=1}^{n} \langle u_{ik}, \xi_k a \xi_j^* y_{ij}^* \rangle \right| ,$$

$$1 \geq \lim_{u,\mathfrak{F}} \left| \sum_{i,j,k=1}^{n} \langle u_{ik}, \xi_k a \xi_j^* y_{ij}^* \rangle \right| = \left| \sum_{i,j,k=1}^{n} \langle y_{ik}, \xi_k a \xi_j^* y_{ij}^* \rangle \right| =$$

$$= \left| \sum_{i,j,k=1}^{n} \langle \xi_j^* y_{ij}^* y_{ik} \xi_k, a \rangle \right| = \left| \left\langle \sum_{i=1}^{n} (y\xi)_i^* (y\xi)_i, a \right\rangle \right| =$$

$$= |\langle \langle y\xi | y\xi \rangle, a \rangle| .$$

Since a is arbitrary,

$$\langle y\xi | y\xi \rangle \in E^{\#} ,$$

$$\|y\xi\|^2 = \|\langle y\xi | y\xi \rangle\| \leq 1 .$$

Since ξ is also arbitrary it follows $y \in (E_{n,n})^{\#}$.

c) follows from b) and Proposition 1.3.6.27.

d) For $x \in E_{n,n}$,

$$\left\langle x, \widetilde{(a,\xi,\eta)} \right\rangle = \langle \langle x\xi|\eta\rangle, a\rangle = \left\langle \sum_{i=1}^{n} \eta_i^* \left(\sum_{j=1}^{n} x_{ij}\xi_j \right), a \right\rangle =$$

$$= \sum_{i,j=1}^{n} \langle \eta_i^* x_{ij}\xi_j, a\rangle = \sum_{i,j=1}^{n} \langle x_{ij}, \xi_j a \eta_i^* \rangle . \quad \blacksquare$$

Corollary 5.6.6.6 *Let T be a nonempty compact hyperstonian space and $n \in \mathbb{N}$. For every $\mu \in (\mathcal{C}(T)^{\pi})_{n,n}$, put*

$$\widetilde{\mu} : \mathcal{C}(T)_{n,n} \longrightarrow \mathbb{K}, \quad x \longmapsto \sum_{i,j=1}^{n} \langle x_{ij}, \mu_{ji} \rangle .$$

Then $\mathcal{C}(T)_{n,n}$ is a W^*-algebra with

$$\{\widetilde{\mu} | \mu \in (\mathcal{C}(T)^{\pi})_{n,n}\}$$

as predual.

The assertion follows from Theorem 4.4.4.22 c $\Rightarrow$ a & d and Theorem 5.6.6.5 c). $\blacksquare$

Theorem 5.6.6.7 (0) *Take $n \in \mathbb{N}$ and consider the C^*-algebra $E_{n,n}$ (Theorem 5.6.6.1 f)).*

a) *If E is finite-dimensional then*

$$\operatorname{Dim}_{\operatorname{Re}} E_{n,n} = \frac{n(n-1)}{2} \operatorname{Dim} E + n \operatorname{Dim}_{\operatorname{Re}} E .$$

b) *E simple $\Rightarrow$ $E_{n,n}$ simple.*

c) *$E^c = \mathbb{K}1 \Rightarrow (E_{n,n})^c = \mathbb{K}1$.*

a) An element $a \in E_{n,n}$ is selfadjoint iff

$$a_{ij}^* = a_{ji}$$

for all $i, j \in \mathbb{N}_n$. It follows

$$\dim_{\operatorname{Re}} E_{n,n} = \frac{n(n-1)}{2} \dim E + n \dim_{\operatorname{Re}} E .$$

b) Let F be a closed ideal of $E_{n,n}$, $F \neq \{0\}$. Let $a \in F\backslash\{0\}$, $p, q \in \mathbb{N}_n$, and

$$G := \{x \in E | [\delta_{ip}\delta_{qj}x]_{i,j\in\mathbb{N}_n} \in F\} .$$

G is a closed ideal of E (Theorem 5.6.6.1 a)). Let $x, y \in E$ and $r, s \in \mathbb{N}_n$ such that $a_{rs} \neq 0$. Put

$$b := [\delta_{ip}\delta_{jr}x]_{i,j\in\mathbb{N}_n} , \quad c := [\delta_{is}\delta_{qj}y]_{i,j\in\mathbb{N}_n} .$$

Then $(ba)c \in F$ and

$$(ba)_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj} = \sum_{k=1}^{n} \delta_{ip}\delta_{kr}xa_{kj} = \delta_{ip}xa_{rj} ,$$

$$((ba)c)_{ij} = \sum_{k=1}^{n} (ba)_{ik}c_{kj} = \sum_{k=1}^{n} \delta_{ip}xa_{rk}\delta_{ks}\delta_{qj}y = \delta_{ip}\delta_{qj}xa_{rs}y ,$$

so that

$$xa_{rs}y \in G .$$

Since E is quasiunital (Theorem 4.2.8.2), we get

$$a_{rs} \in G .$$

Hence $G \neq \{0\}$ and therefore $G = E$. Thus

$$[\delta_{ip}\delta_{qj}x]_{i,j\in\mathbb{N}_n} \in F ,$$

for every $x \in E$. Since p and q are arbitrary, it follows $F = E_{n,n}$, and so $E_{n,n}$ is simple.

c) is easy to see. ■

Corollary 5.6.6.8 (0) *Take* $n, p \in \mathbb{N}$*:*

a) *The map*

$$(E_{n,n})_{p,p} \longrightarrow E_{np,np}$$

obtained by unbinding the matrices of $E_{n,n}$ *is an isomorphism of* E C^**-algebras.*

b) *If E is the direct sum (product) of the family $(F_\iota)_{\iota\in I}$ of C^* algebras, then the natural map of $E_{n,n}$ into the direct sum (product) of $((F_\iota)_{n,n})_{\iota\in I}$ is an isomorphism of E-C^*-algebras.*

c) *The map*

$$\overset{\circ}{\widehat{E_{n,n}}} \longrightarrow \overset{\circ}{E}_{n,n} \quad (x,y) \longmapsto x+iy$$

is an isomorphism of $\overset{\circ}{E}$-C^*-algebras.

d) *If E is a simple purely real C^*-algebra, then so is $E_{n,n}$.*

d) By Proposition 4.3.5.3 b $\Rightarrow$ a, $\overset{\circ}{E}$ is simple, so by Theorem 5.6.6.7 b), $E_{n,n}$ and $\overset{\circ}{E}_{n,n}$ are simple. By c), $\overset{\circ}{\widehat{E_{n,n}}}$ is simple, so by Proposition 4.3.5.3 a $\Rightarrow$ b, $E_{n,n}$ is a purely real C^*-algebra. ■

Corollary 5.6.6.9 (0) *Take $n\in\mathbb{N}$. $\mathbb{R}_{n,n}$ and $\mathbb{H}_{n,n}$ ($\mathbb{C}_{n,n}$) are simple purely real (real and complex) C^*-algebras and*

$$(\mathbb{R}_{n,n})^c=\mathbb{R}1\,,\quad (\mathbb{H}_{n,n})^c=\mathbb{R}1\,,\quad (\mathbb{C}_{n,n})^c=\mathbb{C}1\,,$$

$$\operatorname{Dim}\operatorname{Re}\mathbb{R}_{n,n}=\frac{n(n+1)}{2}\,,$$

$$\operatorname{Dim}\operatorname{Re}\mathbb{H}_{n,n}=2n^2-n\,.$$

In particular, $\mathbb{R}_{2n,2n}$ and $\mathbb{H}_{n,n}$ are not isomorphic.

$\mathbb{R}$,$\mathbb{C}$, and $\mathbb{H}$ are division algebras (Corollary 2.1.4.17), so they are simple C^*-algebras and we have

$$\mathbb{R}^c=\mathbb{R}\,,\quad \mathbb{H}^c=\mathbb{R}1\,,\quad \mathbb{C}^c=\mathbb{C}\,.$$

By Theorem 5.6.6.7, $\mathbb{R}_{n,n}$, $\mathbb{C}_{n,n}$, and $\mathbb{H}_{n,n}$ are simple and

$$(\mathbb{R}_{n,n})^c=\mathbb{R}1\,,\quad (\mathbb{H}_{n,n})^c=\mathbb{R}1\,,\quad (\mathbb{C}_{n,n})^c=\mathbb{C}1\,,$$

$$\operatorname{Dim}\operatorname{Re}\mathbb{R}_{n,n}=n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}\,,$$

$$\operatorname{Dim}\operatorname{Re}\mathbb{H}_{n,n}=n+4\frac{n(n-1)}{2}=2n^2-n\,.$$

In particular, $\mathbb{R}_{n,n}$ and $\mathbb{H}_{n,n}$ are purely real and $\mathbb{R}_{2n,2n}$ and $\mathbb{H}_{n,n}$ are not isomorphic. ■

Remark. Put

$$a := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathbb{R}_{2,2}\,.$$

Then

$$a^* = -a\,, \quad a^2 = -1\,,$$

so that $\mathbb{R}_{2,2}$ endowed with the maps

$$\mathbb{C} \times \mathbb{R}_{2,2} \longrightarrow \mathbb{R}_{2,2}\,, \quad ((\alpha + i\beta), x) \longmapsto (\alpha 1 + \beta a)x\,,$$

$$\mathbb{R}_{2,2} \times \mathbb{C} \longrightarrow \mathbb{R}_{2,2}\,, \quad (x, (\alpha + i\beta)) \longmapsto x(\alpha 1 + \beta a)$$

is a purely real $\mathbb{C}$–C^*-algebra.

Corollary 5.6.6.10 (0) *Take a function $\varphi : \mathbb{N} \to \mathbb{N} \cup \{0\}$ and let E be the C^*-direct product (sum) of the family $\left(\mathbb{C}_{n,n}^{\varphi(n)}\right)_{n\in\mathbb{N}}$ of complex C^*-algebras, where $\mathbb{C}_{n,n}^0 := \{0\}$, and F a real C^*-algebra such that $\overset{\circ}{F}$ and E are isomorphic. Then there are (uniquely) $f, g, h \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}$ such that h vanishes at the odd natural numbers,*

$$f + 2g + h = \varphi\,,$$

and F is isomorphic to the direct product (sum) of the family

$$\left(\mathbb{R}_{n,n}^{f(n)} \times \mathbb{C}_{n,n}^{g(n)} \times \mathbb{H}_{\frac{n}{2},\frac{n}{2}}^{h(n)}\right)_{n\in\mathbb{N}}$$

of real C^*-algebras.

By Proposition 4.3.5.6 (and Corollary 5.6.6.9), F is the C^*-direct product (sum) of a family $(F_n)_{n\in\mathbb{N}}$ of real C^*-algebras such that $\overset{\circ}{F}$ is isomorphic to $\mathbb{C}_{n,n}^{\varphi(n)}$ for every $n \in \mathbb{N}$. By Corollary 5.5.7.15 and Corollary 5.6.6.3 a), for every $n \in \mathbb{N}$, there are (uniquely defined)

$$f(n), g(n), h(n) \in \mathbb{N} \cup \{0\}$$

such that h vanishes if n is odd,

$$\mathbb{R}_{n,n}^{f(n)} \times \mathbb{C}_{n,n}^{g(n)} \times \mathbb{H}_{\frac{n}{2},\frac{n}{2}}^{h(n)}$$

is isomorphic to F_n, and

$$f(n) + 2g(n) + h(n) = \varphi(n) . \qquad \blacksquare$$

Proposition 5.6.6.11 (0) *Let $n, p \in \mathbb{N}$. The following are equivalent:*

a) $\frac{p}{2n} \in \mathbb{N}$.

b) *The real C^*-algebra $\mathbb{C}_{n,n}$ is isomorphic to a unital C^*-subalgebra of $\mathbb{R}_{p,p}$.*

c) *There is a unital algebra homomorphism $\mathbb{C}_{n,n} \to \mathbb{R}_{p,p}$.*

d) *$\mathbb{H}_{n,n}$ is isomorphic to a unital real C^*-subalgebra of $\mathbb{C}_{p,p}$.*

e) *There is a unital algebra homomorphism $\mathbb{H}_{n,n} \to \mathbb{C}_{p,p}$.*

a $\Rightarrow$b & d. $\mathbb{C}$ and $\mathbb{H}$ are isomorphic to a unital C^*-subalgebra of $\mathbb{R}_{2,2}$ and $\mathbb{C}_{2,2}$, respectively, so that $\mathbb{C}_{n,n}$ and $\mathbb{H}_{n,n}$ are isomorphic to a unital C^*-subalgebra of

$$(\mathbb{R}_{2,2})_{n,n} = \mathbb{R}_{2n,2n} \quad \text{and} \quad (\mathbb{C}_{2,2})_{n,n} = \mathbb{C}_{2n,2n} ,$$

respectively (Corollary 5.6.6.8 a)). By a), $\mathbb{K}_{2n,2n}$ is isomorphic to a unital C^*-subalgebra of $\mathbb{K}_{p,p}$. Hence $\mathbb{C}_{n,n}$ and $\mathbb{H}_{n,n}$ are isomorphic to a unital C^*-subalgebra of $\mathbb{R}_{p,p}$ and $\mathbb{C}_{p,p}$, respectively.

b $\Rightarrow$ c and d $\Rightarrow$ e are trivial.

c $\Rightarrow$ a (resp. e $\Rightarrow$ a). By Corollary 2.1.4.13 a),b), there is a family $(x_{ij})_{i,j \in \mathbb{N}_n}$ in $\mathbb{R}_{n,n}$ such that

$$x_{ij} x_{k\ell} = \delta_{jk} x_{i\ell}$$

for all $i, j, k, \ell \in \mathbb{N}_n$ and

$$\sum_{i=1}^{n} x_{ii} = 1 .$$

Let

$$u : \mathbb{C}_{n,n} \longrightarrow \mathbb{R}_{p,p} \qquad (\text{resp.} \quad u : \mathbb{H}_{n,n} \longrightarrow \mathbb{C}_{p,p})$$

be a unital algebra homomorphism. Put

$$y_{ij} := ux_{ij}$$

for all $i, j \in \mathbb{N}_n$. Then

$$y_{ij}y_{k\ell} = \delta_{jk}y_{i\ell}$$

for all $i, j, k, \ell \in \mathbb{N}_n$ and

$$\sum_{i=1}^{n} y_{ii} = 1 .$$

By Lemma 2.1.4.11,

$$\operatorname{Dim}\operatorname{Im} y_{ii} = \operatorname{Dim}\operatorname{Im} y_{jj}$$

for all $i, j \in \mathbb{N}_n$. It follows

$$\operatorname{Dim}\operatorname{Im} y_{ii} = \frac{p}{n}$$

for all $i \in \mathbb{N}_n$. In particular, we get a unital algebra homomorphism

$$\mathbb{C} \longrightarrow \mathbb{R}_{\frac{p}{n},\frac{p}{n}}, \qquad (\text{resp.} \quad \mathbb{H} \longrightarrow \mathbb{C}_{\frac{p}{n},\frac{p}{n}}) .$$

By Proposition 2.1.4.26,

$$\frac{p}{2n} \in \mathbb{N} . \qquad \blacksquare$$

Remark. The set

$$\left\{ \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

is a unital C^*-subalgebra of $\mathbb{R}_{2,2}$, which is isomorphic to $\mathbb{C}$. The set

$$\left\{ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{C} \right\}$$

si a unital C^*-subalgebra of $\mathbb{C}_{2,2}$, which is isomorphic to $\mathbb{H}$.

Corollary 5.6.6.12 (0) *Let $n, p \in \mathbb{N}$. The following are equivalent:*

a) $\frac{p}{n} \in \mathbb{N}$.

b) *The $\mathbb{H}$-C^*-algebra $\mathbb{H}_{n,n}$ is isomorphic to a unital $\mathbb{H}$-C^* subalgebra of $\mathbb{H}_{p,p}$.*

c) *There is a unital algebra homomorphism* $\mathbb{H}_{n,n} \to \mathbb{H}_{p,p}$.

a $\Rightarrow$ b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. By Proposition 5.6.6.11 a $\Rightarrow$ e, there is a unital algebra homomorphism $\mathbb{H}_{n,n} \to \mathbb{C}_{2p,2p}$. By Proposition 5.6.6.11 e $\Rightarrow$ a,

$$\frac{p}{n} = \frac{2p}{2n} \in \mathbb{N}. \qquad \blacksquare$$

Remark. Of course, the above corollary can be proved directly like Corollary 2.1.4.13 c). In fact, it holds for an arbitrary finite dimensional C^*-algebra (Proposition 6.3.6.19), but it fails in the infinite-dimensional case (Corollary 5.6.4.17).

Proposition 5.6.6.13 (0) *Let* $n, p \in \mathbb{N}$. *The following are equivalent:*

a) $\frac{p}{4n} \in \mathbb{N}$.

b) $\mathbb{H}_{n,n}$ *is isomorphic to a unital* C^**-subalgebra of* $\mathbb{R}_{p,p}$.

c) *There is a unital algebra homomorphism* $\mathbb{H}_{n,n} \to \mathbb{R}_{p,p}$.

a $\Rightarrow$ b. H is isomorphic to a unital C^*-subalgebra of $\mathbb{C}_{2,2}$. By Proposition 5.6.6.11 a $\Rightarrow$ b, $\mathbb{H}$ is isomorphic to a unital C^*-subalgebra of $\mathbb{R}_{4,4}$. Hence $\mathbb{H}_{n,n}$ is isomorphic to a unital C^* subalgebra of

$$(\mathbb{R}_{4,4})_{n,n} = \mathbb{R}_{4n,4n}$$

(Corollary 5.6.6.8 a)). By a), $\mathbb{R}_{4n,4n}$ is isomorphic to a unital C^* subalgebra of $\mathbb{R}_{p,p}$. Thus $\mathbb{H}_{n,n}$ is isomorphic to a unital C^*-subalgebra of $\mathbb{R}_{p,p}$.

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. By Corollary 2.1.4.13 a),b), there is a family $(x_{ij})_{i,j\in\mathbb{N}_n}$ in $\mathbb{R}_{n,n}$ such that

$$x_{ij}x_{k\ell} = \delta_{jk}x_{i\ell}$$

for all $i, j, k, \ell \in \mathbb{N}_n$ and

$$\sum_{i=1}^{n} x_{ii} = 1.$$

Let

$$u : \mathbb{H}_{n,n} \longrightarrow \mathbb{R}_{p,p}$$

be a unital algebra homomorphism. Put

$$y_{ij} := ux_{ij}$$

for all $i, j \in \mathbb{N}_n$. Then

$$y_{ij}y_{k\ell} = \delta_{jk}y_{i\ell}$$

for all $i, j, k, \ell \in \mathbb{N}_n$ and

$$\sum_{i=1}^{n} y_{ii} = 1 .$$

By Lemma 2.1.4.11,

$$\operatorname{Dim}\operatorname{Im} y_{ii} = \operatorname{Dim}\operatorname{Im} y_{jj}$$

for all $i, j \in \mathbb{N}_n$. It follows

$$\operatorname{Dim}\operatorname{Im} y_{ii} = \frac{p}{n}$$

for all $i \in \mathbb{N}_n$. In particular we get a unital algebra homomorphism

$$\mathbb{H} \longrightarrow \mathbb{R}_{\frac{p}{n},\frac{p}{n}} .$$

By Proposition 2.1.4.26,

$$\frac{p}{4n} \in \mathbb{N} . \qquad \blacksquare$$

Remark. The set

$$\left\{ \begin{bmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & -\delta & \gamma \\ \gamma & \delta & \alpha & -\beta \\ \delta & -\gamma & \beta & \alpha \end{bmatrix} \middle| \; \alpha, \beta, \gamma, \delta \in \mathbb{R} \right\}$$

is a unital C^*-subalgebra of $\mathbb{R}_{4,4}$, which is isomorphic to $\mathbb{H}$.

Proposition 5.6.6.14 (0) *Take $n \in \mathbb{N}$. We recursively define*

$$E^{(0)} := E , \quad E^{(p)} := (E^{(p-1)})'$$

for every $p \in \mathbb{N}$ and put

$$\widetilde{a} : E_{n,n}^{(p-1)} \longrightarrow \mathbb{K} , \quad b \longmapsto \sum_{i,j=1}^{n} \langle b_{ij}, a_{ji} \rangle ,$$

$$xa := \left(\sum_{k=1}^{n} x_{ik}a_{kj}\right)_{i,j\in\mathbb{N}_n}, \quad ax := \left(\sum_{k=1}^{n} a_{ik}x_{kj}\right)_{i,j\in\mathbb{N}_n}$$

for all $p \in \mathbb{N}$, $a \in E_{n,n}^{(p)}$, *and* $x \in E_{n,n}$ *and endow* $E_{n,n}^{(p)}$ *with the norm*

$$E_{n,n}^{(p)} \longrightarrow \mathbb{R}_+, \quad a \longmapsto \|\widetilde{a}\|$$

defined inductively for all $p \in \mathbb{N}$ *and with the maps*

$$E_{n,n} \times E_{n,n}^{(p)} \longrightarrow E_{n,n}^{(p)}, \quad (x,a) \longmapsto xa,$$

$$E_{n,n}^{(p)} \times E_{n,n} \longrightarrow E_{n,n}^{(p)}, \quad (a,x) \longmapsto ax.$$

Then, given $p \in \mathbb{N}$:

a) $a \in E_{n,n}^{(p)} \Rightarrow \sup\limits_{i,j\in\mathbb{N}_n} \|a_{ij}\| \leq \|a\| \leq \sum\limits_{i,j=1}^{n} \|a_{ij}\|$.

b) $E_{n,n}^{(p)}$ *is an involutive* $E_{n,n}$*-module (Theorem 5.6.6.1 f)) and the map*

$$E_{n,n}^{(p)} \longrightarrow \left(E_{n,n}^{(p-1)}\right)', \quad a \longmapsto \widetilde{a}$$

is an isometry of involutive $E_{n,n}$*-modules.*

By Proposition 2.3.6.7 and Theorem 5.6.6.1, a) holds, $E_{n,n}^{(p)}$ is an involutive E–module, and the map

$$E_{n,n}^{(p)} \longrightarrow \left(E_{n,n}^{(p-1)}\right)', \quad a \longmapsto \widetilde{a}$$

is an isometry of involutive E–modules.

Take $x \in E_{n,n}$ and $a \in E_{n,n}^{(p)}$. Then for every $b \in E_{n,n}^{(p-1)}$,

$$\langle b, \widetilde{xa}\rangle = \sum_{i,j=1}^{n} \langle b_{ij}, (xa)_{ji}\rangle = \sum_{i,j=1}^{n} \langle b_{ij}, \sum_{k=1}^{n} x_{jk}a_{ki}\rangle =$$

$$= \sum_{i,j,k=1}^{n} \langle b_{ij}, x_{jk}a_{ki}\rangle = \sum_{i,j,k=1}^{n} \langle b_{ij}x_{jk}, a_{ki}\rangle =$$

$$= \sum_{i,k=1}^{n} \left\langle \sum_{j=1}^{n} b_{ij}x_{jk}, a_{ki} \right\rangle = \sum_{i,k=1}^{n} \langle (bx)_{ik}, a_{ki}\rangle = \langle bx, \widetilde{a}\rangle = \langle b, x\widetilde{a}\rangle,$$

so that

$$\widetilde{xa} = x\widetilde{a}\,,$$

$$\|xa\| = \|\widetilde{xa}\| = \|x\widetilde{a}\| \leq \|x\|\,\|\widetilde{a}\| = \|x\|\,\|a\|\,.$$

The relations

$$\widetilde{ax} = \widetilde{a}x\,, \quad \|ax\| \leq \|a\|\,\|x\|$$

are proved similarly.

The other assertions are easy to check. ■

Proposition 5.6.6.15 *Let T be a locally compact space and $n \in \mathbb{N}$. We identify as usual $\mathcal{M}_b(T)$, the involutive Banach space of bounded Radon measures on T with $\mathcal{C}_0(T)'$ and then $\mathcal{M}_b(T)_{n,n}$ with $(\mathcal{C}_0(T)_{n,n})'$ as in Proposition 5.6.6.4 a),b). Take $\mu \in \mathcal{M}_b(T)_{n,n}$, put*

$$\nu := \sum_{i,j=1}^{n} |\mu_{ij}|\,,$$

and let $a \in L^\infty(\nu)_{n,n}$ such that

$$\mu_{ij} = a_{ij}\cdot\nu$$

for all $i,j \in \mathbb{N}_n$ Then the following are equivalent:

a) $\mu \geq 0$.

b) $a \geq 0$.

a $\Rightarrow$ b. For $i,j \in \mathbb{N}_n$,

$$\bar{a}_{ij}\cdot\nu = \overline{\mu}_{ij} = \mu_{ji} = a_{ji}\cdot\nu$$

$$\bar{a}_{ij} = a_{ji}$$

so that

$$a \in \operatorname{Re} L^\infty(\nu)_{n,n}\,.$$

Let $\mathfrak{F}$ be a filter on $(\mathcal{C}_0(T)_{n,n})_+$ converging to a^- in $(L^\infty(\nu)_{n,n})_{L^1(\nu)_{n,n}}$. By Theorem 5.6.6.1 g), it follows successively

$$0 \leq \langle a^-, \mu\rangle = \sum_{i,j=1}^{n} \langle a_{ij}^-, \mu_{ji}\rangle = \sum_{i,j=1}^{n} \int a_{ij}^- a_{ji}\, d\nu =$$

$$= \int \sum_{i=1}^{n} (a^- a)_{ii} d\nu) = -\int \sum_{i=1}^{n} (a^-)_{ii}^2 d\nu \leq 0\,,$$

$$i \in \mathbb{N}_n \Longrightarrow a_{ii}^- = 0\,,$$

$$a^- = 0\,,$$

$$a \geq 0\,.$$

b $\Rightarrow$ a. There is a $b \in L^\infty(\nu)_{n,n}$ such that

$$a = b^* b\,.$$

Then

$$a_{ij} = \sum_{k=1}^{n} (b^*)_{ik} b_{kj} = \sum_{k=1}^{n} \overline{b}_{ki} b_{kj}$$

for all $i, j \in \mathbb{N}_n$. Take $x \in \mathcal{C}_0(T)_{n,n}$ Then

$$\langle x^* x, \mu \rangle = \sum_{i,j=1}^{n} \langle (x^* x)_{ij}, \mu_{ji} \rangle = \sum_{i,j,k=1}^{n} \langle \overline{x}_{ki} x_{kj}, \mu_{ji} \rangle =$$

$$= \int \left(\sum_{i,j,k=1}^{n} \overline{x}_{ki} x_{kj} a_{ji} \right) d\nu = \int \left(\sum_{i,j,k,\ell=1}^{n} \overline{x}_{ki} x_{kj} \overline{b}_{\ell j} b_{\ell i} \right) d\nu =$$

$$= \int \sum_{k,\ell=1}^{n} \left(\sum_{i=1}^{n} \overline{x}_{ki} b_{\ell i} \right) \left(\sum_{j=1}^{n} x_{kj} \overline{b}_{\ell i} \right) d\nu =$$

$$= \int \sum_{k,\ell=1}^{n} \left| \sum_{i=1} \overline{x}_{ki} b_{\ell i} \right|^2 d\nu \geq 0\,.$$

Thus $\mu \geq 0$. ■

Proposition 5.6.6.16 *Let $n \in \mathbb{N}$. For every $x \in \mathbb{K}_{n,n}$ put*

$$\widetilde{x} : \mathbb{K}_{n,n} \longrightarrow \mathbb{K}\,, \quad y \longmapsto \sum_{i,j=1}^{n} y_{ij} x_{ji}$$

(by Example 4.2.3.2, $\widetilde{x} \in (\mathbb{K}_{n,n})'$). Take $x \in \mathbb{K}_{n,n}$.

a) *For every* $p \in \Pr \mathbb{K}_{n,n}$, p *is the right carrier of* x *iff* p *is the right carrier of* $\widetilde{x}$.

b) *For every* $y \in \mathbb{K}_{n,n}$, $(y, |x|)$ *is the polar representation of* x *iff* $(y, \widetilde{|x|})$ *is the polar representation of* $\widetilde{x}$.

c) $|\widetilde{x}| = \widetilde{|x|}$.

a) By Proposition 5.6.6.14 b),

$$xp = x \Longleftrightarrow \widetilde{x}p = \widetilde{x}$$

$$xy = 0 \Longleftrightarrow x\widetilde{y} = 0\,,$$

$$py = 0 \Longleftrightarrow p\widetilde{y} = 0$$

for every $y \in \mathbb{K}_{n,n}$. The assertion follows immediately from this remark.

b) First assume that $(y, |x|)$ is the polar representation of x. By Proposition 4.2.3.2 a $\Rightarrow$ b, $\widetilde{|x|} \in (\mathbb{K}_{n,n})'_+$. By Proposition 5.6.6.14 b)),

$$y\widetilde{|x|} = \widetilde{y|x|} = \widetilde{x}\,.$$

$$y^*y\widetilde{|x|} = \widetilde{y^*y|x|} = \widetilde{|x|}\,.$$

Take $z \in \mathbb{K}_{n,n}$ with

$$\widetilde{|x|}z = 0\,.$$

By Proposition 5.6.6.14 b),

$$|x|z = 0$$

so that

$$y^*yz = 0\,.$$

Hence y^*y is the carrier of $\widetilde{|x|}$. Thus $(y, \widetilde{|x|})$ is the polar representation of $\widetilde{x}$.

Now assume that $(y, \widetilde{|x|})$ is the polar representation of $\widetilde{x}$. By Proposition 5.6.6.14 b) and Theorem 4.4.3.5 b),

$$\widetilde{x} = y\widetilde{|x|} = \widetilde{y|x|}\,, \quad x = y|x|\,,$$

$$\widetilde{|x|} = y^*y\widetilde{|x|} = \widetilde{y^*y|x|}\,, \quad |x| = y^*y|x|\,,$$

$$\widetilde{x} = \widetilde{x}y^*y = \widetilde{xy^*y}\,, \quad x = xy^*y\,,$$

$$\widetilde{|x^*|} = yy^*\widetilde{|x^*|} = \widetilde{yy^*|x^*|}\,, \quad |x^*| = yy^*|x^*|\,,$$

$$\widetilde{x} = yy^*\widetilde{x} = \widetilde{yy^*x}\,, \quad x = yy^*x\,.$$

Take $z \in \mathbb{K}_{n,n}$ with

$$1)\ z|x| = 0\,, \quad 2)\ zx = 0 \qquad (\,1)\ |x^*|z = 0\,, \quad 2)\ xz = 0\,)\,.$$

By Proposition 5.6.6.14 b) and Theorem 4.4.3.5 b),

$$1)\ z\widetilde{|x|} = 0\,, \quad 2)\ z\widetilde{x} = 0 \qquad (\,1)\ \widetilde{|x^*|}z = 0\,, \quad 2)\ \widetilde{x}z = 0\,)$$

Hence $(y, |x|)$ is the polar representation of x.

c) follows from b). ■

Remark. By the above proposition and by the remark of Theorem 4.2.5.6, the relation

$$|\mathrm{re}\, a| \leq |a|$$

does not hold and there is no $\alpha \in \mathbb{R}_+$ such that the relation

$$a, b \in E \Longrightarrow |a + b| \leq \alpha(|a| + |b|)$$

is true for all W^*-algebras E.

Proposition 5.6.6.17 (8) *Take $n \in \mathbb{N}$ and identify E canonically with the C^*-subalgebra $\{[x\delta_{ij}]_{i,j\in\mathbb{N}_n} | x \in E\}$ of $E_{n,n}$.*

a) *Put*

$$\varphi : E_{n,n} \longrightarrow E\,, \quad x \longmapsto \sum_{j=1}^{n} x_{jj}\,.$$

Then φ is an involutive continuous linear map such that:

1) $(x, y) \in E_{n,n} \times E \Rightarrow \varphi(xy) = (\varphi x)y\,,\ \varphi(yx) = y\varphi x\,,$

2) $x \in (E_{n,n})_+ \Rightarrow \varphi x \in E_+\,,$

3) $(x,y) \in \left(\overset{\circ}{\widehat{E_{n,n}}}\right)_+ \Rightarrow (\varphi x, \varphi y) \in \overset{\circ}{E}_+$,

4) $\inf\{\|\varphi x\| | x \in (E_{n,n})_+ , \|x\| = 1\} \geq n^{-\frac{5}{2}}$.

b) *If H is a Hilbert right $E_{n,n}$-module then H endowed with the right multiplication*

$$H \times E \longrightarrow H , \quad (\xi, x) \longmapsto \xi x$$

and with the inner-product

$$H \times H \longrightarrow E , \quad (\xi, \eta) \longmapsto \sum_{i=1}^{n} \langle \xi | \eta \rangle_{ii}$$

is a Hilbert right E-module, denoted by $\widetilde{H}$, and $\mathcal{L}_{E_{n,n}}(H)$ is a unital C^*-subalgebra of $\mathcal{L}_E(\widetilde{H})$.

c) *If G and H are Hilbert right $E_{n,n}$-modules then*

$$\mathcal{L}_{E_{n,n}}(G,H) = \{u \in \mathcal{L}_E(\widetilde{G}, \widetilde{H}) | (\xi, a) \in G \times E_{n,n} \Longrightarrow u(\xi a) = (u\xi)a\} .$$

d) *If all Hilbert right E-modules are self-dual (reflexive) then all Hilbert right $E_{n,n}$-modules are self-dual (reflexive).*

a) 1) is obvious. 2) and 4) follow from Theorem 5.6.6.1 g). Take $(x,y) \in (\overset{\circ}{\widehat{E_{n,n}}})_+$. There is an $(a,b) \in E_{n,n}$ such that

$$(x,y) = (a,b)^*(a,b) = (a^*a + b^*b , a^*b - b^*a) .$$

Then

$$x_{jj} = \sum_{k=1}^{n} (a_{kj}^* a_{kj} + b_{kj}^* b_{kj}) ,$$

$$y_{jj} = \sum_{k=1}^{n} (a_{kj}^* b_{kj} - b_{kj}^* a_{kj})$$

for every $j \in \mathbb{N}_n$ so that

$$\varphi x = \sum_{j,k=1}^{n} (a_{kj}^* a_{kj} + b_{kj}^* b_{kj}) ,$$

$$\varphi y = \sum_{j,k=1}^{n} (a_{kj}^* b_{kj} - b_{kj}^* a_{kj}) ,$$

$$(\varphi x, \varphi y) = \sum_{j,k=1}^{n} (a_{kj}^* a_{kj} + b_{kj}^* b_{kj}, a_{kj}^* b_{kj} - b_{kj}^* a_{kj}) =$$

$$= \sum_{j,k=1}^{n} (a_{kj}, b_{kj})^* (a_{kj}, b_{kj}) \in \overset{\circ}{E}_+ .$$

This proves 3).

b),c), and d) follow from a) and Proposition 5.6.2.5 a),b),c),d). ■

Proposition 5.6.6.18 (8) *We use the notation of Proposition 5.6.6.17 and assume* $E \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. *Moreover, we identify* $E_{n,n}$ *with* $\mathcal{L}_E(E^n)$ *(Theorem 5.6.6.1 c)). Take* $\xi \in E^n$ *with* $\|\xi\| = 1$ *and put*

$$p := \xi\langle\cdot|\xi\rangle .$$

a) p *is a minimal element of* $\Pr E_{n,n} \backslash \{0\}$.

b) *An element* $a \in E_{n,n}$ *belongs to* $pE_{n,n}$ *iff there is an* $\eta \in E^n$ *such that*

$$a = \xi\langle\cdot|\eta\rangle .$$

c) $\{\xi\langle\cdot|e_i\rangle | i \in \mathbb{N}_n\}$ *is a Fourier basis of the Hilbert right* E*-module* $\widetilde{pE_{n,n}}$.

d) *For every* $\xi \in E^n$ *there is an* $x \in \operatorname{Un} E_{n,n}$ *with*

$$(x\xi)_t = 0$$

for every $t \in \mathbb{N}_{n-1}$.

a) By Proposition 5.6.5.2 a),f), $p \in \Pr E_{n,n} \backslash \{0\}$. Let $q \in \Pr E_{n,n}$ with $0 < q \leq p$. By Proposition 5.6.5.2 d),e) and Corollary 4.2.7.6 a $\Rightarrow$ c & d,

$$q\xi\langle\cdot|\xi\rangle = q \circ (\xi\langle\cdot|\xi\rangle) = q = (\xi\langle\cdot|\xi\rangle) \circ q = \xi\langle\cdot|q\xi\rangle .$$

By Proposition 5.6.5.2 l),

$$q\xi = \xi\langle q\xi|\xi\rangle .$$

Since

$$\langle q\xi|\xi\rangle = \langle q^*q\xi|\xi\rangle = \langle q\xi|q\xi\rangle \in \mathbb{R}_+$$

we obtain successively

$$q = \langle q\xi|\xi\rangle p\,,$$

$$\langle q\xi|\xi\rangle = 1\,,$$

$$q = p\,.$$

Hence p is a minimal element of $\Pr E_{n,n}\backslash\{0\}$.

b) For $\eta \in E^n$,

$$\xi\langle\cdot|\eta\rangle = (p\xi)\langle\cdot|\eta\rangle = p\circ(\xi\langle\cdot|\eta\rangle) \in pE_{n,n}\,.$$

Now assume $a \in pE_{n,n}$. Then

$$a = pa = (\xi\langle\cdot|\xi\rangle)\circ a = \xi\langle\cdot|a^*\xi\rangle$$

(Proposition 5.6.5.2 e)).

c) By b), $\{\xi\langle\cdot|e_i\rangle|i \in \mathbb{N}_n\} \subset pE_{n,n}$. For $i,j \in \mathbb{N}_n$,

$$\varphi(\langle\xi\langle\cdot|e_i\rangle|\xi\langle\cdot|e_j\rangle\rangle) = \varphi((e_j\langle\cdot|\xi\rangle)\circ(\xi\langle\cdot|e_i\rangle)) =$$

$$= \varphi(e_j\langle\cdot|e_i\rangle) = \delta_{ij}$$

(Proposition 5.6.5.2 a),f), Theorem 5.6.6.1 k)). Hence $\{\xi\langle\cdot|e_i\rangle|i \in \mathbb{N}_n\}$ is a Fourier set of $\widehat{pE_{n,n}}$.

Take $a \in pE_{n,n}$. By b), there is an $\eta \in E^n$ such that

$$a = \xi\langle\cdot|\eta\rangle\,.$$

For every $i \in \mathbb{N}_n$,

$$\varphi(\langle a|\xi\langle\cdot|e_i\rangle\rangle) = \varphi((e_i\langle\cdot|\xi\rangle)\circ(\xi\langle\cdot|\eta\rangle)) =$$

$$\varphi(e_i\langle\cdot|\eta\rangle) = \varphi\left[\delta_{ij}\eta_k^*\right]_{j,k\in\mathbb{N}_n} = \sum_{j=1}^{n}\delta_{ij}\eta_j^* = \eta_i^*$$

(Proposition 5.6.5.2 a),f), Theorem 5.6.6.1 k)). Thus if

$$\varphi(\langle a|\xi\langle\cdot|e_i\rangle\rangle) = 0$$

for every $i \in \mathbb{N}_n$ then $a = 0$. It follows that $\{\xi\langle\cdot|e_i\rangle | i \in \mathbb{N}_n\}$ is a Fourier basis of $\widetilde{pE_{n,n}}$.

d) Put

$$\eta : \mathbb{N}_n \longrightarrow E\,, \quad t \longmapsto \begin{cases} \frac{\xi_t^*}{|\xi_t|} & \text{if } \xi_t \neq 0 \\ 1 & \text{if } \xi_t = 0\,, \end{cases}$$

$$y := [\delta_{st}\eta_t]_{s,t\in\mathbb{N}_n} \in E_{n,n}\,.$$

Then $y \in \operatorname{Un} E_{n,n}$ and $y\xi \in \mathbb{N}^n$. Hence there is a $z \in \operatorname{Un} \mathbb{R}_{n,n}$ with

$$(zy\xi)_t = 0$$

for every $t \in \mathbb{N}_{n-1}$. Then $x := zy$ has the desired properties. ■

Proposition 5.6.6.19 (8) *Assume E unital and take $n \in \mathbb{N}$. Let G, H be Hilbert right $E_{n,n}$-modules and $\xi \in G$, $\eta \in H$, $\zeta \in E^n$ such that*

$$\langle\zeta|\zeta\rangle = 1\,,$$

$$\langle\xi|\xi\rangle = \langle\eta|\eta\rangle = \zeta\langle\cdot|\zeta\rangle \in \operatorname{Pr} E_{n,n}\,.$$

Put

$$\varphi : E_{n,n} \longrightarrow E\,, \quad x \longmapsto \sum_{i=1}^{n} x_{ii}$$

and

$$x_i := \zeta\langle\cdot|e_i\rangle \in E_{n,n}\,,$$

$$\xi_i := \xi x_i\,, \quad \zeta_i := \eta x_i$$

for every $i \in \mathbb{N}_n$.

a) $i \in \mathbb{N}_n \Rightarrow \langle\xi|\xi\rangle x_i = x_i$.

b) $i, j \in \mathbb{N}_n \Rightarrow \varphi\langle\xi_i|\xi_j\rangle = \varphi\langle\eta_i|\eta_j\rangle = \delta_{ij}$.

c) $\zeta_1, \zeta_2 \in E^n \Rightarrow \varphi(\zeta_1\langle\cdot|\zeta_2\rangle) = \langle\zeta_2^*|\zeta_1^*\rangle$.

d) $\eta\langle\cdot|\xi\rangle = \sum_{i=1}^{n} \eta_i\varphi\langle\cdot|\xi_i\rangle$.

e) *If* $\mathbb{K} = \mathbb{R}$ *and* $E = \mathbb{C}$ $(E = \mathbb{H})$ *then*

$$\eta\langle\cdot|\xi\rangle = \sum_{\ell=1}^{n}(\eta_\ell \mathrm{re}\,\varphi\langle\cdot|\xi_\ell\rangle + \eta_\ell i\mathrm{re}\,\varphi\langle\cdot|\xi_\ell i\rangle)$$

$$\left(\eta\langle\cdot|\xi\rangle = \sum_{\ell=1}^{n}\left(\begin{array}{l}\eta_\ell \mathrm{re}\,\varphi\langle\cdot|\xi_\ell\rangle + \eta_\ell i\mathrm{re}\,\varphi\langle\cdot|\xi_\ell i\rangle + \\ +\eta_\ell j\mathrm{re}\,\varphi\langle\cdot|\xi_\ell j\rangle + \eta_\ell k\mathrm{re}\,\varphi\langle\cdot|\xi_\ell k\rangle\end{array}\right)\right)$$

a) $\langle\xi|\xi\rangle x_i = (\zeta\langle\cdot|\zeta\rangle)\circ(\zeta\langle\cdot|e_i\rangle) =$

$= \zeta\langle\zeta|\zeta\rangle\langle\cdot|e_i\rangle = \zeta\langle\cdot|e_i\rangle = x_i$

(Proposition 5.6.5.2 f)).

b) By a),

$$\varphi\langle\xi_i|\xi_j\rangle = \varphi\langle\xi x_i|\xi x_j\rangle = \varphi(x_j^*\langle\xi|\xi\rangle x_i) =$$

$$= \varphi(x_j^* x_i) = \varphi((e_j\langle\cdot|\zeta\rangle)\circ(\zeta\langle\cdot|e_i\rangle)) =$$

$$= \varphi(e_j\langle\zeta|\zeta\rangle\langle\cdot|e_i\rangle) = \varphi(e_j\langle\cdot|e_i\rangle) = \delta_{ij}$$

(Proposition 5.6.1.2 a), Proposition 5.6.5.2 f), Theorem 5.6.6.1 k)),

$$\varphi\langle\eta_i|\eta_j\rangle = \delta_{ij}\,.$$

c) By Theorem 5.6.6.1 k),

$$\varphi(\zeta_1\langle\cdot|\zeta_2\rangle) = \varphi\left(\left[\zeta_{1i}\zeta_{2j}^*\right]_{i,j\in\mathbb{N}_n}\right) =$$

$$= \sum_{i=1}^{n}\zeta_{1i}\zeta_{2i}^* = \langle\zeta_2^*|\zeta_1^*\rangle\,.$$

d) For $\xi' \in G$, by c),

$$\sum_{i=1}^{n} x_i\varphi\langle\xi'|\xi_i\rangle = \sum_{i=1}^{n} x_i\varphi(x_i^*\langle\xi'|\xi\rangle) =$$

$$= \sum_{i=1}^{n} x_i\varphi((e_i\langle\cdot|\zeta\rangle)\circ\langle\xi'|\xi\rangle) =$$

$$= \sum_{i=1}^{n} x_i \varphi(e_i \langle \cdot | \langle \xi | \xi' \rangle \zeta \rangle) =$$

$$= \sum_{i=1}^{n} \zeta \langle \cdot | e_i \rangle \langle (\langle \xi | \xi' \rangle \zeta)^* | e_i^* \rangle =$$

$$= \sum_{i=1}^{n} \zeta \langle \cdot | e_i \rangle (\langle \xi | \xi' \rangle \zeta)_i^* = \zeta \langle \cdot | \sum_{i=1}^{n} (\langle \xi | \xi' \rangle \zeta)_i e_i \rangle =$$

$$= \zeta \langle \cdot | \langle \xi | \xi' \rangle \zeta \rangle = (\zeta \langle \cdot | \zeta \rangle) \circ \langle \xi' | \xi \rangle = \langle \eta | \eta \rangle \langle \xi' | \xi \rangle$$

(Proposition 5.6.1.2 a), Proposition 5.6.5.2 a),e)), where we identified the element $(\langle \xi | \xi' \rangle \zeta)_i^*$ of E with the corresponding diagonal element of $E_{n,n}$. By Proposition 5.6.3.9 a $\Rightarrow$ b,

$$\sum_{i=1}^{n} \eta_i \varphi \langle \xi' | \xi_i \rangle = \eta \sum_{i=1}^{n} x_i \varphi \langle \xi' | \xi_i \rangle =$$

$$\eta \langle \eta | \eta \rangle \langle \xi' | \xi \rangle = \eta \langle \xi' | \xi \rangle ,$$

$$\sum_{i=1}^{n} \eta_i \varphi \langle \cdot | \xi_i \rangle = \eta \langle \cdot | \xi \rangle .$$

e) follows from d) and Proposition 5.6.5.12 b) (and Proposition 5.6.2.5 a),e)). ■

Proposition 5.6.6.20 (8) *Let* $n \in \mathbb{N}$.

a) *If* $\mathcal{P}$ *is a subset of* $\operatorname{Pr} \mathbb{H}_{n,n} \backslash \{0\}$ *such that*

$$pq = \delta_{pq} p$$

for all $p, q \in \mathcal{P}$ *then*

$$\operatorname{Card} \mathcal{P} \leq n .$$

b) *For every* $x \in \operatorname{Re} \mathbb{H}_{n,n}$,

$$\operatorname{Card} \sigma(x) \leq n .$$

c) *If E is a Gelfand C^*-subalgebra of $\mathbb{H}_{n,n}$ then*

$$\operatorname{Dim} E \leq n .$$

a) By Theorem 5.6.6.1 b), for every $p \in \mathcal{P}$ there is a $\xi_p \in \mathbb{H}^n$ such that

$$p\xi_p \neq 0 .$$

Let $(\alpha_p)_{p\in\mathcal{P}} \in \mathbb{H}^{(\mathcal{P})}$ such that

$$\sum_{p\in\mathcal{P}} (p\xi_p)\alpha_p = 0 .$$

Then for every $q \in \mathcal{P}$,

$$0 = q\left(\sum_{p\in\mathcal{P}} (p\xi_p)\alpha_p\right) = \sum_{p\in\mathcal{P}} (q(p\xi_p))\alpha_p = (q\xi_q)\alpha_q ,$$

$$\alpha_q = 0 .$$

Hence $(p\xi_p)_{p\in\mathcal{P}}$ is linearly independent and therefore

$$\operatorname{Card} \mathcal{P} \leq n .$$

b) Since the complexification of $\mathbb{H}_{n,n}$ is isomorphic to $\mathbb{C}_{2n,2n}$ (Corollary 5.6.6.3 b)), $\sigma(x)$ is finite. By Theorem 4.1.4.6 a $\Rightarrow$ b & c, there is a family $(p_\alpha)_{\alpha\in\sigma(x)}$ in $\operatorname{Pr}\mathbb{H}_{n,n}\backslash\{0\}$, such that

$$\alpha, \beta \in \sigma(x) \Longrightarrow p_\alpha p_\beta = \delta_{\alpha\beta} p_\alpha .$$

By a),

$$\operatorname{Card} \sigma(x) \leq n .$$

c) follows from b) and Corollary 4.1.2.6. ■

Corollary 5.6.6.21 *Let*

$$(\lambda_n)_{n\in\mathbb{N}} , (\mu_n)_{n\in\mathbb{N}} , (\nu_n)_{n\in\mathbb{N}} \in (\mathbb{N} \cup \{0\})^{(\mathbb{N})}$$

and put

$$E := \left(\prod_{n\in\mathbb{N}} \mathbb{R}_{n,n}^{\lambda_n}\right) \times \left(\prod_{n\in\mathbb{N}} \mathbb{C}_{n,n}^{\mu_n}\right) \times \left(\prod_{n\in\mathbb{N}} \mathbb{H}_{n,n}^{\nu_n}\right) .$$

a) *For every* $x \in \operatorname{Re} E$,

$$\operatorname{Card} \sigma(x) \leq \sum_{n \in \mathbb{N}} n(\lambda_n + \mu_n + \nu_n) .$$

b) *If* F *is Gelfand* C^**-subalgebra of* E *then*

$$\operatorname{Dim} F \leq \sum_{n \in \mathbb{N}} n(\lambda_n + \mu_n + \nu_n) .$$

a) follows from Proposition 5.6.6.20 b).

b) follows from a) and Corollary 4.1.2.6. ■

Proposition 5.6.6.22 (8) *Let* E *be a unital* C^**-algebra,* $n \in \mathbb{N}$, *and*

$$e_{ij} := [\delta_{i,k}\delta_{j,\ell}]_{k,\ell \in \mathbb{N}_n} \in E_{n,n}$$

for all $i, j \in \mathbb{N}_n$. *For* T *a set,* $\xi \in \ell^2(T, E_{n,n})$ *(Theorem 5.6.6.1 f)), and* $i, j \in \mathbb{N}_n$ *define* $\xi_{ij} \in \ell^2(T, E)$ *by*

$$\sum_{i,j=1}^{n} \xi_{ij} e_{ij} = \xi .$$

Take S, T *sets and for every*

$$u \in \left(\mathcal{L}_E(\ell^c(S, E), \ell^2(T, E))\right)_{n,n}$$

put

$$\widetilde{u} : \ell^2(S, E_{n,n}) \longrightarrow \ell^2(T, E_{n,n}) , \quad \xi \longmapsto \sum_{i,j,k=1}^{n} (u_{ij}\xi_{jk}) e_{ik} .$$

a) *If* $u \in (\mathcal{L}_E(\ell^2(S, E)\,,\, \ell^2(T, E)))_{n,n}$ *and* $a \in E_{n,n}$ *then*

$$\widetilde{u} \in \mathcal{L}_{E_{n,n}}\left(\ell^2(S, E_{n,n})\,,\, \ell^2(T, E_{n,n})\right) , \; \widetilde{u}^* = \widetilde{u^*} ,$$

$$\widetilde{ua} = \widetilde{u}a , \quad \widetilde{au} = a\widetilde{u}$$

(Example 5.6.4.2 c), Proposition 5.6.1.8 g)).

b) *The map*

$$\left(\mathcal{L}_E(\ell^2(S, E), \ell^2(T, E))\right)_{n,n} \longrightarrow \mathcal{L}_{E_{n,n}}\left(\ell^2(S, E_{n,n}), \ell^2(T, E_{n,n})\right) , \quad u \longmapsto \widetilde{u}$$

is bijective.

a) Take $\xi \in \ell^2(S, E_{n,n})$ and $\eta \in \ell^2(T, E_{n,n})$. Then for $i, j \in \mathbb{N}_n$,

$$\langle \widetilde{u}\xi | \eta \rangle_{ij} = \left(\sum_{t \in T} \eta^*(t)(\widetilde{u}\xi)(t) \right)_{ij} =$$

$$= \sum_{t \in T} \sum_{k=1}^{n} (\eta^*)_{ik}(t)(\widetilde{u}\xi)_{kj}(t) =$$

$$= \sum_{t \in T} \sum_{k=1}^{n} \eta^*_{ki}(t) \left(\sum_{\ell=1}^{n} u_{k\ell}\xi_{\ell j} \right)(t) =$$

$$= \sum_{k,\ell=1}^{n} \langle u_{k\ell}\xi_{\ell j} | \eta_{ki} \rangle = \sum_{k,\ell=1}^{n} \langle \xi_{\ell j} | u^*_{k\ell}\eta_{ki} \rangle =$$

$$= \sum_{k,\ell=1}^{n} \langle \xi_{\ell j} | (u^*)_{\ell k}\eta_{ki} \rangle = \sum_{\ell=1}^{n} \langle \xi_{\ell j} \mid (\widetilde{u^*}\eta)_{\ell i} \rangle =$$

$$= \sum_{\ell=1}^{n} \sum_{t \in T} (\widetilde{u^*}\eta)^*_{\ell i}(t)\xi_{\ell j}(t) =$$

$$= \sum_{t \in T} \sum_{\ell=1}^{n} \left((\widetilde{u^*}\eta)^* \right)_{i\ell}(t)\xi_{\ell j}(t) =$$

$$= \sum_{t \in T} \left((\widetilde{u^*}\eta)^*(t)\xi(t) \right)_{ij} = \langle \xi | \widetilde{u^*}\eta \rangle_{ij}.$$

Hence

$$\widetilde{u} \in \mathcal{L}_{E_{n,n}} \left(\ell^2(S, E_{n,n}), \ell^2(T, E_{n,n}) \right), \quad \widetilde{u}^* = \widetilde{u^*}.$$

Moreover,

$$(\widetilde{ua}\xi)_{ij} = \sum_{k=1}^{n} (ua)_{ik}\xi_{kj} = \sum_{k,\ell=1}^{n} (u_{i\ell}a_{\ell k})\xi_{kj} =$$

$$= \sum_{k,\ell=1}^{n} u_{i\ell}(a_{\ell k}\xi_{kj}) = \sum_{\ell=1}^{n} u_{i\ell}(a\xi)_{\ell j} = (\widetilde{u}(a\xi))_{ij} = ((\widetilde{u}a)\xi)_{ij},$$

$$(\widetilde{au}\xi)_{ij} = \sum_{k=1}^{n}(au)_{ik}\xi_{kj} = \sum_{k,\ell=1}^{n}(a_{i\ell}u_{\ell k})\xi_{kj} =$$

$$= \sum_{k,\ell=1}^{n} a_{i\ell}(u_{\ell k}\xi_{kj}) = \sum_{\ell=1}^{n} a_{i\ell}(\widetilde{u}\xi)_{\ell j} = a(\widetilde{u}\xi)_{ij} = ((a\widetilde{u})\xi)_{ij}\,,$$

so that

$$\widetilde{ua}\xi = (\widetilde{u}a)\xi\,, \quad \widetilde{au}\xi = (a\widetilde{u})\xi\,,$$

and

$$\widetilde{ua} = \widetilde{u}a\,, \quad \widetilde{au} = a\widetilde{u}\,.$$

b) Take

$$v \in \mathcal{L}_{E_{n,n}}(\ell^2(S, E_{n,n})\,,\ \ell^2(T, E_{n,n}))\,.$$

For $i, j, k, \ell \in \mathbb{N}_n$ put

$$v_{ij,k\ell} : \ell^2(S, E) \longrightarrow \ell^2(T, E)\,, \quad \eta \longmapsto (v(\eta e_{k\ell}))_{ij}\,.$$

For $\xi \in \ell^2(S, E_{n,n})$ and $i, j \in \mathbb{N}_n$,

$$(v\xi)_{ij} = \left(v\left(\sum_{k,\ell=1}^{n}\xi_{k\ell}e_{k\ell}\right)\right)_{ij} = \sum_{k,\ell=1}^{n} v_{ij,k\ell}\xi_{k\ell}\,.$$

Take $\xi \in \ell^2(S, E_{n,n})$ and $p, q \in \mathbb{N}_n$. Then for $k, \ell \in \mathbb{N}_n$,

$$(\xi e_{pq})_{k\ell} = \left(\sum_{i,j=1}^{n}\xi_{ij}e_{ij}e_{pq}\right)_{k\ell} = \left(\sum_{i=1}^{n}\xi_{ip}e_{iq}\right)_{k\ell} =$$

$$= \sum_{i=1}^{n}\xi_{ip}\delta_{ik}\delta_{\ell q} = \xi_{kp}\delta_{\ell q}\,.$$

It follows

$$\sum_{i,j,k=1}^{n} v_{ij,kq}\xi_{kp}e_{ij} = \sum_{i,j,k,\ell=1}^{n}(v_{ij,k\ell}\xi_{kp}\delta_{\ell q})e_{ij} =$$

$$= v(\xi e_{pq}) = \sum_{i,j,k,\ell=1}^{n}(v_{ij,k\ell}\xi_{k\ell})e_{ij}e_{pq} = \sum_{i,k,\ell=1}^{n} v_{ip,k\ell}\xi_{k\ell}e_{iq}$$

and

$$\sum_{k=1}^{n} v_{ij,kq}\xi_{kp} = \delta_{jq} \sum_{k,\ell=1}^{n} v_{ip,k\ell}\xi_{k\ell}$$

for all $i, j \in \mathbb{N}_n$. Take $\eta \in \ell^2(T, E)$ and $i, j, a, b \in \mathbb{N}_n$ and put

$$\xi := \eta e_{ab} .$$

Then

$$\sum_{k=1}^{n} v_{ij,kq}\delta_{ka}\delta_{pb}\eta = \delta_{jq} \sum_{k,\ell=1}^{n} v_{ip,k\ell}\delta_{ka}\delta_{\ell b}\eta ,$$

$$\delta_{pb} v_{ij,aq} = \delta_{jq} v_{ip,ab} .$$

Put

$$u_{ij} := v_{i1,j1}$$

for all $i, j \in \mathbb{N}_n$. Then

$$v_{ip,ab} = \delta_{pb} v_{ij,aj} .$$

Since j is arbitrary,

$$v_{ip,ab} = \delta_{pb} u_{ia} .$$

We get for $\xi \in \ell^2(S, E_{n,n})$ and $i, j \in \mathbb{N}_n$

$$(v\xi)_{ij} = \sum_{k,\ell=1}^{n} v_{ij,k\ell}\xi_{k\ell} = \sum_{k,\ell=1}^{n} \delta_{j\ell} u_{ik}\xi_{k\ell} =$$

$$= \sum_{k=1}^{n} u_{ik}\xi_{kj} = (\tilde{u}\xi)_{ij}$$

so that

$$\tilde{u} = v$$

and the map is surjective. Since it is obviously injective it is bijective. ■

5.6.7 Type I W^*-algebras

Throughout this subsection E denotes a W^*-algebra.

Proposition 5.6.7.1 *Let $p \in \Pr E$, $F := pEp$, and $(p_\iota)_{\iota \in I}$ a family in $\Pr E$ such that*

$$0 < p_\iota \leq p$$

for every $\iota \in I$. For every $\iota \in I$ denote by H_ι (by K_ι) the von Neumann right E-module (right F-module) associated to $p_\iota E$ (to $p_\iota F$) (Propositions 4.4.2.2 and 5.6.2.3) and put

$$H := \overset{w}{\bigoplus_{\iota \in I}} H_\iota, \quad K := \overset{w}{\bigoplus_{\iota \in I}} K_\iota.$$

a) *H (resp. K) is a von Neumann right E-module (right F-module).*

b) *$K = \{\xi \in H \mid \xi p = \xi\}$.*

c) *If $u \in \mathcal{L}_E(H)$ then $u(K) \subset K$. Put*

$$u_K : K \longrightarrow K, \quad \xi \longmapsto u\xi$$

for every $u \in \mathcal{L}_F(H)$; then $u_K \in \mathcal{L}_F(K)$ with $(u_K)^* = (u^*)_K$.

d) *The map*

$$\mathcal{L}_E(H) \longrightarrow \mathcal{L}_F(K), \quad u \longmapsto u_K$$

is an isomorphism of W^*-algebras.

a) follows from Theorem 5.6.4.7 a) (and Proposition 5.6.2.3).

b) For every $\xi \in K$ and $\iota \in I$, $\xi_\iota \in K_\iota = p_\iota F$, so that $\xi_\iota = \xi_\iota p$. It follows $\xi p = \xi$. Conversely, take $\xi \in H$ with $\xi p = \xi$. For every $\iota \in I$,

$$\xi_\iota = \xi_\iota p = p_\iota \xi p = p_\iota p \xi p \in p_\iota F = K_\iota.$$

Hence $\xi \in K$.

c) By b), for every $\xi \in K$, $\xi = \xi p$ and so

$$u\xi = u(\xi p) = (u\xi)p \in K$$

(Proposition 5.6.1.8 d)). Hence, by b), $u(K) \subset K$.

For $\xi, \eta \in K$,

$$\langle u_K \xi | \eta \rangle = \langle u\xi | \eta \rangle = \langle \xi | u^* \eta \rangle = \langle \xi | (u^*)_K \eta \rangle .$$

Hence

$$u_K \in \mathcal{L}_F(K), \quad (u_K)^* = (u^*)_K .$$

d) By a) and Theorem 5.6.3.5 b), $\mathcal{L}_E(H)$ and $\mathcal{L}_F(K)$ are W^*-algebras with preduals $\ddot{H}$ and $\ddot{K}$, respectively. By c), the map

$$\varphi : \mathcal{L}_E(H) \longrightarrow \mathcal{L}_F(K), \quad u \longmapsto u_K$$

is well-defined and is involutive. It is obviously an algebra homomorphism.

Step 1 φ is injective

For every $\xi \in H$ and $J \subset I$ put

$$\xi_J : I \longrightarrow E, \quad \iota \longmapsto \begin{cases} \xi_\iota & \text{if } \iota \in J \\ 0 & \text{if } \iota \in I \backslash J , \end{cases}$$

$$\pi_J : H \longrightarrow H, \quad \xi \longmapsto \xi_J .$$

For every $\iota \in I$ put

$$f_\iota : I \longrightarrow E, \quad \lambda \longmapsto \delta_{\iota\lambda} p_\iota ;$$

obviously $f_\iota \in K$. Take $u \in \operatorname{Ker} \varphi$ and $\xi \in H$. Then

$$uf_\iota = u_K f_\iota = 0$$

and so

$$u\xi_{\{\iota\}} = u(f_\iota \xi_\iota) = (uf_\iota)\xi_\iota = 0$$

for every $\iota \in I$ (Proposition 5.6.1.8 d)). It follows

$$u\xi_J = 0$$

and

$$\pi_J u \pi_J = 0$$

for every $J \in \mathfrak{P}_f(I)$. By Theorem 5.6.4.7 c),

$$u = \lim_{J, \mathfrak{F}_I} \pi_J u \pi_J = 0 .$$

Thus φ is injective.

Step 2 $\varphi(\mathcal{L}_E(H))$ is a W^*-subalgebra of $\mathcal{L}_F(K)$

Take $u \in \mathcal{L}_E(H)$, $a \in \ddot{F}$, and $\xi, \eta \in K$. By Corollary 4.4.4.12 c), F is a W^*-subalgebra of E and so there is a $b \in \ddot{E}$ with $b|F = a$ (Corollary 4.4.4.9). From

$$\langle \varphi u, \widetilde{(a, \xi, \eta)} \rangle = \langle \langle (\varphi u)\xi | \eta \rangle, a \rangle = \langle \langle (\varphi u)\xi | \eta \rangle, b \rangle =$$

$$= \langle \langle u\xi | \eta \rangle, b \rangle = \langle u, \widetilde{(b, \xi, \eta)} \rangle$$

it follows by Theorem 5.6.3.5 d), that φ is a W^*-homomorphism. By Corollary 4.4.4.8 b), $\varphi(\mathcal{L}_E(H))$ is a W^*-subalgebra of $\mathcal{L}_F(K)$.

Step 3 φ is surjective

Put

$$\rho_J : K \longrightarrow K , \quad \xi \longmapsto \xi_J$$

for every $J \subset I$. Take $u \in \mathcal{L}_F(K)$ and $\iota \in I$. The map

$$v : H \longrightarrow H , \quad \xi \longmapsto (uf_\iota)\xi_\iota$$

belongs to $\widehat{\mathcal{L}}_E(H)$. By a) and Proposition 5.6.2.4, it belongs to $\mathcal{L}_E(H)$. For $\xi \in K$,

$$v\xi = (uf_\iota)\xi_\iota = u(f_\iota \xi_\iota) = u\xi_{\{\iota\}} = u\rho_{\{\iota\}}\xi$$

so that

$$u\rho_{\{\iota\}} = v_K \in \varphi(\mathcal{L}_E(H)) .$$

Moreover,

$$\rho_{\{\lambda\}} u \rho_{\{\iota\}} = (\pi_{\{\lambda\}} v)_K \in \varphi(\mathcal{L}_E(H))$$

for every $\lambda \in I$ (Theorem 5.6.4.7 c)). It follows

$$\rho_J u \rho_J \in \varphi(\mathcal{L}_E(H))$$

for every $J \in \mathfrak{P}_f(I)$. By Theorem 5.6.4.7 c),

$$u = \lim_{J,\mathfrak{F}_I} \rho_J u \rho_J$$

in $(\mathcal{L}_F(K))_{\bar{K}}$. By Step 2,

$$u \in \varphi(\mathcal{L}_E(H)),$$

i.e. φ is surjective. ■

Theorem 5.6.7.2 *Let $p \in \Pr E$ and let $(x_\iota)_{\iota \in I}$ be a family in E such that*

$$x_\iota^* x_\lambda = \delta_{\iota\lambda} p$$

for all $\iota, \lambda \in I$ and

$$\sum_{\iota \in I}^{E} x_\iota x_\iota^* = 1 .$$

For every $\iota \in I$ denote by H_ι the von Neumann right E-module associated to pE (Proposition 5.6.2.3) and put

$$H := \bigoplus_{\iota \in I}^{W} H_\iota , \quad \zeta := (x_\iota^*)_{\iota \in I} .$$

a) *$\zeta \in H$, $\langle \zeta | \zeta \rangle = 1$, and $\zeta \langle \cdot | \zeta \rangle$ is the identity map of H.*

b) *For every $x \in E$ the map*

$$\widetilde{x} : H \longrightarrow H , \quad \xi \longmapsto \zeta x \langle \xi | \zeta \rangle$$

belongs to $\mathcal{L}_E(H)$ with $\widetilde{x}^* = \widetilde{x^*}$. Define

$$\varphi : E \longrightarrow \mathcal{L}_E(H) , \quad x \longmapsto \widetilde{x} .$$

c) *For every $u \in \mathcal{L}_E(H)$ put*

$$\widetilde{u} := \langle u\zeta | \zeta \rangle$$

and define

$$\psi : \mathcal{L}_E(H) \longrightarrow E , \quad u \longmapsto \widetilde{u} .$$

Then φ and ψ are isomorphisms of W^*-algebras and $\psi = \varphi^{-1}$.

d) *If* $\ddot{\varphi}$ *and* $\ddot{\psi}$ *denote the pretranspose of* φ *and* ψ, *respectively (Theorem 5.6.4.7 a), Theorem 5.6.3.5 b)), then*

$$\ddot{\varphi}(\widetilde{a,\xi,\eta}) = \langle\xi|\zeta\rangle a\langle\zeta|\eta\rangle\,,$$

$$\ddot{\psi}a = \widetilde{(a,\zeta,\zeta)}$$

for all $a \in \ddot{E}$ *and* $\xi, \eta \in H$.

e) $x \in E^c$, $\xi \in H \Longrightarrow \widetilde{x}\xi = \xi x$.

f) *If* $\iota, \lambda \in I$ *and* $\xi \in H$ *then*

$$\widetilde{x_\iota x_\lambda^*}\xi = (\delta_{\mu\iota}\xi_\lambda)_{\mu\in I}\,.$$

g) *If* pEp *is commutative then the map*

$$E^c \longrightarrow pEp\,, \quad x \longmapsto xp$$

is an isomorphism of W^**-algebras (Corollary 4.4.2.5).*

a) By Corollary 4.1.2.22 b $\Rightarrow$ c,

$$\zeta_\iota = p\zeta_\iota \in pE$$

for every $\iota \in I$. Since

$$1 = \sum_{\iota\in I}^{\ddot{E}} x_\iota x_\iota^* = \sum_{\iota\in I}^{\ddot{E}} \zeta_\iota^*\zeta_\iota$$

it follows $\zeta \in H$ and $\langle\zeta|\zeta\rangle = 1$. Moreover, for $\xi \in H$ and $\iota \in I$,

$$((\zeta\langle\cdot|\zeta\rangle)\xi)_\iota = \zeta_\iota\langle\xi|\zeta\rangle = x_\iota^* \sum_{\lambda\in I}^{\ddot{E}} x_\lambda\xi_\lambda = \sum_{\lambda\in I}^{\ddot{E}} x_\iota^* x_\lambda\xi_\lambda = \sum_{\lambda\in I}^{\ddot{E}} \delta_{\iota\lambda}p\xi_\lambda = \xi_\iota\,,$$

i.e.

$$(\zeta\langle\cdot|\zeta\rangle)\xi = \xi\,.$$

b) By a), $\widetilde{x}$ is well-defined. For $\xi, \eta \in H$,

$$\langle\widetilde{x}\xi|\eta\rangle = \langle\zeta x\langle\xi|\zeta\rangle|\eta\rangle = \langle\zeta|\eta\rangle x\langle\xi|\zeta\rangle =$$

$$= \langle\xi|\zeta x^*\langle\eta|\zeta\rangle\rangle = \langle\xi|\widetilde{x^*}\eta\rangle\,.$$

Thus $\widetilde{x} \in \mathcal{L}_E(H)$ and $\widetilde{x}^* = \widetilde{x^*}$.

c) For $x, y \in E$ and $\xi \in H$, by a),

$$(\varphi x)(\varphi y)\xi = \widetilde{x}\widetilde{y}\xi = \zeta x\langle \widetilde{y}\xi|\zeta\rangle = \zeta x\langle \zeta y\langle \xi|\zeta\rangle|\zeta\rangle =$$

$$= \zeta x\langle \zeta|\zeta\rangle y\langle \xi|\zeta\rangle = \zeta xy\langle \xi|\zeta\rangle = \widetilde{xy}\xi = (\varphi(xy))\xi\,,$$

so that

$$(\varphi x)(\varphi y) = \varphi(xy)$$

and φ is an algebra homomorphism. By b), it is involutive.

For $x \in E$, by a),

$$\psi\varphi x = \langle \widetilde{x}\zeta|\zeta\rangle = \langle \zeta x\langle \zeta|\zeta\rangle|\zeta\rangle =$$

$$= \langle \zeta|\zeta\rangle x\langle \zeta|\zeta\rangle = x\,.$$

For $u \in \mathcal{L}_E(H)$ and $\xi \in H$, by a),

$$(\varphi\psi u)\xi = \widetilde{\psi u}\xi = \zeta(\psi u)\langle \xi|\zeta\rangle = \zeta\langle u\zeta|\zeta\rangle\langle \xi|\zeta\rangle =$$

$$= \zeta\langle u(\zeta\langle \xi|\zeta\rangle)|\zeta\rangle = \zeta\langle u\xi|\zeta\rangle = u\xi\,,$$

so that

$$\varphi\psi u = u\,.$$

Hence φ and ψ are isomorphisms of W^*-algebras and $\psi = \varphi^{-1}$.

d) For $x \in E$

$$\langle \varphi x, \widetilde{(a, \xi, \eta)}\rangle = \langle \langle \widetilde{x}\xi|\eta\rangle, a\rangle = \langle \langle \zeta x\langle \xi|\zeta\rangle|\eta\rangle, a\rangle =$$

$$= \langle \langle \zeta|\eta\rangle x\langle \xi|\zeta\rangle, a\rangle = \langle x, \langle \xi|\zeta\rangle a\langle \zeta|\eta\rangle\rangle\,.$$

Thus

$$\ddot{\varphi}\widetilde{(a, \xi, \eta)} = \langle \xi|\zeta\rangle a\langle \zeta|\eta\rangle\,.$$

For $u \in \mathcal{L}_E(H)$

$$\langle \psi u, a\rangle = \langle \langle u\zeta|\zeta\rangle, a\rangle = \langle u, \widetilde{(a, \zeta, \zeta)}\rangle\,.$$

Thus

$$\ddot{\psi}a = \widetilde{(a, \zeta, \zeta)}\,.$$

e) By a),

$$\tilde{x}\xi = \zeta x\langle\xi|\zeta\rangle = \zeta\langle\xi|\zeta\rangle x = \xi x\,.$$

f) We have

$$x_\lambda^*\langle\xi|\zeta\rangle = x_\lambda^* \sum_{\mu\in I}^{\ddot{E}} x_\mu\xi_\mu = \sum_{\mu\in I}^{\ddot{E}} x_\lambda^* x_\mu \xi_\mu =$$

$$= \sum_{\mu\in I}^{\ddot{E}} \delta_{\lambda\mu} p\xi_\mu = p\xi_\lambda = \xi_\lambda\,,$$

$$\zeta x_\iota = (x_\mu^* x_\iota)_{\mu\in I} = (\delta_{\mu\iota} p)_{\mu\in I}\,,$$

$$\widetilde{x_\iota x_\lambda^*}\xi = \zeta x_\iota x_\lambda^*\langle\xi|\zeta\rangle = (\delta_{\mu\iota}p)_{\mu\in I}\xi_\lambda = (\delta_{\mu\iota}\xi_\lambda)_{\mu\in I}\,.$$

g) It is easy to see that the map is an involutive algebra homomorphism. Take $x \in E^c$ with $xp = 0$. Then

$$x_\iota^* x = x x_\iota^* = x p x_\iota^* = 0$$

for every $\iota \in I$, so that

$$x = \left(\sum_{\iota\in I}^{\ddot{E}} x_\iota x_\iota^*\right) x = \sum_{\iota\in I}^{\ddot{E}} x_\iota x_\iota^* x = 0\,.$$

Hence the map is injective.

Put

$$F := pEp\,.$$

For every $\iota \in I$ denote by K_ι the von Neumann right F–module associated to F (Proposition 5.6.2.3) and define

$$K := \bigoplus_{\iota\in I}^{W} K_\iota\,.$$

Let $y \in F$ and put

$$u : K \longrightarrow K\,, \quad \xi \longmapsto y\xi = \xi y\,.$$

Then

$$\langle u\xi|\eta\rangle = \langle \xi y|\eta\rangle = \langle \xi|\eta\rangle y = y\langle \xi|\eta\rangle = \langle \xi|\eta y^*\rangle$$

for all $\xi, \eta \in K$, so that $u \in \mathcal{L}_F(K)$. For every $v \in \mathcal{L}_F(K)$ and $\xi \in K$,

$$uv\xi = (v\xi)y = v(\xi y) = vu\xi\,,$$

$$uv = vu\,, \quad u \in \mathcal{L}_F(K)^c\,.$$

By c) and Proposition 5.6.7.1 d), there is an $x \in E^c$ with

$$\widetilde{x}\xi = u\xi$$

for every $\xi \in K$. By e),

$$x\xi = \widetilde{x}\xi = u\xi = y\xi$$

for every $\xi \in K$. Take $\iota \in I$ and

$$\xi := (\delta_{\iota\lambda}p)_{\lambda\in I}\,.$$

Then

$$(\delta_{\iota\lambda}xp)_{\lambda\in I} = x\xi = y\xi = (\delta_{\iota\lambda}yp)_{\lambda\in I} = (\delta_{\iota\lambda}p)_{\lambda\in I}\,,$$

$$xp = y\,.$$

Hence the map is surjective. By the above, it is an isomorphism of W^*-algebras. ■

Proposition 5.6.7.3 *Let $p, q \in \Pr E$ such that pEp is commutative and $0 < q \leq p$. Further let H be a von Neumann right E-module, A a Fourier basis of H, and B a Fourier set of H such that*

$$\langle \xi|\xi\rangle \leq p\,, \quad q \leq \langle \eta|\eta\rangle \leq p$$

for all $\xi \in A$ and $\eta \in B$.

a) $\xi, \eta \in A \cup B \Longrightarrow \langle \xi|\eta\rangle \in pEp\,,\ \langle \xi|\eta\rangle\langle \eta|\xi\rangle = \langle \eta|\xi\rangle\langle \xi|\eta\rangle\,.$

b) $\eta \in B \Longrightarrow \langle \eta|\eta\rangle = \overset{\cdot\cdot}{\sum_{\xi\in A}} \langle \eta|\xi\rangle\langle\xi|\eta\rangle$.

c) $\operatorname{Card} B \leq \operatorname{Card} A$.

a) By Proposition 5.6.3.9 a $\Rightarrow$ b and Proposition 5.6.1.2 a),

$$\langle \xi|\eta\rangle = \langle \xi\langle\xi|\xi\rangle|\eta\langle\eta|\eta\rangle\rangle = \langle\eta|\eta\rangle\langle\xi|\eta\rangle\langle\xi|\xi\rangle =$$

$$= p\langle\eta|\eta\rangle\langle\xi|\eta\rangle\langle\xi|\xi\rangle p \in pEp\,.$$

Since pEp is commutative, it follows

$$\langle\xi|\eta\rangle\langle\eta|\xi\rangle = \langle\eta|\xi\rangle\langle\xi|\eta\rangle\,.$$

b) By a) and Theorem 5.6.3.13 f),

$$\langle\eta|\eta\rangle = \overset{\cdot\cdot}{\sum_{\xi\in A}} \langle\xi|\eta\rangle\langle\eta|\xi\rangle = \overset{\cdot\cdot}{\sum_{\xi\in A}} \langle\eta|\xi\rangle\langle\xi|\eta\rangle\,.$$

c) Case 1 A is finite

Let K be the vector subspace of H generated by $A \cup B$. By a),

$$\langle\zeta_1|\zeta_2\rangle \in pEp$$

for all $\zeta_1, \zeta_2 \in K$. Take $x' \in \sigma(pEp)$ with $\langle q, x'\rangle \neq 0$ and put

$$A' := \{\xi \in A \mid \langle\langle\xi|\xi\rangle, x'\rangle \neq 0\}\,.$$

The map

$$K \times K \longrightarrow \mathbb{K}\,, \quad (\zeta_1, \zeta_2) \longmapsto \langle\langle\zeta_1|\zeta_2\rangle, x'\rangle$$

being a positive sesquilinear form, we get by Schwarz inequality,

$$|\langle\langle\zeta|\xi\rangle, x'\rangle|^2 \leq \langle\langle\zeta|\zeta\rangle, x'\rangle\langle\langle\xi|\xi\rangle, x'\rangle = 0$$

for every $\zeta \in K$ and $\xi \in A\backslash A'$. Define

$$\widetilde{\zeta} : A' \longrightarrow \mathbb{K}\,, \quad \xi \longmapsto \langle\langle\zeta|\xi\rangle, x'\rangle$$

for every $\zeta \in K$. Then $\widetilde{\zeta} \in \ell^2(A')$ for every $\zeta \in K$ and

$$\langle\widetilde{\zeta_1}|\widetilde{\zeta_2}\rangle = \sum_{\xi\in A'} \langle\langle\zeta_1|\xi\rangle, x'\rangle\overline{\langle\langle\zeta_2|\xi\rangle, x'\rangle} =$$

$$= \sum_{\xi \in A} \langle \langle \xi | \zeta_2 \rangle, x' \rangle \langle \langle \zeta_1 | \xi \rangle, x' \rangle =$$

$$= \left\langle \sum_{\xi \in A} \langle \xi | \zeta_2 \rangle \langle \zeta_1 | \xi \rangle, x' \right\rangle = \langle \langle \zeta_1 | \zeta_2 \rangle, x' \rangle$$

for all $\zeta_1, \zeta_2 \in K$ (Theorem 5.6.3.13 f)). In particular,

$$\langle \tilde{\eta}_1 | \tilde{\eta}_2 \rangle = \langle \langle \eta_1 | \eta_2 \rangle, x' \rangle = \delta_{\eta_1 \eta_2} \langle \langle \eta_1 | \eta_1 \rangle, x' \rangle = \delta_{\eta_1 \eta_2}$$

for all $\eta_1, \eta_2 \in B$. Thus $(\tilde{\eta})_{\eta \in B}$ is linearly independent. Since $(\tilde{\xi})_{\xi \in A'}$ generates $\ell^2(A')$ (in fact, it is an orthonormal basis of $\ell^2(A')$),

$$\operatorname{Card} B \leq \operatorname{Card} A' \leq \operatorname{Card} A .$$

Case 2 A is infinite

Take $a \in \ddot{E}_+$ with $\langle q, a \rangle \neq 0$ and put

$$f : A \longrightarrow \mathfrak{P}(B), \quad \xi \longmapsto \{\eta \in B \mid \langle \langle \xi | \eta \rangle \langle \eta | \xi \rangle, a \rangle \neq 0\} .$$

By a) and Theorem 5.6.3.13 c), for every $\xi \in A$,

$$\sum_{\eta \in f(\xi)} \langle \langle \xi | \eta \rangle \langle \eta | \xi \rangle, a \rangle = \left\langle \sum_{\eta \in f(\xi)}^{\ddot{E}} \langle \xi | \eta \rangle \langle \eta | \xi \rangle, a \right\rangle =$$

$$= \left\langle \sum_{\eta \in f(\xi)}^{\ddot{E}} \langle \eta | \xi \rangle \langle \xi | \eta \rangle, a \right\rangle \leq \langle \langle \xi | \xi \rangle, a \rangle < \infty ,$$

so that $f(\xi)$ is countable.

Take $\eta \in B$. By Theorem 5.6.3.13 f),

$$\sum_{\xi \in A} \langle \langle \xi | \eta \rangle \langle \eta | \xi \rangle, a \rangle = \left\langle \sum_{\xi \in A}^{\ddot{E}} \langle \xi | \eta \rangle \langle \eta | \xi \rangle, a \right\rangle =$$

$$= \langle \langle \eta | \eta \rangle, a \rangle \geq \langle q, a \rangle \neq 0 .$$

Hence

$$\eta \in \bigcup_{\xi \in A} f(\xi) , \quad B \subset \bigcup_{\xi \in A} f(\xi) ,$$

$$\operatorname{Card} B \leq \operatorname{Card} (A \times \aleph_0) = \operatorname{Card} A . \qquad \blacksquare$$

Corollary 5.6.7.4 *Let $p \in \Pr E$ be such that pEp is commutative. Further let H be a von Neumann right E-module and A, B Fourier bases of H such that*

$$\langle \xi_1 | \xi_1 \rangle = \langle \xi_2 | \xi_2 \rangle \leq p\,,$$

$$\langle \eta_1 | \eta_1 \rangle = \langle \eta_2 | \eta_2 \rangle \leq p$$

for all $\xi_1, \xi_2 \in A$ and $\eta_1, \eta_2 \in B$. Then

$$\operatorname{Card} A = \operatorname{Card} B$$

and

$$\langle \xi | \xi \rangle = \langle \eta | \eta \rangle$$

for all $(\xi, \eta) \in A \times B$.

$$\operatorname{Card} A = \operatorname{Card} B$$

follows from Proposition 5.6.7.3 c). By Proposition 5.6.7.3 a),b) and Proposition 5.6.3.9 a $\Rightarrow$ b, for all $\eta \in B$ and $\zeta \in A$,

$$\langle \eta | \eta \rangle \langle \zeta | \zeta \rangle = \sum_{\xi \in A}^{\dot{E}} \langle \eta | \xi \rangle \langle \xi | \eta \rangle \langle \zeta | \zeta \rangle =$$

$$= \sum_{\xi \in A}^{\dot{E}} \langle \eta | \xi \langle \zeta | \zeta \rangle \rangle \langle \xi | \eta \rangle = \sum_{\xi \in A}^{\dot{E}} \langle \eta | \xi \rangle \langle \xi | \eta \rangle = \langle \eta | \eta \rangle\,,$$

so that

$$\langle \eta | \eta \rangle \leq \langle \zeta | \zeta \rangle\,.$$

By symmetry,

$$\langle \xi | \xi \rangle = \langle \eta | \eta \rangle$$

for all $(\xi, \eta) \in A \times B$. ■

Proposition 5.6.7.5 *Let $p \in \Pr E$ such that pEp is commutative. Further let H be a von Neumann right E-module, A a Fourier basis of H such that*

$$\langle \xi | \xi \rangle \leq p$$

for every $\xi \in A$, and $(u_{\iota\lambda})_{(\iota,\lambda)\in I\times I}$ a family in $\mathcal{L}_E(H)$ such that

1) $\iota, \lambda \in I \Rightarrow u_{\iota\lambda}^* = u_{\lambda\iota}$,

2) $\iota, \lambda, \mu, \nu \in I \Rightarrow u_{\iota\lambda} u_{\mu\nu} = \delta_{\lambda\mu} u_{\iota\nu}$,

3) $\exists \mu, \nu \in I$, $u_{\mu\nu} \neq 0$.

a) $\iota, \lambda \in I \Rightarrow u_{\iota,\iota} \in \Pr \mathcal{L}_E(H)$, $\|u_{\iota\lambda}\| = 1$.

b) *There is a Fourier family $(\eta_\iota)_{\iota\in I}$ in H such that*

$$\eta_\iota = u_{\iota\lambda}\eta_\lambda, \quad \langle \eta_\iota | \eta_\iota \rangle = \langle \eta_\lambda | \eta_\lambda \rangle \leq p$$

for all $\iota, \lambda \in I$.

c) $\operatorname{Card} I \leq \operatorname{Card} A$.

a) By 1) and 2),

$$u_{\iota\iota} \in \Pr \mathcal{L}_E(H),$$

$$\|u_{\iota\iota}\| = \|u_{\iota\lambda} u_{\lambda\iota}\| = \|u_{\iota\lambda} u_{\iota\lambda}^*\| = \|u_{\iota\lambda}\|^2 =$$

$$= \|u_{\iota\lambda}^* u_{\iota\lambda}\| = \|u_{\lambda\iota} u_{\iota\lambda}\| = \|u_{\lambda\lambda}\|.$$

By 3),

$$\|u_{\iota\lambda}\| = 1.$$

b) Take $\mu \in I$ and $\zeta \in \operatorname{Im} u_{\mu\mu} \backslash \{0\}$. By a), Theorem 5.6.3.13 f), and Proposition 5.6.3.4 c),

$$\zeta = u_{\mu\mu}\zeta = u_{\mu\mu}\left(\sum_{\xi\in A}^{H} \xi\langle\zeta|\xi\rangle\right) = \sum_{\xi\in A}^{H} (u_{\mu\mu}\xi)\langle\zeta|\xi\rangle.$$

Hence there is a $\xi \in A$ with $u_{\mu\mu}\xi \neq 0$. From

$$\langle \xi - u_{\mu\mu}\xi | u_{\mu\mu}\xi \rangle = \langle u_{\mu\mu}(\xi - u_{\mu\mu}\xi) | \xi \rangle = 0$$

it follows

$$p \geq \langle \xi | \xi \rangle = \langle \xi - u_{\mu\mu}\xi | \xi - u_{\mu\mu}\xi \rangle + \langle u_{\mu\mu}\xi | u_{\mu\mu}\xi \rangle \geq$$

$$\geq \langle u_{\mu\mu}\xi | u_{\mu\mu}\xi \rangle ,$$

$$\langle u_{\mu\mu}\xi | u_{\mu\mu}\xi \rangle \in pEp .$$

By Proposition 5.6.3.10 a),c),d), there is an $x \in pEp$ such that if we put

$$\eta := (u_{\mu\mu}\xi)x$$

then

$$\langle \eta | \eta \rangle \in \Pr E \backslash \{0\} .$$

Moreover,

$$\eta = (u_{\mu\mu}\xi)x = u_{\mu\mu}\eta \in \operatorname{Im} u_{\mu\mu} ,$$

$$\langle \eta | \eta \rangle = x^* \langle u_{\mu\mu}\xi | u_{\mu\mu}\xi \rangle x \in pEp .$$

Put

$$\eta_\iota := u_{\iota\mu}\eta$$

for every $\iota \in I$. Then for all $\iota, \lambda \in I$,

$$u_{\iota\lambda}\eta_\lambda = u_{\iota\lambda}u_{\lambda\mu}\eta = u_{\iota\mu}\eta = \eta_\iota ,$$

$$\langle \eta_\iota | \eta_\lambda \rangle = \langle u_{\iota\mu}\eta | u_{\lambda\mu}\eta \rangle = \langle \eta | u_{\mu\iota}u_{\lambda\mu}\eta \rangle =$$

$$= \delta_{\iota\lambda}\langle \eta | u_{\mu\mu}\eta \rangle = \delta_{\iota\lambda}\langle \eta | \eta \rangle \leq p .$$

c) follows from b) and Proposition 5.6.7.3 c). ■

Proposition 5.6.7.6 *Let $p, q, r \in \Pr E$ such that pEp is commutative and*

$$0 < q \leq p , \quad 0 < r \leq p .$$

Further, let $(x_\iota)_{\iota \in I}$, $(y_\mu)_{\mu \in M}$ be families in E such that

$$x_\iota^* x_\lambda = \delta_{\iota\lambda} q\,, \quad y_\mu^* y_\nu = \delta_{\mu\nu} r$$

for all $\iota, \lambda \in I$ *and* $\mu, \nu \in M$ *and*

$$\sum_{\iota \in I}^{\ddot{E}} x_\iota x_\iota^* = \sum_{\mu \in M}^{\ddot{E}} y_\mu y_\mu^* = 1\,.$$

Then

$$\operatorname{Card} I = \operatorname{Card} M\,, \quad q = r\,.$$

Put

$$H := \bigoplus_{\iota \in I}^{W} qE\,, \quad K := \bigoplus_{\mu \in M}^{W} rE\,.$$

By Theorem 5.6.7.2 c), E is isomorphic to $\mathcal{L}_E(H)$ and $\mathcal{L}_E(K)$. Put

$$u_{\iota\lambda} := x_\iota x_\lambda^*$$

for all $\iota, \lambda \in I$. Then

$$u_{\iota\lambda}^* = x_\lambda x_\iota^* = u_{\lambda\iota}\,,$$

$$u_{\iota\lambda} u_{\mu\nu} = x_\iota x_\lambda^* x_\mu x_\nu^* = \delta_{\lambda\mu} u_{\iota\nu}\,,$$

$$0 \neq \|q\| = \|x_\iota^* x_\iota\| = \|x_\iota\|^2 = \|x_\iota x_\iota^*\| = \|u_{\iota\iota}\|$$

for all $\iota, \lambda, \mu, \nu \in I$. By Proposition 5.6.7.5 c),

$$\operatorname{Card} I \leq \operatorname{Card} M\,.$$

By symmetry,

$$\operatorname{Card} I = \operatorname{Card} M\,.$$

By Theorem 5.6.7.2 g), qEq and rEr are isomorphic, so that

$$q = r\,.$$

■

Proposition 5.6.7.7 *Let* I *be a set and* $\mu \in I$. *Put*

$$H := \bigoplus_{\iota \in \aleph}^{W} E$$

and define

$$u_\iota : H \longrightarrow H\,, \quad \xi \longmapsto (\delta_{\nu\iota} \xi_\mu)_{\nu \in I}$$

for every $\iota \in I$.

a) $u_\iota \in \mathcal{L}_E(H)$ *for every* $\iota \in I$ *and*

$$u_\iota^* \eta = (\delta_{\nu\mu}\eta_\iota)_{\nu\in I}$$

for every $\eta \in H$.

b) $\iota, \lambda \in I \Rightarrow u_\iota^* u_\lambda = \delta_{\iota\lambda} u_\mu$.

c) $u_\mu \in \Pr \mathcal{L}_E(H)$.

d) $\sum\limits_{\iota\in I}^{H} u_\iota u_\iota^* = 1$.

a) For $\xi, \eta \in H$,

$$\langle u_\iota \xi | \eta \rangle = \langle (\delta_{\nu\iota}\xi_\mu)_{\nu\in I} \mid (\eta_\nu)_{\nu\in I} \rangle = \sum_{\nu\in I}^{E} \eta_\nu^* \delta_{\nu\iota}\xi_\mu =$$

$$= \eta_\iota^* \xi_\mu = \sum_{\nu\in I}^{E} \delta_{\nu\mu}\eta_\iota^* \xi_\nu = \langle (\xi_\nu)_{\nu\in I} \mid (\delta_{\nu\mu}\eta_\iota)_{\nu\in I} \rangle .$$

Hence $u_\iota \in \mathcal{L}_E(H)$ and

$$u_\iota^* \eta = (\delta_{\nu\mu}\eta_\iota)_{\nu\in I} .$$

b) By a),

$$u_\iota^* u_\lambda \xi = u_\iota^* (\delta_{\nu\lambda}\xi_\mu)_{\nu\in I} = (\delta_{\nu\mu}\delta_{\iota\lambda}\xi_\mu)_{\nu\in I} =$$

$$= \delta_{\iota\lambda}(\delta_{\nu\mu}\xi_\mu)_{\nu\in I} = \delta_{\iota\lambda} u_\mu \xi$$

for every $\xi \in H$, so that

$$u_\iota^* u_\lambda = \delta_{\iota\lambda} u_\mu .$$

c) follows from a) and b).

d) By a), for $\iota \in I$, $\xi, \eta \in H$, and $a \in \ddot{E}$,

$$u_\iota u_\iota^* \xi = u_\iota (\delta_{\nu\mu}\xi_\iota)_{\nu\in I} = (\delta_{\nu\iota}\xi_\iota)_{\nu\in I} ,$$

$$\langle u_\iota u_\iota^* \xi | \eta \rangle = \sum_{\nu\in I}^{E} \eta_\nu^* \delta_{\nu\iota}\xi_\iota = \eta_\iota^* \xi_\iota$$

$$\sum_{\iota\in I}\langle\langle u_\iota u_\iota^*\xi|\eta\rangle, a\rangle = \sum_{\iota\in I}\langle\eta_\iota^*\xi_\iota, a\rangle =$$

$$= \langle\langle\xi|\eta\rangle, a\rangle = \langle 1, \widetilde{(a,\xi,\eta)}\rangle\,.$$

Hence

$$\sum_{\iota\in I}^{H} u_\iota u_\iota^* = 1\,.$$ ■

Definition 5.6.7.8 *The W^*-algebra E ist called* ***homomogeneous*** *if there exist a $p\in \Pr E$ and a family $(x_\iota)_{\iota\in I}$ in E such that pEp is commutative,*

$$x_\iota^* x_\lambda = \delta_{\iota\lambda} p$$

for all $\iota,\lambda\in I$, and

$$\sum_{\iota\in I}^{E} x_\iota x_\iota^* = 1\,.$$

Every commutative W^* algebra is homomogeneous. We have

$$\|p\| = \|x_\iota^* x_\iota\| = \|x_\iota\|^2 = \|x_\iota x_\iota^*\|$$

for every $\iota\in I$, so that $p\neq 0$.

Theorem 5.6.7.9 *For every cardinal number $\aleph$ and for every commutative W^*-algebra F put*

$$\widetilde{(\aleph,F)} := \mathcal{L}_F\left(\overset{W}{\bigoplus_{\iota\in\aleph}} F\right)\,.$$

a) *$\widetilde{(\aleph,F)}$ is a homogeneous W^*-algebra and F is isomorphic to $\widetilde{(\aleph,F)}^c$.*

b) *For every homogeneous W^*-algebra E there is a pair $(\aleph,F)$, such that $\aleph$ is a cardinal number, F is a commutative W^*-algebra, and $\widetilde{(\aleph,F)}$ is isomorphic to E.*

c) *If $\aleph,\aleph'$ are cardinal numbers and F,F' are commutative W^*-algebras such that*

$$\widetilde{(\aleph,F)} = \widetilde{(\aleph',F')}$$

then $\aleph=\aleph'$ and F and F' are isomorphic.

d) *Let F be a commutative W^*-algebra, $\aleph$ a cardinal number, $n \in \mathbb{N}$ and*

$$\aleph' := \mathrm{Card}(\aleph \times \mathbb{N}_n)\,.$$

Then $\widetilde{(\aleph, F)}_{n,n}$ is isomorphic to $\widetilde{(\aleph', F)}$. In particular $E_{n,n}$ is homogeneous for every homogeneous W^*-algebra E.

a) follows from Proposition 5.6.7.7 and Theorem 5.6.7.2 g).

b) There exist a $p \in \Pr E$ and a family $(x_\iota)_{\iota \in I}$ in E such that $F := pEp$ is commutative,

$$x_\iota^* x_\lambda = \delta_{\iota\lambda} p$$

for all $\iota, \lambda \in I$, and

$$\sum_{\iota \in I}^{\ddot{E}} x_\iota x_\iota^* = 1\,.$$

By Theorem 5.6.7.2.c) and Proposition 5.6.7.1 d), E is isomorphic to

$$\mathcal{L}_F\left(\overset{W}{\underset{\iota \in I}{\bigoplus}} F\right) = \widetilde{\widetilde{(\mathrm{Card}\, I, F)}}\,.$$

c) By Proposition 5.6.7.7, there is a family $(u_\iota)_{\iota \in \aleph}$ in $\widetilde{(\aleph, F)}$ such that for all $\iota, \lambda \in I$,

$$u_\iota^* u_\lambda = \delta_{\iota\lambda} p\,,$$

where $p \in \Pr \widetilde{(\aleph, F)} \backslash \{0\}$. Put

$$u_{\iota\lambda} := u_\iota u_\lambda^*$$

for every $\iota, \lambda \in \aleph$. Then for $\iota, \lambda, \mu, \nu \in \aleph$,

$$u_{\iota\lambda}^* = u_\lambda u_\iota^* = u_{\lambda\iota}\,,$$

$$u_{\iota\lambda} u_{\mu\nu} = u_\iota u_\lambda^* u_\mu u_\nu^* = u_\iota \delta_{\lambda\mu} p u_\nu^* = \delta_{\lambda\mu} u_\iota u_\nu^* = \delta_{\lambda\mu} u_{\iota\nu}\,,$$

$$\|u_{\iota\iota}\| = \|u_\iota u_\iota^*\| = \|u_\iota\|^2 = \|u_\iota^* u_\iota\| = \|p\| \neq 0\,.$$

By Proposition 5.6.7.5 c) (and Proposition 5.6.4.6 f)),

$$\aleph \leq \aleph'\,.$$

By symmetry,

$$\aleph = \aleph' .$$

By a), F and F' are isomorphic.

d) It is easy to see that

$$\bigoplus_{\iota\in\aleph\times\mathbb{N}_n}^{W} F \text{ and } \bigoplus_{\iota\in\mathbb{N}_n}^{W}\bigoplus_{\iota\in\aleph}^{W} F$$

are isomorphic, so that the assertion follows from Proposition 5.6.4.16 d). ■

Proposition 5.6.7.10 *Assume E is the C^*-direct product of the family $(E_\lambda)_{\lambda\in L}$ of W^*-algebras and let I be a set. Put*

$$H := \bigoplus_{\iota\in I}^{W} E$$

and

$$H_\lambda := \bigoplus_{\iota\in I}^{W} E_\lambda$$

for every $\lambda \in L$ and denote by $\mathcal{L}$ the C^*-direct product of the family $(\mathcal{L}_{E_\lambda}(H_\lambda))_{\lambda\in L}$.

a) *The elements of H may be identified with the functions ξ defined on $I \times L$ such that*

$$\xi(\cdot,\lambda) \in H_\lambda$$

for every $\lambda \in L$ and such that

$$\sup_{\lambda\in L} \|\xi(\cdot,\lambda)\| < \infty .$$

b) *For $\xi, \eta \in H$,*

$$\langle\xi|\eta\rangle = (\langle\xi(\cdot,\lambda) \mid \eta(\cdot,\lambda)\rangle)_{\lambda\in L} .$$

c) *For every $u \in \mathcal{L}$ put*

$$\widetilde{u} : H \longrightarrow H , \quad \xi \longmapsto (u_\lambda \xi(\iota,\lambda))_{(\iota,\lambda)\in I\times L} .$$

Then $\widetilde{u} \in \mathcal{L}_E(H)$ for every $u \in \mathcal{L}$ and the map

$$\mathcal{L} \longrightarrow \mathcal{L}_E(H) , \quad u \longmapsto \widetilde{u}$$

is an isomorphism of W^*-algebras. ■

Definition 5.6.7.11 *E is called a type I W^*-algebra if it is isomorphic to the C^*-direct product of a family of homomogeneous W^*-algebras.*

This is not the usual definition of a type I W^*-algebra (in the complex case), but is equivalent to it.

Theorem 5.6.7.12 *Let $\mathfrak{F}$ be a class of commutative W^*-algebras such that any commutative W^*-algebra is isomorphic to exactly one element of $\mathfrak{F}$. Let further $\mathcal{F}$ be the class of families $(F_\aleph)_{\aleph\in I}$ in $\mathfrak{F}$ for which I is a set of cardinal numbers. For every $F = (F_\aleph)_{\aleph\in I} \in \mathcal{F}$ denote by $\widetilde{F}$ the C^*- direct product of the family*

$$\left(\mathcal{L}_{F_\aleph}\left(\overset{W}{\underset{\iota\in\aleph}{\bigoplus}} F_\aleph\right)\right)_{\aleph\in I}$$

of W^*-algebras.

a) *$\widetilde{F}$ is a type I W^*-algebra for every $F \in \mathcal{F}$.*

b) *For every type I W^*-algebra E there is exactly one $F \in \mathcal{F}$ such that E is isomorphic to $\widetilde{F}$.*

c) *$E_{n,n}$ is a type I W^*-algebra for every type I W^*-algebra E and for every $n \in \mathbb{N}$.*

a) follows from Theorem 5.6.7.9 a).

b) By definition, E is isomorphic to the C^*-direct product of a family $(G_\lambda)_{\lambda\in L}$ of homogeneous W^*-algebra. By Theorem 5.6.7.9 b), for every $\lambda \in L$, there is a cardinal number $\aleph_\lambda$ and an $F'_\lambda \in \mathfrak{F}$ such that G_λ is isomorphic to

$$\mathcal{L}_{F'_\lambda}\left(\overset{W}{\underset{\iota\in\aleph_\lambda}{\bigoplus}} F'_\lambda\right).$$

Put

$$I := \{\aleph_\lambda \mid \lambda \in L\}.$$

For every $\aleph \in I$ put

$$I_\aleph := \{\lambda \in L \mid \aleph_\lambda = \aleph\}$$

and denote by $F_\aleph$ the element of $\mathfrak{F}$ which is isomorphic to the C^*-direct product of the family $(F'_\lambda)_{\lambda\in I_\aleph}$. By Proposition 5.6.7.10 c),

$$\mathcal{L}_{F_\aleph}\left(\bigvee_{\iota\in\aleph}^{W} F_\aleph\right)$$

is isomorphic to the C^*-direct product of the family

$$\left(\mathcal{L}_{F'_\lambda}\left(\bigvee_{\iota\in\aleph}^{W} F'_\lambda\right)\right)_{\lambda\in I_\aleph}.$$

Hence, if we put

$$F := (F_\aleph)_{\aleph\in I},$$

then $F \in \mathcal{F}$ and $\widetilde{F}$ is isomorphic to E.

Let $F := (F_\aleph)_{\aleph\in I}$ and $G := (G_\aleph)_{\aleph\in L}$ be elements of $\mathcal{F}$ such that $\widetilde{F}$ and $\widetilde{G}$ are isomorphic. Then there is a bijection $\varphi : I \to L$ such that

$$\mathcal{L}_{F_\aleph}\left(\bigvee_{\iota\in\aleph}^{W} F_\aleph\right) \quad \text{and} \quad \mathcal{L}_{G_{\varphi(\aleph)}}\left(\bigvee_{\iota\in\varphi(\aleph)}^{W} G_{\varphi(\aleph)}\right)$$

are isomorphic for every $\aleph \in I$. By Theorem 5.6.7.9 c),

$$\aleph = \varphi(\aleph), \quad F_\aleph = G_\aleph$$

for every $\aleph \in I$, i.e. $F = G$.

c) follows immediately from Theorem 5.6.7.9 d). ■

Name Index

Alaoglu, L. 1.2.8.1
Arens, R.F. 1.5.2.10, 2.2.7.13
Arzelà, C. 1.1.2.16
Ascoli, G. 1.1.2.14, 1.1.2.16
Atkinson, F.V. 3.1.3.7, 3.1.3.11, 3.1.3.12, 3.1.3.21, 5.3.3.16
Autonne, L. 2.3.1.3
Banach, S. 1.1.1.2, 1.2.8.2, 1.3.1.2, 1.3.2, 1.3.3.1,
1.3.4.1, 1.3.4.10, 1.4.1.2, 1.4.2.3, 1.4.2.19
Beurling, A. 2.2.5.4
Bourbaki, N. 1.2.8.1
Branges, L. de 1.3.5.14
Calkin, J.W. 5.4.3.5
Carleman, T. 3.1.3.1
Cauchy, A. 1.3.10.6
Choquet, G. 5.4.3.5
Dedekind, R. 1.7.2.1
Dieudonné, J. 1.2.8.2, 3.1.3.9
Dixmier, J. 4.4.4.4
Drewnowski, L. 4.3.2.13
Dworetzky, A. 1.1.6.14
Dye, H.A. 4.1.3.7
Eberlein, W.F. 1.3.7.15
Effros, E.G. 4.2.4.15
Enflo, P. 3.1.1.7
Ford, J.W.M. 2.4.2.4
Fourier, J.-B.-J. 2.4.6.2
Fréchet, M. 1.1.1.2, 1.1.2.13, 5.2.5.2
Fredholm, E. 3.1.6.23
Frobenius, G.F. 2.1.4.21
Fuglede, B. 4.1.4.1
Fukamiya, M. 4.2.1.1
Gelfand, I.M. 1.4.1.9, 2.2.5.4, 2.2.5.5, 2.4.1.2, 2.4.5.7, 4.1.1.1, 4.1.2.5
4.1.3.1, 4.2.6.6, 5.4.1.2, 5.4.2.5
Gohberg, I. 3.1.3.12
Goldstine, H.H. 1.3.6.8

Goodearl, K.R. 2.2.1.19. 4.1.1.1, 5.4.2.13
Gowers, W.T. 1.2.1.12
Gram, J.P. 5.5.1.18
Grothendieck, A. 1.6.1.1, 3.1.6.25, 4.2.8.13
Hahn, H. 1.1.1.2, 1.2.1.3, 1.3.3.1, 1.3.6.1, 1.3.6.3, 1.3.8.1, 1.4.1.4
Hamilton, W.R. 2.1.4.17
Hellinger, E. 5.2.5.4
Helly, E. 1.1.1.2, 1.3.3.13
Hilbert, D. 2.1.3.1
Hirschfeld, R.A. 2.2.5.6
Jacobson, N. 2.1.3.10
James, R.C. 1.3.8.1
Jordan, C. 5.1.1.6
Kadison, R.K. 4.3.3.20
Kaplanski, I. 4.1.2.1, 4.2.6.5, 4.4.2.24, 5.6.1.1
Kelley, J.L. 4.2.1.1
Kojima, ?? 1.2.3.11
Kolmogoroff, A. 1.1.1.2
Kottman, C.A. E 1.3.5
Krein, M.G. 1.3.1.10, 1.3.7.3
Laguerre, E.N. 2.2.3.5
Laurent, P.A. 1.3.10.8
Lax, P.D. 5.3.1.3
Le Page, C. 2.2.3.8, 2.2.4.3, 2.2.5.6
Lindenstrauss, J. 1.2.5.13
Liouville, J. 1.3.10.6
Lomonsov, V.I. 3.1.5.10
Löwing, H. 5.1.1.1
Mackey, G.W. 1.3.7.2
Mazur, S. 2.2.5.5
Mihlin, S.G. 3.1.3.12
Milgram, A.N. 5.3.1.3
Milman, D.P. 1.3.1.10
Minkowski, H. 1.1.1.2, 1.1.3.4
Murray, F.J. 1.2.5.8
Nagumo, M. 2.2.1.1
Naimark, M.A. 4.1.1.1, 4.1.2.5, 4.1.3.1, 4.2.6.6, 5.4.1.2, 5.4.2.5, 5.5.1.24
Neumann, C. 2.2.3.5

Neumann, J. von 1.1.1.2, 3.1.3.1, 5.1.1.1, 5.1.1.6, 5.2.4
Nikodym, O. 4.3.2.14, 5.2.1.2
Noether, F. 3.1.3.1
Palmer, T.W. 4.1.1.1
Paschke, W.L. 5.6.1.1, 5.6.2.2, 5.6.2.6, 5.6.2.11, 5.6.3.3, 5.6.3.5
Pedersen, G.K. 4.2.4.16
Peter, F. 2.2.1.15
Pettis, P.J. 1.3.8.4, 1.3.8.5
Phillips, R.S. 1.2.5.14, E 1.3.3, 2.1.4.9
Pierce, B. 2.1.1.1, 2.1.3.6
Plancherel, M. 5.5.4.1
Putnam, I.F. 4.1.4.1
Rellich, F. 5.1.1.1
Rickart, C.E. 4.1.1.20, 4.1.2.12
Rieffel, M.A. 5.6.1.1
Riesz, F. 1.1.1.2, 1.1.4.4, 1.2.1.1, 1.2.2.5, 2.2.5.1, 3.1.1.1, 3.1.3.8, 3.1.3.17, 3.1.5.1, 5.2.1.2, 5.2.5.2, 5.3.3.20
Rogers, C.A. 1.1.6.14
Rosenblum, M. 4.1.4.1
Russo, B. 4.1.3.7
Sakai, S. 4.4, 4.4.1.1, 4.4.3.5
Schauder, J.P. 3.1.1.22
Schmidt, E. 1.1.1.2, 5.5.1.18
Schur, I. 1.2.3.11, 1.2.3.12, 1.3.6.11
Schwartz, L. 2.4.6.5, 2.4.6.8
Segal, I.E. 4.2.6.2, 4.2.6.5, 4.2.8.2, 5.4.1.2
Shirali, S. 2.4.2.4
Sierpiński, W. 1.1.2.17
Šilow, G. 2.2.4.27
Šmulian, V. 1.3.7.3, 1.3.7.15
Steinhaus, H.D. 1.4.1.2
Stone, M.H. 1.3.4.10, 1.3.5.16, 2.3.3.12, 4.1.2.5
Størmer, E. 4.2.6.3
Toeplitz, O. 1.2.3.4, 2.3.1.3, 5.2.5.4
Vaught, R.L. 4.2.1.1
Vitushkin, A.G. 2.4.3.7
Volterra, V. 2.2.4.22
Weierstrass, K. 1.3.5.16

Weyl, H. 2.2.1.15
Wielandt, H. 2.2.5.8
Wiener, N. 2.4.5.7
Yood, B. 3.1.3.11, 3.1.3.12, 4.1.1.13
Żelazko, W. 2.2.5.6

Subject Index

NT means Notation and Terminology

$(\mathcal{A},\mathcal{B},\mathcal{C})$–multiplication 1.5.1.1
absolute value of a number 1.1.1.1
absolute value of a measure NT
absolutely convex 1.2.7.1
absolutely convex closed hull 1.2.7.6
absolutely convex hull 1.2.7.4
absolutely summable family 1.1.6.9
acts irreducibly 5.3.2.19
acts non–degenerately 5.3.2.19
additive group NT
adherence, point of NT
adherent point NT
adjoint 2.3.1.1, 5.6.1.8
adjoint differential operator 3.2.2.3
adjoint kernel 3.1.6.5
adjoint operator 5.3.1.4
adjoint sesquilinear form 2.3.3.1
adjoint sesquilinear map 2.3.3.1
adjointable 5.6.1.7
algebra 2.1.1.1
algebra, Calkin 3.1.1.13
algebra, complex 2.1.1.1
algebra, degenerate 2.1.1.1
algebra, division 2.1.2.1
algebra, Gelfand 2.4.1.1
algebra, Gelfand unital 2.4.2.1
algebra, involutive 2.3.1.3
algebra, involutive Gelfand 2.4.2.1
algebra, involutive unital Gelfand 2.4.2.1
algebra, normed 2.2.1.1
algebra, real 2.1.1.1
algebra, semi–simple 2.1.3.18
algebra, strongly symmetric 2.3.1.26

algebra, symmetric 2.3.1.26
algebra, unital 2.1.1.3
algebra, unital Gelfand 2.4.1.1
algebra homomorphism 2.1.1.6
algebra homomorphism associated to A 5.4.2.3
algebra homomorphism associated to x' 5.4.1.2
algebra homomorphism associated to $(x'_\iota)_{\iota\in I}$ 5.4.2.2
algebra homomorphism, unital 2.1.1.6
algebra isomorphism 2.1.1.6
algebra isomorphism, unital 2.1.1.6
algebraic dimension 1.1.2.18
algebraic dual 1.1.1.1
algebraic eigenspace 5.3.3.20
algebraic eigenvector 5.3.3.20
algebraic isomorphism, associated 1.2.4.6
algebraic multiplicity 5.3.3.20
algebras, isomorphism of involutive 2.3.1.3
analytic function 1.3.10.1
approximate unit 2.2.1.15
approximate unit of a C^*-algebra, canonical 4.2.8.2
Arens multiplication, left 1.5.2.10, 2.2.7.13
Arens mutliplication, right 1.5.2.10, 2.2.7.13
associated algebraic isomorphism 1.2.4.6
associated quadratic form 2.3.3.1
associated quadratic map 2.3.3.1
associated unital C^*-algebra 4.1.1.13
atom 4.3.2.20
atomic 4.3.2.20
atomless 4.3.2.20
Baire function 1.7.2.12
Baire set 1.7.2.12
ball, unit 1.1.1.2
Banach algebra 2.2.1.1
Banach algebra, involutive 2.3.2.1
Banach algebra, quasiunital 2.2.1.15
Banach algebra, unital 2.2.1.1
Banach categories, functor of 1.5.2.1
Banach categories, functor of unital 1.5.2.1

Banach category 1.5.1.1
Banach category, unital 1.5.1.1
Banach space 1.1.1.2
Banach space, complex 1.1.1.2
Banach space, involutive 2.3.2.1
Banach space, ordered 1.7.1.4
Banach space, real 1.1.1.2
Banach subalgebra generated by 2.2.1.9
Banach system 1.5.1.1
Banach system, bidual of a 1.5.1.9
Banach system, dual of a 1.5.1.9
Banach systems, isometric 1.5.2.1
band 1.7.2.1
basis, Fourier 5.6.3.11
basis, orthonormal 5.5.1.1
basis of $\ell^2(T)$, canonical orthonormal 5.5.3.1
Bergman kernel 5.2.5.9
Bessel's identity 5.5.1.7
Bessel's inequality 5.5.1.8
bicommutant 2.1.1.16
bidual of a Banach system 1.5.1.9
bidual of a normed space 1.3.6.1
bijective NT
bilinear map 1.2.9.1
binomial theorem 2.2.3.12
bitranspose 1.3.6.15
bound, lower 1.7.2.1
bound, upper 1.7.2.1
bounded map 1.1.1.2
bounded operator 1.2.1.3
bounded operator, lower 1.2.1.18
bounded sequence 1.1.1.2
bounded set 1.1.1.2
boundedness, principle of uniform 1.4.1.2
boundedness theorem, Nikodym's 4.3.2.14
C^* algebra 4.1.1.1
C^*-algebra, canonical approximate unit of a 4.2.8.2
C^*-algebra, canonical order of a 4.2.1.2

C^*-algebra, complex 4.1.1.1
C^*-algebra, complex unital 4.1.1.1
C^*-algebra, Gelfand 4.1.1.1
C^*-algebra, purely real 4.1.1.8
C^*-algebra, real 4.1.1.1
C^*-algebra, real unital 4.1.1.1
C^*-algebra, simple 4.3.5.1
C^*-algebra, unital 4.1.1.1
C^*-algebra associated, unital 4.1.1.13
Calkin algebra 3.1.1.13
Calkin category 3.1.1.12
canonical approximate unit of a C^*-algebra 4.2.8.2
canonical involution of E^F 2.3.1.1
canonical metric of a normed space 1.1.1.2
canonical norm of a pre–Hilbert space 5.1.1.2
canonical norm of $\mathcal{L}(E,F)$ 1.2.1.9
canonical order of a C^*-algebra 4.2.1.2
canonical orthonormal basis of $\ell^2(T)$ 5.5.3.1
canonical projection of the tridual of E 1.3.6.19
canonical scalar product of $\mathbb{K}^n$ 5.1.2.4
cardinal number NT
cardinality, topological NT
carrier 4.3.3.1
carrier, left 4.3.3.1
carrier, right 4.3.3.1
carrier of a function NT
carrier of a Radon measure NT
category, Banach 1.5.1.1
C^*-direct product 4.1.1.6
C^*-direct sum 4.1.1.6
character 2.4.1.1
characteristic function of a set 1.1.2.1
C^*-hull 4.1.1.22
class NT
closed graph theorem 1.4.2.19
closed involutive subalgebra generated by 2.3.2.14, 2.3.2.15
closed involutive unital subalgebra generated by 2.3.2.14, 2.3.2.15
closed subalgebra generated by 2.2.1.9

closed unital subalgebra generated by 2.2.1.9
closed vector subspace generated by 1.1.5.5
C^*-module, Hilbert right 5.6.1.4
codimension 1.2.4.1
codomain NT
cokernel of a linear map 1.2.4.5
commutant 2.1.1.16
commutative 2.1.1.1
commutative monoid E 2.1.1
compact, relatively 1.1.2.9
compact operator 3.1.1.1
compatible, simultaneously 1.5.1.1
compatible (left and right) multiplications 1.5.1.1
complement of a subspace 1.2.5.3
complemented subspace 1.2.5.3
complete, $\mathbb{C}$-order 4.3.2.3
complete, order 1.7.2.1
complete norm 1.1.1.2
complete normed space 1.1.1.2
complete ordered set 1.7.2.1
completion of a normed algebra 2.2.1.13
completion of a normed space 1.3.9.1
completion of a pre-Hilbert space 5.1.1.7
complex algebra 2.1.1.1
complex Banach space 1.1.1.2
complex C^*-algebra 4.1.1.1
complex C^*-algebra, unital 4.1.1.1
complex Hilbert space 5.1.1.2
complex normed algebra 2.2.1.1
complex normed space 1.1.1.2
complex pre-Hilbert space 5.1.1.1
complex unital C^*-algebra 4.1.1.1
complex universal representation 5.4.2.16
complex W^*-algebra 4.4.1.1
complexification of algebras 2.1.5.7
complexification of Banach algebras 2.2.1.19
complexification of Hilbert spaces 5.3.1.8
complexification of involutive algebras 2.3.1.40

complexification of involutive vector spaces 2.3.1.38
complexification of right C^*-modules 5.6.1.6
complexification of vector spaces 2.1.5.1
composition of functors 1.5.2.1
composition of maps NT
compression of a representation 5.4.2.8
cone 1.3.7.4
cone, sharp 1.3.7.4
conjugacy class 2.2.2.7
conjugate exponent of 1.2.2.1
conjugate exponents 1.2.2.1
conjugate exponents, weakly 1.2.2.1
conjugate involution 2.3.1.3
conjugate linear map 1.3.7.10
conjugate number 1.1.1.1
continuous, order 1.7.2.3
convergence, radius of 1.1.6.22
convex 1.2.7.1
convex, absolutely 1.2.7.1
convex closed hull 1.2.7.6
convex closed hull, absolutely 1.2.7.6
convex hull 1.2.7.4
convex hull, absolutely 1.2.7.4
convolution 2.2.2.7, 2.2.2.10
$\mathbb{C}$-order complete 4.3.2.3
$\mathbb{C}$-order σ-complete 4.3.2.3
C^*-subalgebra 4.1.1.1
C^*-subalgebra, unital 4.1.1.1
C^*-subalgebra generated by 4.1.1.1
C^*-subalgebra generated by, hereditary 4.3.4.1
C^*-subalgebra generated by, unital 4.1.1.1
cyclic element 5.4.1.1
cyclic representation 5.4.1.1
cyclic vector 5.3.2.19, 5.4.1.1
cyclic vector associated to x' 5.4.1.2
decomposition, spectral 4.3.2.19, 5.3.4.7
degenerate algebra 2.1.1.1
derivative 1.1.6.24

diagonalization of u 5.5.6.1
differentiable 1.1.6.24
differential operator, adjoint 3.2.2.3
differential operator, selfadjoint 3.2.2.3
dimension, algebraic 1.1.2.18
dimension, Hilbert 5.5.2.2
Dirac measure 1.2.7.14
direct integral of E with respect to μ 5.5.2.19
direct sum 1.2.5.3
directed, downward 1.1.6.1
directed, upward 1.1.6.1
disjoint family of sets 1.2.3.9
distance of a point from a set 1.1.4.1
division algebra 2.1.2.1
domain NT
downward directed 1.1.6.1
dual, algebraic 1.1.1.1
dual of a Banach system 1.5.1.9
dual of a normed space 1.2.1.3
dual space 1.3.1.11
E-algebra 2.2.7.1
E-algebra, involutive 2.3.6.1
E-algebra, involutive unital 2.3.6.1
E-algebra, unital 2.2.7.1
E-algebras, homomorphism of 2.2.7.1
E-algebras, homomorphism of involutive 2.3.6.1
E-algebras, homomorphism of involutive unital 2.3.6.1
E-algebras, homomorphism of unital 2.2.7.1
E-C^*-algebra 5.6.1.10
E-C^*-algebra, unital 5.6.1.10
E-C^*-algebras, isomorphism of 5.6.1.10
eigenspace 3.1.4.1
eigenspace, algebraic 5.3.3.20
eigenvalue 3.1.4.1
eigenvector 3.1.4.1
eigenvector, algebraic 5.3.3.20
E-module 2.2.7.1
E-module, Hilbert 5.6.1.4

E-module, Hilbert right 5.6.1.4
E-module, inner-product right 5.6.1.1
E-module, involutive 2.3.6.1
E-module, involutive unital 2.3.6.1
E-module, semi-inner-product right 5.6.1.1
E-module, weak semi-inner-product right 5.6.1.1
E-module, unital 2.2.7.1
E-module, unital Hilbert 5.6.1.4
E-module, von Neumann (right) 5.6.3.2
E-modules, homomorphism of 2.2.7.1
E-modules, homomorphism of involutive 2.3.6.1
equicontinuous 1.1.2.14
equivalence class NT
equivalence class of a point NT
equivalence of GNS-triples 5.4.1.2
equivalence of representations 5.4.1.1
equivalence relation NT
equivalent GNS-triples 5.4.1.2
equivalent norms 1.1.1.2
equivalent representations 5.4.1.1
essential spectrum 3.1.3.24
E-submodule 2.2.7.1
Euclidean norm 1.1.5.2
evaluation 1.2.1.8
evaluation functor 1.5.2.1
evaluation operator of a normed space 1.3.6.3
E-valued spectral measure 4.3.2.16
exact set 1.7.2.12
expansion, Fourier 5.5.1.15
exponential function 2.2.3.5
exponents, conjugate 1.2.2.1
exponents, weakly conjugate 1.2.2.1
extreme point 1.2.7.9
face of a convex set 1.2.7.9
factorization of a linear map 1.2.4.6
faithful, order 4.2.2.18
faithful representation 5.4.1.1
family NT

family, absolutely summable 1.1.6.9
family, sum of a 1.1.6.2
family, summable 1.1.6.2
family of sets, disjoint 1.2.3.9
filter, lower section 1.1.6.1
filter, upper section 1.1.6.1
filter of cofinite subsets NT
finite-dimensional 1.1.2.18
$\mathcal{F}$-invariant 3.1.4.4
Fourier basis 5.6.3.11
Fourier expansion 5.5.1.15
Fourier integral 2.4.6.2
Fourier-Plancherel operator 5.5.4.1
Fourier set 5.6.3.11
Fourier transform 2.4.6.2
Fréchet-Riesz Theorem 5.2.5.2
Fredholm alternative 3.1.6.23
Fredholm operator 3.1.3.1
Fredholm operator, index of a 3.1.3.1
free ultrafilter NT
function NT
function, Baire 1.7.2.12
function, step NT
functional calculus 4.1.3
functor 1.5.2.1
functor, identity 1.5.2.1
functor, inclusion 1.5.2.16
functor, isometric 1.5.2.1
functor, quotient 1.5.2.17
functor, transpose of a 1.5.2.3
functor of (unital) Banach categories 1.5.2.1
functor of (unital) A-categories 1.5.2.1
functor of (left, right) A-modules 1.5.2.1
functors, composition of 1.5.2.1
Gelfand, Theorem of 2.2.5.4
Gelfand algebra 2.4.1.1
Gelfand algebra, involutive 2.4.2.1
Gelfand algebra, involutive unital 2.4.2.1

Gelfand algebra, spectrum of a 2.4.1.1
Gelfand algebra, unital 2.4.1.1
Gelfand C^* algebra 4.1.1.1
Gelfand–Mazur, Theorem of 2.2.5.5
Gelfand transform 2.4.1.2
GNS-construction 5.4.1.2
GNS triple of E associated to x' 5.4.1.2
GNS-triples, equivalence of 5.4.1.2
GNS-triples, equivalent 5.4.1.2
Gram–Schmidt orthonormalization 5.5.1.18
graph NT, 1.4.2.18
Green function 3.2.1.2
group, additive NT
Hahn–Banach Theorem 1.3.3.1
hereditary 4.3.4.1
hereditary C^*-subalgebra generated by 4.3.4.1
Hermitian sesquilinear map 2.3.3.3
Hilbert dimension 5.5.2.2
Hilbert E-module 5.6.1.4
Hilbert E-module, unital 5.6.1.4
Hilbert right C^*-module 5.6.1.4
Hilbert right E-module 5.6.1.4
Hilbert right E-modules, isomorphic 5.6.1.7
Hilbert space 5.1.1.2
Hilbert space, complex 5.1.1.2
Hilbert space, complexification 5.3.1.8
Hilbert space, involutive 5.5.7.1
Hilbert space, real 5.1.1.2
Hilbert space associated to A 5.4.2.3
Hilbert space associated to x' 5.4.1.2
Hilbert space associated to $(x'_\iota)_{\iota\in I}$ 5.4.2.2
Hilbert space of square summable sequences 5.1.2.3
Hilbert sum of a family of Hilbert spaces 5.1.3.1
Hilbert sum of a family of representations 5.4.2.1
Hölder inequality 1.2.2.5
homogeneous W^*-algebra 5.6.7.8
homomorphism of C^*-algebra 4.1.1.20
homomorphism of E-algebras 2.2.7.1

homomorphism of E-modules 2.2.7.1
homomorphism of involutive E–algebras 2.3.6.1
homomorphism of involutive E-modules 2.3.6.1
homomorphism of involutive unital E–algebras 2.3.6.1
homomorphism of unital E–algebras 2.2.7.1
hyperstonian space 1.7.2.12
ideal 2.1.1.1
ideal, left 2.1.1.1
ideal, maximal proper 2.1.1.1
ideal, maximal proper left 2.1.1.1
ideal, maximal proper right 2.1.1.1
ideal, proper 2.1.1.1
ideal, proper left 2.1.1.1
ideal, proper right 2.1.1.1
ideal, regular maximal proper 2.1.3.17
ideal, regular maximal proper left 2.1.3.17
ideal, regular maximal proper right 2.1.3.17
ideal, right 2.1.1.1
ideal generated by 2.1.1.2
idempotent 2.1.3.6
identical representation 5.5.1.23
identity functor 1.5.2.1
identity map NT
identity operator 1.2.1.3
iff NT
image of a linear map 1.2.4.5
imaginary part 1.1.1.1, 2.3.1.22
inclusion functor 1.5.2.16
inclusion map NT
index of a Fredholm operator 3.1.3.1
index of U 3.1.3.21
induced norm 1.1.1.2
infimum 1.7.2.1
infinite-dimensional 1.1.2.18
infinite matrix 1.2.3.1
injective NT
inner multiplication 1.5.1.1
inner-product 5.6.1.1

inner-product right E-module 5.6.1.1
integral of E with respect to μ, direct 5.5.2.19
interior point NT
invariant vector subspace 3.1.4.4
inverse of a bijective map NT
inverse of a morphism 1.5.1.6
inverse of an element in a unital algebra 2.1.2.4
inverse operators, principle of 1.4.2.4
invertible 1.5.1.5, 2.1.2.1
invertible, left 1.5.1.5
invertible, right 1.5.1.5
involution 2.3.1.1
involution, conjugate 2.3.1.3
involution of E^F, canonical 2.3.1.1
involutive algebra 2.3.1.3
involutive algebra, complexification of an 2.3.1.40
Proposition involutive algebra, strongly symmetric 2.3.1.26
involutive algebra, symmetric 2.3.1.26
involutive algebras, isomorphism of 2.3.1.3
involutive Banach algebra 2.3.2.1
involutive Banach space 2.3.2.1
involutive Banach unital algebra associated to 2.3.2.9
involutive E-algebra 2.3.6.1
involutive E-module 2.3.6.1
involutive Gelfand algebra 2.4.2.1
involutive Hilbert space 5.5.7.1
involutive map 2.3.1.1
involutive normed algebra 2.3.2.1
involutive normed space 2.3.2.1
involutive normed unital algebra associated to 2.3.2.9
involutive set 2.3.1.1
involutive space 2.3.1.1
involutive subalgebra generated by 2.3.1.18
involutive unital algebra associated to 2.3.1.9
involutive unital E-algebra 2.3.6.1
involutive unital E-module 2.3.6.1
involutive unital Gelfand algebra 2.4.2.1
involutive unital subalgebra generated by 2.3.1.18

involutive vector space 2.3.1.3
involutive vector spaces, isomorphism of 2.3.1.3
involutive vector subspace generated by 2.3.1.18
irreducible representation 5.4.1.1
irreducibly, acts 5.3.2.19
isometric Banach systems 1.5.2.1
isometric functor 1.5.2.1
isometric normed algebras 2.2.1.1
isometric normed spaces 1.2.1.12
isometric normed unital algebras 2.2.1.1
isometry of Hilbert spaces, partial 5.3.2.25
isometry of normed algebras 2.2.1.1
isometry of W^*-algebras 4.4.4.5
isometry of normed spaces 1.2.1.12
isometry of normed unital algebras 2.2.1.1
isomorphic algebras 2.1.1.6
isomorphic Hilbert right E-modules 5.6.1.7
isomorphic normed algebras 2.2.1.1
isomorphic normed spaces 1.2.1.12
isomorphic normed unital algebras 2.2.1.1
isomorphic unital algebras 2.1.1.6
isomorphism, algebra 2.1.1.6
isomorphism associated to a linear map, algebraic 1.2.4.6
isomorphism of E-C^*-algebras 5.6.1.10
isomorphism of involutive algebras 2.3.1.3
isomorphism of involutive vector spaces 2.3.1.3
isomorphism of normed algebras 2.2.1.1
isomorphism of normed spaces 1.2.1.12
isomorphism of normed unital algebras 2.2.1.1
kernel, Bergman 5.2.5.9
kernel of a linear map 1.2.4.5
Kronecker's symbol 1.2.2.6
lattice 1.7.2.1
lattice, vector 1.7.2.1
Laurent series 1.3.10.8, 1.3.10.9
Lax–Milgram Theorem 5.3.1.3
left Arens multiplication 1.5.2.10, 2.2.7.13
left carrier 4.3.3.1

left ideal 2.1.1.1
left ideal, maximal proper 2.1.1.1
left ideal, proper 2.1.1.1
left ideal, regular maximal proper 2.1.3.17
left ideal generated by 2.1.1.2
left invertible 1.5.1.5
left multiplication 1.5.1.1, 5.6.1.4
left shift 1.2.2.9, E 1.2.11
left (unital) A-module 1.5.1.10
linear form 1.1.1.1
linear form, positive 1.7.1.9
linear map, conjugate 1.3.7.10
lower bound 1.7.2.1
lower bounded operator 1.2.1.18
lower section filter 1.1.6.1
L^2-distributions, in the sense of 3.2.2.3
map NT
map, bilinear 1.2.9.1
map, bounded 1.1.1.2
map, conjugate linear 1.3.7.10
map, identity NT
map, inclusion NT
map, inverse of a bijective NT
map, involutive 2.3.1.1
map, nuclear 1.6.1.1
map, quotient 1.2.4.1
maps, composition of NT
matrix, infinite 1.2.3.1
maximal proper ideal 2.1.1.1
maximal proper ideal, regular 2.1.3.17
maximal proper left ideal 2.1.1.1
maximal proper left ideal, regular 2.1.3.17
maximal proper right ideal 2.1.1.1
maximal proper right ideal, regular 2.1.3.17
mean ergodic theorem 5.2.4.3
measure, Dirac 1.2.7.14
measure, E-valued spectral 4.3.2.19
measure, Radon NT

measure of x, spectral 4.3.2.19
measure space, σ-finite 3.1.6.14
metric of a normed space, canonical 1.1.1.2
module 2.2.7.1
module, involutive 2.3.6.1
module, involutive unital 2.3.6.1
module, unital 2.2.7.1
modules, homomorphism of 2.2.7.1
modules, homomorphism of involutive 2.3.6.1
modulo NT
modulus 4.2.5.1, 4.4.3.5
monoid E 2.1.1
monoid, commutative E 2.1.1
morphism 1.5.1.1
morphism, inverse of a 1.5.1.6
multipliable sequence 2.2.4.33
multiplication 2.1.1.1
multiplication, $(\mathcal{A},\mathcal{B},\mathcal{C})$- 1.5.1.1
multiplication, compatible (left and right) 1.5.1.1
multiplication, inner 1.5.1.1
multiplication, left 1.5.1.1, 5.6.1.4
multiplication, left (right) Arens 1.5.2.10, 2.2.7.13
multiplication, right 1.5.1.1, 5.6.1.1
multiplication operator 2.2.2.22
multiplicity 3.1.4.1
multiplicity, algebraic 5.3.3.20
negative 1.7.1.1
negative part 4.2.2.9, 4.2.8.13
Nikodym's boundedness theorem 4.3.2.14
nilpotent 2.1.1.1
non-degenerate representation 5.4.1.1
non-degenerately, acts 5.3.2.19
norm 1.1.1.2
norm, complete 1.1.1.2
norm, Euclidean 1.1.5.2
norm, induced 1.1.1.2
norm, quotient 1.2.4.2
norm, supremum 1.1.2.2, 1.1.5.2

norm of an operator 1.2.1.3
norm of a pre-Hilbert space, canonical 5.1.1.2
norm of $\mathcal{L}(E, F)$, canonical 1.2.1.9
norm topology 1.1.1.2
normal 2.3.1.3
normed algebra 2.2.1.1
normed algebra, completion of a 2.2.1.13
normed algebra, complex 2.2.1.1
normed algebra, involutive 2.3.2.1
normed algebra, quasiunital 2.2.1.15
normed algebra, real 2.2.1.1
normed algebras, isometric 2.2.1.1
normed algebras, isometry of 2.2.1.1
normed algebras, isomorphic 2.2.1.1
normed algebras, isomorphism of 2.2.1.1
normed space 1.1.1.2
normed space, bidual of a 1.3.6.1
normed space, complete 1.1.1.2
normed space, completion of a 1.3.9.1
normed space, complex 1.1.1.2
normed space, involutive 2.3.2.1
normed space, ordered 1.7.1.4
normed space, real 1.1.1.2
normed spaces, isometric 1.2.1.12
normed spaces, isometry of 1.2.1.12
normed spaces, isomorphic 1.2.1.12
normed spaces, isomorphism of 1.2.1.12
normed unital algebra 2.2.1.1
normed unital algebras, isometric 2.2.1.1
normed unital algebras, isometry of 2.2.1.1
normed unital algebras, isomorphic 2.2.1.1
normed unital algebras, isomorphism of 2.2.1.1
norms, equivalent 1.1.1.2
nuclear map 1.6.1.1
number, cardinal NT
number, ordinal NT
object of a Banach system 1.5.1.1
onto NT

open mapping principle 1.4.2.3
operator 1.2.1.3
operator, adjoint 5.3.1.4
operator, adjoint differential 3.2.2.3
operator, bounded 1.2.1.3
operator, compact 3.1.1.1
operator, Fourier-Plancherel 5.5.4.1
operator, Fredholm 3.1.3.1
operator, identity 1.2.1.3
operator, index of a Fredholm 3.1.3.1
operator, lower bounded 1.2.1.18
operator, multiplication 2.2.2.22
operator, order of an 3.1.3.18
operator, selfadjoint differential 3.2.2.3
operators, principle of inverse 1.4.2.4
order complete 1.7.2.1
order continuous 1.7.2.3
order faithful 4.2.2.18
order of a pole 1.3.10.9
order relation of a C^*-algebra, canonical 4.2.1.2
order summable 1.7.2.10
order σ-complete 1.7.2.1
order σ-continuous 1.7.2.3
order σ-faithful 4.2.2.18
ordered Banach space 1.7.1.4
ordered normed space 1.7.1.4
ordered set, complete 1.7.2.1
ordered set, totally NT
ordered set, σ-complete 1.7.2.1
ordered vector space 1.7.1.1
ordinal number NT
orthogonal 5.2.2.1
orthogonal projection 4.1.2.18, 5.2.3.2
orthogonal set of A 5.2.2.1
orthogonal sets 5.2.2.1
orthogonal vectors 5.2.2.1
orthonormal basis 5.5.1.1
orthonormal basis of $\ell^2(T)$, canonical 5.5.3.1

orthonormal family 5.5.1.1
orthonormal set 5.5.1.1
orthonormalization, Gram–Schmidt 5.5.1.18
parallelogram law 2.3.3.2
Parseval's Equation 5.5.1.15
partial isometry of Hilbert spaces 5.3.2.25
partition of a set NT
p–norm 1.1.2.5, 1.1.5.2
point, adherent NT
point, extreme 1.2.7.9
point, interior NT
point of adherence NT
point spectrum 3.1.4.1
polar 1.3.5.1
polar representation 4.2.6.9, 4.4.3.1, 4.4.3.5
polarization identity 2.3.3.2
pole (of order) 1.3.10.9
positive 1.7.1.1, 2.3.3.3, 2.3.4.1
positive linear form 1.7.1.9, 2.3.4.1
positive part 4.2.2.9, 4.2.8.13
power series 1.1.6.22
precompact 1.1.2.9
predual of a Banach space 1.3.1.11
predual of a W^*–algebra 4.4.1.1, 4.4.4.4
pre–Hilbert space 5.1.1.1
pre–Hilbert space, canonical norm of a 5.1.1.2
pre–Hilbert space, completion of a 5.1.1.7
pre–Hilbert space, complex 5.1.1.1
pre–Hilbert space, real 5.1.1.1
prepolar 1.3.5.1
pretranspose of an operator 1.3.4.9, 4.4.4.8
principal part 1.3.10.8, 1.3.10.9
principle of inverse operators 1.4.2.4
principle of open mapping 1.4.2.3
principle of uniform boundedness 1.4.1.2
product 2.1.1.1
product, C^*–direct 4.1.1.6
product of a family of sets NT

product of a sequence 2.2.4.33
product, scalar 5.1.1.1
product associated to f, scalar 5.1.2.9
projection 1.2.5.7
projection, orthogonal 4.1.2.18, 5.2.3.2
projection of the tridual of E, canonical 1.3.6.19
proper ideal 2.1.1.1
proper ideal, maximal 2.1.1.1
proper ideal, regular maximal 2.1.3.17
proper left ideal 2.1.1.1
proper left ideal, maximal 2.1.1.1
proper left ideal, reguar maximal 2.1.3.17
proper right ideal 2.1.1.1
proper right ideal, maximal 2.1.1.1
proper right ideal, regular maximal 2.1.3.17
pure state 2.3.5.1
pure state space 2.3.5.1
purely real C^*-algebra 4.1.1.8
Pythagoras' Theorem 5.2.2.3
quadratic form, associated 2.3.3.1
quadratic map, associated 2.3.3.1
quasinilpotent 2.2.4.20
quasiunital 2.2.1.15
quaternion 2.1.4.17
quotient functor 1.5.2.17
quotient map NT, 1.2.4.1
quotient norm 1.2.4.2
quotient space 1.2.4.2
quotient A-category 1.5.2.17
quotient A-module 1.5.2.17
Raabe's ratio test 2.2.3.11
radical 2.1.3.18
radius of convergence 1.1.6.22
Radon measure NT
Radon-Nikodym Theorem 4.4.3.15
range of values NT
real algebra 2.1.1.1
real Banach space 1.1.1.2

real C^*-algebra 4.1.1.1
real C^*-algebra, purely 4.1.1.8
real C^*-algebra, unital 4.1.1.1
real Hilbert space 5.1.1.2
real normed space 1.1.1.2
real part 1.1.1.1, 2.3.1.3
real pre-Hilbert space 5.1.1.1
real W^*-algebra 4.4.1.1
reduces u 5.2.3.11
reflexive 1.3.8.1
regular maximal proper ideal 2.1.3.17
regular maximal proper left ideal 2.1.3.17
regular maximal proper right ideal 2.1.3.17
relatively compact 1.1.2.9
representation 5.4.1.1
representation, associated to A 5.4.2.3
representation, associated to x' 5.4.1.2
representation, associated to $(x_\iota)_{\iota\in I}$ 5.4.2.2
representation, complex universal 5.4.2.6
representation, compression of a 5.4.2.8
representation, cyclic 5.4.1.1
representation, faithful 5.4.1.1
representation, identical 5.5.1.23
representation, irreducible 5.4.1.1
representation, non-degenerate 5.4.1.1
representation, unital 5.4.1.1
representation, universal 5.4.2.3
representation, 0- 5.4.1.1
representations, equivalence of 5.4.1.1
representations, equivalent 5.4.1.1
representations, Hilbert sum of 5.4.2.1
residue 1.3.10.8, 1.3.10.9
resolvent 2.1.3.1
resolvent equation 2.1.3.9
Riesz, theorem of 2.2.5.1
right Arens multiplication 1.5.2.10, 2.2.7.13
right carrier 4.3.3.1
right C^*-module, Hilbert 5.6.1.4

right E-module, Hilbert 5.6.1.4
right E-module, inner-product 5.6.1.1
right E-module, semi-inner-product 5.6.1.1
right E-module, von Neumann 5.6.3.2
right E-module, weak semi-inner-product 5.6.1.1
right ideal 2.1.1.1
right ideal, maximal proper 2.1.1.1
right ideal, proper 2.1.1.1
right ideal, regular maximal proper 2.1.3.17
right ideal generated by 2.1.1.2
right invertible 1.5.1.5
right multiplication 1.5.1.1, 5.6.1.1
right shift 1.2.2.9, E 1.2.11
right (unital) A-module, 1.5.1.10
right W^*-module, von Neumann 5.6.3.2
scalar 1.1.1.1
scalar product 5.1.1.1
scalar product associated to f 5.1.2.9
scalar product of $\mathbb{K}^n$, canonical 5.1.2.4
Schwartz space of rapidly decreasing C^∞-functions 2.4.6.5
Schwarz inequality 2.3.3.9, 5.1.1.2
section filter, lower 1.1.6.1
section filter, upper 1.1.6.1
selfadjoint 2.3.1.1
selfadjoint differential operator 3.2.2.3
self-dual 5.6.2.2
self-normal 2.3.1.3
semi-inner-product right E module 5.6.1.1
semi-inner-product right E-module, weak 5.6.1.1
seminorm 1.1.1.2
semi-simple algebra 2.1.3.18
separating vector 5.3.2.19, 5.4.4.1
sequence NT
series, Laurent 1.3.10.8, 1.3.10.9
series, power 1.1.6.22
sesquilinear form 2.3.3.1
sesquilinear form, adjoint 2.3.3.1
sesquilinear map 2.3.3.1

sesquilinear map, adjoint 2.3.3.1
sesquilinear map, Hermitian 2.3.3.3
set, Baire 1.7.2.12
set, bounded 1.1.1.2
set, complete ordered 1.7.2.1
set, exact 1.7.2.12
set, partition of a NT
set, totally ordered NT
set, μ-null NT
set, σ-complete ordered 1.7.2.1
sharp cone 1.3.7.4
shift, left 1.2.2.9
shift, right 1.2.2.9
simple C^*-algebra 4.3.5.1
simultaneously compatible 1.5.1.1
space, Banach 1.1.1.2
space, bidual of a normed 1.3.6.1
space, complete normed 1.1.1.2
space, completion of a normed 1.3.9.1
space, complex Banach 1.1.1.2
space, complex Hilbert 5.1.1.2
space, complex normed 1.1.1.2
space, complex pre-Hilbert 5.1.1.1
space, dual 1.3.1.11
space, Hilbert 5.1.1.2
space, *hyperstonian* 1.7.2.12
space, involutive 2.3.1.1
space, involutive Banach 2.3.2.1
space, involutive normed 2.3.2.1
space, involutive vector 2.3.1.3
space, normed 1.1.1.2
space, ordered Banach 1.7.1.4
space, ordered normed 1.7.1.4
space, ordered vector 1.7.1.1
space, pre-Hilbert 5.1.1.1
space, pure state 2.3.5.1
space, quotient 1.2.4.2
space, real Banach 1.1.1.2

space, real Hilbert 5.1.1.2
space, real normed 1.1.1.2
space, real pre-Hilbert 5.1.1.1
space, state 2.3.5.1
space, Stone 1.7.2.12
space, subspace of a normed 1.1.1.2
space, vector 1.1.1.1
space, σ-Stone 1.7.2.12
space of square summable sequences, Hilbert 5.1.2.3
spaces, isometric normed 1.2.1.12
spaces, isometry of normed 1.2.1.12
spaces, isomorphic normed 1.2.1.12
spaces, isomorphism of involutive vector 2.3.1.3
spaces, isomorphism of normed 1.2.1.12
spectral decomposition 4.3.2.19, 5.3.4.7
spectral measure, E-valued 4.3.2.16
spectral measure of x 4.3.2.19
spectral radius 2.1.3.1
spectrum, essential 3.1.3.24
spectrum, point 3.1.4.1
spectrum of an element 2.1.3.1
spectrum of a Gelfand algebra 2.4.1.1
square summable sequences, Hilbert space of 5.1.2.3
state 2.3.5.1
state, pure 2.3.5.1
state space 2.3.5.1
state space, pure 2.3.5.1
step function NT
Stone space 1.7.2.12
strongly symmetric involutive algebra 2.3.1.26
subalgebra 2.1.1.1
subalgebra, unital 2.1.1.3
subalgebra generated by 2.1.1.4
subalgebra generated by, involutive 2.3.1.18
subspace, complemented 1.2.5.3
subspace generated by, closed vector 1.1.5.5
subspace of a normed space 1.1.1.2
sum, C^* direct 4.1.1.6

sum, direct 1.2.5.3
sum of a family 1.1.6.2
sum of a family in F_A 5.6.3.6
sum of representations, Hilbert 5.4.2.1
summable, absolutely 1.1.6.9
summable, order 1.7.2.10
summable family 1.1.6.2
summable in F_A 5.6.3.6
support of a function NT
support of a Radon measure NT
supremum 1.7.2.1
supremum norm 1.1.2.2, 1.1.5.2
surjective NT
symbol, Kronecker's 1.2.2.6
symmetric involutive algebra 2.3.1.26
symmetric involutive algebra, strongly 2.3.1.26
theorem, mean ergodic 5.2.4.3
Theorem of Alaoglu–Bourbaki 1.2.8.1
Theorem of Banach 1.3.1.2
Theorem of Banach–Steinhaus 1.4.1.2
Theorem of closed graph 1.4.2.19
Theorem of Fréchet–Riesz 5.2.5.2
Theorem of Gelfand 2.2.5.4
Theorem of Gelfand–Mazur 2.2.5.5
Theorem of Hahn–Banach 1.3.3.1
Theorem of Laurent 1.3.10.8
Theorem of Lax–Milgram 5.3.1.3
Theorem of Liouville 1.3.10.6
Theorem of Krein–Milman 1.3.1.10
Theorem of Krein–Šmulian 1.3.7.3
Theorem of Minkowski 1.1.3.4
Theorem of Murray 1.2.5.8
Theorem of Pythagoras 5.2.2.3
Theorem of Radon–Nikodym 4.4.3.15
Theorem of Riesz 2.2.5.1
Theorem of Weierstrass–Stone 1.3.5.16
topological cardinality NT
topological zero-divisor 2.2.4.24

topology, norm 1.1.1.2
topology, weak 1.3.6.9
totally ordered set NT
transpose kernel 3.1.6.5
transpose of a functor 1.5.2.3
transpose of an operator 1.3.4.1
transpose unital category of $\mathcal{L}$ 1.5.2.2
transposition functor of $\mathcal{L}$ 1.5.2.2
triangle inequality 1.1.1.2
tridual of a Banach system 1.5.1.9
tridual of a normed space 1.3.6.1
type I W^*-algebra 5.6.7.11
u-invariant 3.1.4.4
ultrafilter, free NT
uniform boundedness, principle of 1.4.1.2
unit 1.5.1.1, 1.5.1.4, 2.1.1.1
unit, approximate 2.2.1.15
unit ball 1.1.1.2
unit of an inner multiplication 1.5.1.1
unital algebra 2.1.1.3
unital algebra, normed 2.2.1.1
unital algebra associated to 2.1.1.8
unital algebra associated to, involutive 2.3.1.9
unital algebra homomorphism 2.1.1.6
unital algebra isomorphism 2.1.1.6
unital algebras, isometric normed 2.2.1.1
unital algebras, isometry of normed 2.2.1.1
unital algebras, isomorphic 2.1.1.6
unital algebras, isomorphic normed 2.2.1.1
unital algebras, isomorphism of normed 2.2.1.1
unital Banach algebra 2.2.1.1
unital Banach algebra associated to 2.2.1.4
unital Banach algebras, isomorphism of 2.2.1.1
unital Banach category 1.5.1.1
unital Banach subalgebra generated by 2.2.1.9
unital C^*-algebra 4.1.1.1
unital C^*-algebra, complex 4.1.1.1
unital C^*-algebra, real 4.1.1.1

unital C^*-algebra, associated to a C^*-algebra 4.1.1.13
unital C^*-subalgebra 4.1.1.1
unital C^*-subalgebra generated by 4.1.1.1
unital E-algebra 2.2.7.1
unital E-algebra, involutive 2.3.6.1
unital E-C^*-algebra 5.6.1.10
unital E-module 2.2.7.1
unital E-module, involutive 2.3.6.1
unital Gelfand algebra 2.4.1.1
unital Gelfand algebra, involutive 2.4.2.1
unital Hilbert E-module 5.6.1.4
unital involutive algebra associated to 2.3.1.9
unital involutive Banach algebra associated to 2.3.2.9
unital involutive normed algebra associated to 2.3.2.9
unital left A-module 1.5.1.10
unital normed algebra associated to 2.2.1.4
unital normed algebras, isomorphism of 2.2.1.1
unital representation 5.4.1.1
unital right A-module 1.5.1.10
unital subalgebra 2.1.1.3
unital subalgebra generated by 2.1.1.4
unital subalgebra generated by, involutive 2.3.1.18
unital W^*-subalgebra 4.4.4.5
unital W^*-subalgebra generated by 4.4.4.5
unital A-category 1.5.1.14
unital A-module 1.5.1.12
unital (A, Δ)-module 1.5.1.12
unitary 2.3.1.3
universal representation 5.4.2.3
universal representation, complex 5.4.2.6
upper bound 1.7.2.1
upper section filter 1.1.6.1
upward directed 1.1.6.1
vector, cyclic 5.3.2.19, 5.4.1.1
vector, separating 5.3.2.19, 5.4.1.1
vector associated to x', cyclic 5.4.1.2
vector lattice 1.7.2.1
vector space 1.1.1.1

vector space, involutive 2.3.1.3
vector spaces, isomorphisms of involutive 2.3.1.3
Volterra integral equation 2.2.4.22
von Neumann (right) E-module 5.6.3.2
von Neumann (right) W^*-module 5.6.3.2
W^*-algebra 4.4.1.1
W^* algebra, complex 4.4.1.1
W^*-algebra, homogeneous 5.6.7.8
W^*-algebra, predual of a 4.4.1.1, 4.4.4.4
W^*-algebra, real 4.4.1.1
W^*-algebra, type I 5.6.7.11
W^*-algebras, isometry of 4.4.4.5
W^*-homomorphism 4.4.4.5
W^*-module, von Neumann (right) 5.6.3.2
W^*-subalgebra 4.4.4.5
W^*-subalgebra, unital 4.4.4.5
W^*-subalgebra generated by 4.4.4.5
W^*-subalgebra generated by, unital 4.4.4.5
weak semi-inner-product right E-module 5.6.1.1
weak topology 1.3.6.9
weakly conjugate exponents 1.2.2.1
zero-divisor 2.1.1.1
zero-divisor, topological 2.2.4.24
Λ-categories, functor of (unital) 1.5.2.1
Λ-category 1.5.1.14
Λ-category, quotient 1.5.2.17
Λ-category, unital 1.5.1.14
Λ-module 1.5.1.12
Λ-module, left (right) 1.5.1.10
Λ-module, quotient 1.5.2.17
Λ-module, unital 1.5.1.12
Λ-module, unital left (right) 1.5.1.10
Λ-modules, functor of left (right) 1.5.2.1
Λ-subcategory 1.5.2.16
Λ-submodule 1.5.2.16
(Λ, Δ)-module 1.5.1.12
(Λ, Δ)-module, unital 1.5.1.12
μ-null set NT

σ–complete, $\mathbb{C}$–order 4.3.2.3
σ–complete order 1.7.2.1
σ–complete ordered set 1.7.2.1
σ–continuous, order 1.7.2.3
σ–faithful, order 4.2.2.18
σ–finite measure space 3.1.6.14
σ–Stone space 1.7.2.12
0–representation 5.4.1.1

Symbol Index

NT means Notation and Terminology

$|a|$ 4.4.3.5
a^* 2.3.1.30
A^* 2.3.1.1
$A^\perp$ 5.2.2.1
$\overline{A}$ NT
$\mathring{A}$ NT
${}^\circ A, A^\circ$ 1.3.5.1
A^c, A^{cc}, A^{ccc} 2.1.1.16
$\mathcal{A}', \mathcal{A}'', \mathcal{A}'''$ 1.5.1.9
$A_{A'}$ 1.3.6.9
aa', $a'a$ 2.2.7.8
ab 2.1.4.23
ax 2.2.7.23
$a'x''$ 2.2.7.11
$a''x'$ 1.5.2.8
$\widetilde{(a,\xi)}$ 5.6.3.2
$\widetilde{(a,\xi,\eta)}$ 5.6.3.2
$\mathcal{A}/\mathcal{B}$ 1.5.2.17
$A+B$ 1.2.4.1
$A\backslash B$ NT
$A\triangle B$ NT
$A\times B$ NT
$A+z$ 1.2.4.1
$\mathcal{B}$ 1.1.2.4
$\mathbb{C}$ NT
c 1.1.2.3, 2.1.4.3
c_0 1.1.2.3, 2.1.4.3
$c(T)$ 1.1.2.3, 2.1.4.3
$c_0(T)$ 1.1.2.3, 2.1.4.3
$\mathcal{C}(T)$ 1.1.2.4, 2.1.4.4
$\mathcal{C}(T,E)$ 1.1.2.8
$\mathcal{C}_0(T)$ 1.2.2.10, 2.1.4.4

Card NT
Coker 1.2.4.5
d_A 1.1.4.1
Det NT
Dim 1.1.2.18
$\mathcal{D}(k,\mu,\nu)$ 3.1.6.15
$\mathcal{D}_0(k,\nu)$ 3.1.6.1
E' 1.2.1.3
E'' 1.3.6.1
E''' 1.3.6.1
$\overset{\circ}{E}$ 2.1.5.1, 2.1.5.7, 2.3.1.38, 2.3.1.40, 5.3.1.8
e_A 1.1.2.1
e_t 1.1.2.1
e_A^T 1.1.2.1
e_t^T 1.1.2.1
e^x 2.2.3.5
$E_a(u)$ 3.1.3.18
$E_b(u)$ 3.1.3.18
$E_{m,n}$ 2.1.4.23, 2.3.1.30
$E_{n,n}$ 2.1.4.24, 2.3.1.31, 5.6.6.1
E^T 1.1.2.1
$E^{(T)}$ 1.1.2.1
E^π 1.7.2.3
E^σ 1.7.2.3
$\ddot{E}$ 4.4.4.4
E_+ 1.7.1.1, 4.2.1.1
$E^\#$ 1.1.1.2
$E_+^\#$ 1.7.1.4
$E''_\dashv$, $E''_\vdash$ 2.2.7.15
$E(x)$ 2.3.2.15
$E(x,1)$ 2.3.2.15
$E \overset{x}{\to} F$ 1.5.1.1
$E \underset{A}{\overset{x}{\to}} F$ 1.5.1.1
E/F 1.2.4.1, 2.1.1.13, 2.3.1.42
$\dot{F}$ 4.3.1.7
$F(E)$ 5.5.7.10

f' 1.1.6.24
$\widetilde{f}$ 2.3.3.1
$\mathcal{F}_A$ 1.2.6.1
$\mathcal{F}(E)$ 3.1.3.1
$\mathcal{F}(E,F)$ 3.1.3.1
$\mathfrak{F}_I$ 1.1.6.1
$f|S$ NT
$f(a,\cdot)$ NT
$f(\cdot,b)$ NT
$f(A)$ NT
$f(x)$ NT, 4.1.3.1, 4.1.3.2, 4.3.2.5
f^{-1} NT
$\overset{-1}{f}(B)$ NT
$\overset{-1}{f}(y)$ NT
$f: X \to Y$ NT
$f: X \to Y,\ x \mapsto T(x)$ NT
$F[s,t]$ NT
$F[t]$ NT
$F \oplus G$ 1.2.5.3
$\{f = g\}$ NT
$\{f \neq g\}$ NT
$\{f > \alpha\}$ NT
$g \circ f$ NT, 1.5.2.1
$\mathfrak{H}_A$ 1.7.2.3
$\mathbb{H}$ 2.1.4.17, 2.3.1.46, 4.1.1.31
$\overset{\circ}{H}$ 5.6.1.6
$\widehat{H}$ 5.6.2.2
$\ddot{H}$ 5.6.3.2
$\dddot{H}$ 5.6.3.2
im 1.1.1.1, 2.3.1.22
Im 1.2.4.5
Ind u 3.1.3.1
Ind U 3.1.3.21
j_E 1.3.6.3, 1.5.2.1
j_{EF} 1.5.2.1
$\mathbb{K}$ 1.1.1.1
$\mathbb{K}1$ 2.1.1.3

$\mathbb{K}[\cdot], \mathbb{K}[\cdot,\cdot]$ 1.1.1.1
$\mathcal{K}(E)$ 3.1.1.1
$\mathcal{K}(E,F)$ 3.1.1.1
$\mathcal{K}_E(G,H)$ 5.6.5.3
$\mathcal{K}_E(H)$ 5.6.5.3
kx 3.1.6.1, 3.1.6.15
k' 3.1.6.5
k^* 3.1.6.5
$k^\nu x$ 3.1.6.1, 3.1.6.15
$\overset{\cap}{k}$ 1.2.3.1
$\overset{\cup}{k}$ 1.2.3.1
Ker 1.2.4.5
$\mathcal{L}$ 1.2.1.3, 1.5.1.1, 2.1.4.6
$\mathcal{L}_\Omega$ 1.5.1.1
$\mathcal{L}_f$ 1.2.1.3
$\mathcal{L}^1$ 1.6.1.1, 1.6.1.3
$\mathcal{L}^1_\Omega$ 1.6.1.13
$\mathcal{L}_E(G,H)$ 5.6.1.7
$\widehat{\mathcal{L}}_E(G,H)$ 5.6.2.2
$\mathcal{L}_E(H)$ 5.6.1.7
ℓ^p 1.1.2.5
$\ell^2(I,F)$ 5.6.4.2
$\ell^p(T)$ 1.1.2.5
$\ell^2(T)$ 5.5.7.1
ℓ^0 1.1.2.3
$\ell^0(T)$ 1.1.2.3
ℓ^∞ 1.1.2.2, 2.1.4.3
$\ell^\infty(T)$ 1.1.2.2, 2.1.4.3
$\ell^{p,q}(S,T)$ 1.2.3.2
$\ell_0^{\infty,q}(S,T)$ 1.2.3.2
log 2.2.3.9, 4.2.4.4
$\mathcal{M}_b$ 1.1.2.26
$\mathcal{M}_{F,F}$ 5.6.4.20
$\mathcal{M}_{F,G}$ 5.6.4.19
$\mathbb{N}$ NT
$\mathbb{N}_n$ 1.1.3.3

$N_{x'}$ 2.3.4.1
No 2.3.1.3
$\mathcal{N}_{F,F}$ 5.6.4.16
$\mathcal{N}_{F,G}$ 5.6.4.15, 5.6.4.19
Pr 4.1.2.18
$\mathfrak{P}$ 1.1.2.1
$\mathfrak{P}_f$ 1.1.2.1
$\mathbb{Q}$ NT
$\mathbb{R}$ NT
$\overline{\mathbb{R}}$ NT
re 1.1.1.1, 2.3.1.3
Re 2.3.1.1
Re $E^{\#}$ 2.3.2.1
$r(x)$, $r_E(x)$ 2.1.3.1
Sn 2.3.1.3
$\mathcal{S}(\mathbb{R}^n)$ 2.4.6.5
Supp f NT
Supp μ NT
$\mathbf{T}$ 2.4.4.1
u' 1.3.4.1
u'' 1.3.6.15
u^* 2.3.1.1, 3.2.2.3, 5.3.1.4, 5.6.1.8. 5.6.4.12, 5.6.4.14
$\overset{\circ}{u}$ 2.1.5.11, 2.3.1.41
Un 2.3.1.3
$U_\alpha(t)$ 1.1.1.2
$U_\alpha^T(t)$ 1.1.1.2
vu 5.6.4.12, 5.6.4.14
$X/\sim$ NT
x^n 2.1.1.1
x^α 4.2.4.1, 4.2.4.4
x^{-n} 2.1.2.5
x^0 2.1.1.3
x^{-1} 1.5.1.6, 2.1.2.4
x^* 2.3.1.1
$\overline{x}$ 5.5.7.1
$\widehat{x}$, $\overset{E}{\widehat{x}}$ 2.4.1.1
$\overset{\vee}{x}$ 2.3.5.1
$|x|$ 4.2.5.1

x^+, x^- 4.2.2.9
x'^+, x'^- 4.2.8.13
$(x_\iota)_{\iota\in I}$ NT
xa 2.2.7.23
$x'a''$ 1.5.2.8
$x''a'$ 2.2.7.11
xx' , $x'x$ 1.5.2.5
$x\xi$ 5.6.1.4
$\langle x, x'\rangle$, $\langle x', x\rangle$ 1.2.1.3
$\langle\cdot|x\rangle y$ 5.3.2.12
$x * y$ 2.2.2.7
$x'' \dashv y''$, $x'' \vdash y''$ 1.5.2.10, 2.2.7.13
$x \otimes y$ 3.1.1.25
$\overbrace{(x', \xi, \eta)}$ 5.6.3.1
$\{x \mid P(x)\}$ NT
$\{x \in X \mid P(x)\}$ NT
$\langle\cdot, x'\rangle y$ 1.3.3.3
$\mathbb{Z}$ NT
$z + A$ 1.2.4.1
$\overline{\alpha}$ 1.1.1.1
$|\alpha|$ 1.1.1.1
αA 1.2.4.1
$\binom{\alpha}{n}$ 2.2.3.10
$]\alpha, \beta[$, $]\alpha, \beta]$, $[\alpha, \beta[$, $[\alpha, \beta]$ NT
Δ NT
δ_{st} 1.2.2.6
δ_t 1.2.7.14
$\delta(s, t)$ 1.2.2.6
$|\mu|$ NT
$\mu * \nu$ 2.2.2.10
$\vec{\xi}$ 5.5.7.1
ξx 5.6.1.1
$\xi\langle\cdot|\eta\rangle$ 5.6.5.1
π_A 5.2.1.2

$\prod_{\iota\in E} E_\iota$ 2.1.4.1, 2.3.1.4
$\prod_{\iota\in I} X_\iota$ NT

$\prod_{n\in A} x_n$, $\prod_{n\in\mathbb{N}} x_n$ 2.2.4.33
$\sigma(E)$, $\sigma_0(E)$ 2.4.1.1
$\sigma(T)$ 2.4.4.1
$\sigma(x)$, $\sigma_E(x)$ 2.1.3.1
$\sigma^e(u)$ 3.1.3.24
$\sigma_p(u)$ 3.1.4.1

$\sum_{t\in T} x(t)$ 1.1.2.1
$\sum_{n=p}^{q} x_n$ 1.1.6.2
$\sum_{\iota\in I} x_\iota$ 1.1.6.2
$\sum_{\iota\in I}\langle\cdot, x'_\iota\rangle y_\iota$ 1.3.3.3
$\sum_{n=0}^{\infty} \alpha^n x_n$ 1.1.6.22
$\sum_{\iota\in I}^{A} \xi_\iota$ 5.6.3.6
$\sum_{\iota\in I}^{\leq} x_\iota$ 1.7.2.10
$\tau(E)$, $\tau_0(E)$ 2.3.5.1
1 1.2.1.3, 2.1.1.3
1_E 1.2.1.3, 1.5.1.5
$\frac{1}{\alpha - x}$ 2.2.3.5
$[1+x]^\alpha$ 2.2.3.13
0^0 2.1.1.3
$+$ 1.2.4.1
$\times$ NT
$\setminus$ NT
$\langle\cdot,\cdot\rangle$ 1.2.1.3
$\langle\cdot|\cdot\rangle$ 5.1.1.1, 5.6.1.1
$\{\cdot|\cdot\}$ NT
$\{\cdot = \cdot\}$, $\{\cdot \neq \cdot\}$, $\{\cdot > \cdot\}$ NT
$\equiv \pmod p$ NT
$\dashv, \vdash$ 1.5.2.10, 2.2.7.13
$\perp$ 5.2.2.1
$\vee, \wedge$ 1.7.2.1
$\|\cdot\|$ 1.1.1.2, 1.2.1.3, 5.1.1.1, 5.6.1.4
$\|\cdot\|_p$ 1.1.2.5
$\|\cdot\|_1$ 1.6.1.1

$\|\cdot\|_0$ 1.1.2.3
$\|\cdot\|_\infty$ 1.1.2.2
$\forall, \exists, \exists!$ NT
$\circ$ NT, 1.5.2.1
$\bigoplus$ 5.1.3.1, 5.1.3.3, 5.6.4.1
$\overset{w}{\bigoplus}$ 5.6.4.6
$\oplus$ 1.2.5.3
$[\cdot,\cdot],]\cdot,\cdot[, [\cdot,\cdot[,]\cdot,\cdot]$ NT
$\int x\, d\mu$ 4.3.2.17